Retrocells Redux

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Bimodules

•The bicategory $\mathcal{B}im$ has rings $R, S, T, \ldots$ as objects, bimodules $M : R \xrightarrow{\bullet} S$ as 1-cells, and S-R-linear maps as 2-cells
Composition is $\otimes$

$$\begin{tikzcd} & S \arrow[dr, "N"] & \\ R \arrow[ur, "M"] \arrow[rr, "N \otimes_S M"'] & & T \end{tikzcd}$$

• $\mathcal{B}im$ is biclosed, $\otimes$ has right adjoints in each variable

$$\frac{M \longrightarrow N \obslash_T P}{\dfrac{N \otimes_S M \longrightarrow P}{N \longrightarrow P \oslash_R M}}$$

$N \obslash_T P = Hom_T(N, P)$, $P \oslash_R M = Hom_R(M, P)$

Biclosed

Many bicategories are biclosed

- $\mathcal{B}im$: Rings, bimodules, linear maps
- $\mathcal{P}rof$: Categories, profunctors, natural transformations
- $\mathbf{V}$-$\mathcal{P}rof$: $\mathbf{V}$
  – with colimits preserved by $\otimes$
  – biclosed
  – limits
- $\mathcal{S}pan(\mathbf{A})$: $\mathbf{A}$ with pullbacks and locally cartesian closed
- $\mathcal{M}at(\mathbf{V})$: $\mathbf{V}$
  – with coproducts preserved by $\otimes$
  – biclosed
  – products

$\mathcal{P}rof$ biclosed

$$
\begin{tikzcd}
\mathbf{A} \arrow[rr, dashed, "Q \obslash P"] \arrow[dr, "P"'] & & \mathbf{B} \arrow[dl, "Q"] \\
& \mathbf{C} &
\end{tikzcd}
$$

$$Q \obslash P(A, B)$$

$$= \{Q(B, -) \xrightarrow{n.t.} P(A, -)\}$$

$\mathbf{V}$-$\mathcal{P}rof$ similar

$\mathcal{S}pan$ is biclosed

$\mathcal{S}pan = \mathcal{S}pan(\mathbf{Set})$

$$
\begin{tikzcd}
 & S \arrow[dl] \arrow[dr] & \\
A & & C
\end{tikzcd}
\qquad
\begin{tikzcd}
 & T \arrow[dl] \arrow[dr] & \\
B & & C
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[rr, dashed, "T \oslash S"] \arrow[dr, "S"'] & & B \arrow[dl, "T"] \\
 & C &
\end{tikzcd}
$$

$$T \oslash S = \{(a, b, \langle f_c \colon T_{bc} \longrightarrow S_{ac} \rangle_{c \in C})\}$$

with the obvious projections to A and B

$\mathcal{S}pan(\mathbf{A})$ is biclosed

$\mathbf{A}$ has pullbacks and is locally cartesian closed

For spans $R\colon A \xrightarrow{\bullet} B$ and $T\colon B \xrightarrow{\bullet} C$ we can compute $T \otimes_B R$ as

$$\begin{tikzcd}
R \arrow[d] & P \arrow[l] \arrow[d] \arrow[dl, phantom, very near start, "\llcorner"] & \\
A \times B & A \times T \arrow[l, "A\times\tau_0"] \arrow[r, "A\times\tau_1"'] & A \times C
\end{tikzcd}$$

$$T \otimes_B (\ )\colon \mathbf{A}/(A \times B) \xrightarrow{(A\times\tau_0)^*} \mathbf{A}/(A \times T) \xrightarrow{\sum_{A\times\tau_1}} \mathbf{A}/(A \times C)$$

$\sum_{A\times\tau_1}$ always has a right adjoint $(A \times \tau_1)^*$

$(A \times \tau_0)^*$ will have a right adjoint $\prod_{A\times\tau_0}$

$\mathcal{M}at(\mathbf{V})$ is biclosed

$$
\begin{array}{c}
[W_{bc}] \otimes_B [X_{ab}] \longrightarrow [V_{ac}] \\ \hline
[\sum_b W_{bc} \otimes X_{ab}] \longrightarrow [V_{ac}] \\ \hline
\langle \sum_b (W_{bc} \otimes X_{ab}) \longrightarrow V_{ac} \rangle_{a,c} \\ \hline
\langle W_{bc} \otimes X_{ab} \longrightarrow V_{ac} \rangle_{a,b,c} \\ \hline
\langle X_{ab} \longrightarrow W_{bc} \oslash V_{ac} \rangle_{a,b,c} \\ \hline
\langle X_{ab} \longrightarrow \prod_c (W_{bc} \oslash V_{ac}) \rangle_{a,b} \\ \hline
[X_{a,b}] \longrightarrow [\prod_c (W_{bc} \oslash V_{ac})]
\end{array}
$$

$$
\begin{tikzcd}
A \arrow[rr, "{[X_{ab}]}", "\bullet" marking] \arrow[dr, "{[V_{ac}]}"', "\bullet" marking] & & B \arrow[dl, "{[W_{bc}]}", "\bullet" marking] \\
& C &
\end{tikzcd}
$$

$$[W_{bc}] \oslash [V_{ac}] = [\prod_c (W_{bc} \oslash V_{ac})]$$

Scandal

Good bicategories (all of the above) are the vertical part of naturally occurring double categories:

$\mathbb{R}\mathrm{ing}$, $\mathbb{C}\mathrm{at}$, $\mathbf{V}$-$\mathbb{C}\mathrm{at}$, $\mathbb{S}\mathrm{pan}\mathbf{A}$, $\mathbf{V}$-$\mathbb{S}\mathrm{et}$

But the internal homs $\oslash$ and $\obslash$ are not double functors!

Double categories

- A *double category* is a "category with two sorts of morphisms"

- **Example:** $\mathbb{R}\mathrm{ing}$

$$\begin{tikzcd} R \arrow[r, "f"] \arrow[d, "M"'] \arrow[dr, phantom, "\overset{\alpha}{\Longrightarrow}"] & S \arrow[d, "N"] \\ R' \arrow[r, "f'"'] & S' \end{tikzcd}$$

$\mathbb{C}\mathrm{at}$

- **Example:** $\mathbb{C}\mathrm{at}$

$$\begin{tikzcd} \mathbf{A} \arrow[r, "F"] \arrow[d, "P"'] \arrow[dr, phantom, "\overset{\phi}{\Longrightarrow}"] & \mathbf{B} \arrow[d, "Q"] \\ \mathbf{C} \arrow[r, "G"'] & \mathbf{D} \end{tikzcd}$$

$$P : \mathbf{A}^{op} \times \mathbf{C} \longrightarrow \mathbf{Set}$$
$$Q : \mathbf{B}^{op} \times \mathbf{D} \longrightarrow \mathbf{Set}$$
$$\phi : P(-, =) \longrightarrow Q(F-, G=)$$

$\mathbf{V}$-$\mathbb{S}\mathrm{et}$

- Example: $\mathbf{V}$-$\mathbb{S}\mathrm{et}$ (Vertical bicategory is $\mathcal{M}at(\mathbf{V})$)

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "{[V_{a,c}]}"', "\bullet" marking] \arrow[dr, phantom, "\overset{[\phi_{a,c}]}{\Longrightarrow}"] & B \arrow[d, "{[W_{b,d}]}", "\bullet" marking] \\ C \arrow[r, "g"'] & D \end{tikzcd}$$

$$\phi_{a,c} \colon V_{a,c} \longrightarrow W_{f(a),g(c)}$$

$\mathbb{S}\mathrm{pan}$

- **Example:** $\mathbb{S}\mathrm{pan}\,\mathbf{A}$

$$
\begin{tikzcd}
 & A \arrow[rr, "f"] & & B \\
S \arrow[ur, "\sigma_0"] \arrow[dr, "\sigma_1"'] \arrow[rr, dashed, "h"] & & T \arrow[ur, "\tau_0"'] \arrow[dr, "\tau_1"] & \\
 & C \arrow[rr, "g"'] & & D
\end{tikzcd}
$$

Left homs

- $\mathbb{A}$ has *left homs* if $y \bullet (\ )$ has a right adjoint $y \mathbin{\bullet\!\!\!\!\backslash} (\ )$ in $\mathcal{V}ert\mathbb{A}$

$$\begin{tikzcd} A \arrow[rr, "x"] \arrow[dr, "z"'] & & B \arrow[dl, "y"] \\ & C & \end{tikzcd} \qquad \frac{y \bullet x \to z}{x \to y \mathbin{\bullet\!\!\!\!\backslash} z} \qquad \text{in } \mathcal{V}ert\mathbb{A}$$

Mike Shulman, "Framed bicategories and monoidal fibrations" (TAC 2008)
Roald Koudenburg, "On pointwise Kan extensions in double categories" (TAC 2014)

Respecting boundaries

- $y \mathbin{\backslash\!\!\!\bullet} z$ is covariant in z and contravariant in y: for cells β, γ in $\mathcal{V}ert(\mathbb{A})$

$$y' \xrightarrow{\ \beta\ } y,\, z \xrightarrow{\ \gamma\ } z' \quad \rightsquigarrow \quad y \mathbin{\backslash\!\!\!\bullet} z \xrightarrow{\ \beta \mathbin{\backslash\!\!\!\bullet} \gamma\ } y' \mathbin{\backslash\!\!\!\bullet} z'$$

Respecting boundaries

- $y \,\backslash\!\!\!\bullet\, z$ is covariant in z and contravariant in y: for cells β, γ in $\mathcal{V}ert(\mathbb{A})$

$$y' \xrightarrow{\beta} y,\; z \xrightarrow{\gamma} z' \quad \rightsquigarrow \quad y \,\backslash\!\!\!\bullet\, z \xrightarrow{\beta \,\backslash\!\!\!\bullet\, \gamma} y' \,\backslash\!\!\!\bullet\, z'$$

- We have evaluation $\epsilon : y \bullet (y \,\backslash\!\!\!\bullet\, z) \longrightarrow z$

$$\begin{tikzcd}
A \arrow[r, equal] \arrow[d, "y \,\backslash\!\!\bullet\, z"', "\bullet" marking] & A \arrow[r, equal] \arrow[d, "y \,\backslash\!\!\bullet\, z", "\bullet" marking] & A \arrow[r, equal] \arrow[dd, "z", "\bullet" marking] & A \arrow[dd, "z'", "\bullet" marking] \\
B \arrow[r, equal] \arrow[d, "y'"', "\bullet" marking] \arrow[ur, phantom, "="] & B \arrow[d, "y", "\bullet" marking] \arrow[r, phantom, "\overset{\epsilon}{\Rightarrow}"] & {} \arrow[r, phantom, "\overset{\gamma}{\Rightarrow}"] & {} \\
C \arrow[r, equal] \arrow[ur, phantom, "\overset{\beta}{\Rightarrow}"] & C \arrow[r, equal] & C \arrow[r, equal] & C
\end{tikzcd}$$

$$y \,\backslash\!\!\!\bullet\, z \longrightarrow y' \,\backslash\!\!\!\bullet\, z'$$

Respecting boundaries

- But for cells β and γ in $\mathbb{A}$

$$y' \xrightarrow{\beta} y,\; z \xrightarrow{\gamma} z' \quad \rightsquigarrow \quad y \mathbin{\setminus\!\!\!\bullet} z \xrightarrow[\bullet]{\beta \mathbin{\setminus\!\!\!\bullet} \gamma} y' \mathbin{\setminus\!\!\!\bullet} z' \qquad ?$$

-

$$\begin{tikzcd}
? & A \arrow[r, equal] \arrow[d, "\bullet\, y \mathbin{\setminus\!\!\!\bullet} z"] & A \arrow[r, "a"] \arrow[dd, "\bullet\, z"] & A' \arrow[dd, "\bullet\, z'"] \\
B' \arrow[r, "b"] \arrow[d, "\bullet\, y'"'] \arrow[dr, phantom, "\overset{\beta}{\Rightarrow}"] & B \arrow[d, "\bullet\, y"] \arrow[r, phantom, "\overset{\epsilon}{\Rightarrow}"] & {} \arrow[r, phantom, "\overset{\gamma}{\Rightarrow}"] & {} \\
C' \arrow[r, "c"'] & C \arrow[r, equal] & C \arrow[r, "c'"'] & C'' \\
? \arrow[rrr, dash, "y \mathbin{\setminus\!\!\!\bullet} z \to y' \mathbin{\setminus\!\!\!\bullet} z'"'] & & & ?
\end{tikzcd}$$

Globular universal

$$\forall \quad \begin{tikzcd} A \arrow[r, equal] \arrow[d, "x"', "\bullet" marking] & A \arrow[dd, "z", "\bullet" marking] \\ B \arrow[d, "y"', "\bullet" marking] \arrow[r, phantom, "\overset{\alpha}{\Rightarrow}"] & {} \\ C \arrow[r, equal] & C \end{tikzcd} \qquad \exists! \quad \begin{tikzcd} A \arrow[r, equal] \arrow[d, "x"', "\bullet" marking] \arrow[r, phantom, shift right=6ex, "\overset{\beta}{\Rightarrow}"] & A \arrow[d, "y \mathbin{\bullet\!\!\!\backslash} z", "\bullet" marking] \\ B \arrow[r, equal] & B \end{tikzcd}$$

s.t.

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, "x"', "\bullet" marking] \arrow[r, phantom, shift right=6ex, "\overset{\beta}{\Rightarrow}"] & A \arrow[r, equal] \arrow[d, "y \mathbin{\bullet\!\!\!\backslash} z", "\bullet" marking] & A \arrow[dd, "z", "\bullet" marking] \\ B \arrow[r, equal] \arrow[d, "y"', "\bullet" marking] \arrow[r, phantom, shift right=6ex, "="] & B \arrow[d, "y", "\bullet" marking] \arrow[r, phantom, "\overset{\epsilon}{\Rightarrow}"] & {} \\ C \arrow[r, equal] & C \arrow[r, equal] & C \end{tikzcd} \quad = \quad \begin{tikzcd} A \arrow[r, equal] \arrow[d, "x"', "\bullet" marking] & A \arrow[dd, "z", "\bullet" marking] \\ B \arrow[d, "y"', "\bullet" marking] \arrow[r, phantom, "\overset{\alpha}{\Rightarrow}"] & {} \\ C \arrow[r, equal] & C \end{tikzcd}$$

More universal

$$\forall \quad \begin{tikzcd} A' \arrow[r, "f"] \arrow[d, "x"'] & A \arrow[dd, "z"] \\ B \arrow[d, "y"'] \arrow[r, phantom, "\overset{\alpha}{\Rightarrow}"] & \\ C \arrow[r, equal] & C \end{tikzcd} \qquad \exists! \quad \begin{tikzcd} A' \arrow[r, "f"] \arrow[d, "x"'] \arrow[dr, phantom, "\overset{\beta}{\Rightarrow}"] & A \arrow[d, "y \backslash\!\!\bullet z"] \\ B \arrow[r, equal] & B \end{tikzcd}$$

s.t.

$$\begin{tikzcd} A' \arrow[r, "f"] \arrow[d, "x"'] \arrow[dr, phantom, "\overset{\beta}{\Rightarrow}"] & A \arrow[r, equal] \arrow[d, "y \backslash\!\!\bullet z"] & A \arrow[dd, "z"] \\ B \arrow[r, equal] \arrow[d, "y"'] \arrow[dr, phantom, "="] & B \arrow[d, "y"] \arrow[r, phantom, "\overset{\epsilon}{\Rightarrow}"] & \\ C \arrow[r, equal] & C \arrow[r, equal] & C \end{tikzcd} \quad = \quad \begin{tikzcd} A' \arrow[r, "f"] \arrow[d, "x"'] & A \arrow[dd, "z"] \\ B \arrow[d, "y"'] \arrow[r, phantom, "\overset{\alpha}{\Rightarrow}"] & \\ C \arrow[r, equal] & C \end{tikzcd}$$

Strong universality

Strong universal property:

$$
\begin{tikzcd}
A' \arrow[rr, "f"] \arrow[dr, "y \bullet x"', "\bullet" marking] & \arrow[d, phantom, "\overset{\alpha}{\Rightarrow}"] & A \arrow[dl, "z", "\bullet" marking] \\
& C &
\end{tikzcd}
\qquad \Bigg| \qquad
\begin{tikzcd}
A' \arrow[rr, "f"] \arrow[dr, "x"', "\bullet" marking] & \arrow[d, phantom, "\overset{\beta}{\Rightarrow}"] & A \arrow[dl, "y \mathbin{\backslash\!\!\!\bullet} z", "\bullet" marking] \\
& B &
\end{tikzcd}
$$

Definition

A double category is *vertically biclosed* if it has left and right homs, $\mathbin{\backslash\!\!\!\bullet}$ and $\mathbin{\bullet\!\!\!/}$, satisfying the strong universal properties

Companions

- In a double category $\mathbb{A}$, a vertical arrow $v : A \xrightarrow{\bullet} B$ is a *companion* of a horizontal arrow $f : A \longrightarrow B$ if there are *binding cells* α and β such that

$$
\begin{tikzcd}
A \arrow[r, "1_A"] \arrow[d, "\mathrm{id}_A"', "\bullet" marking] \arrow[dr, phantom, "\overset{\alpha}{\Rightarrow}"] & A \arrow[r, "f"] \arrow[d, "v", "\bullet" marking] \arrow[dr, phantom, "\overset{\beta}{\Rightarrow}"] & B \arrow[d, "\mathrm{id}_B", "\bullet" marking] \\
A \arrow[r, "f"'] & B \arrow[r, "1_A"'] & B
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\mathrm{id}_A"', "\bullet" marking] \arrow[dr, phantom, "\overset{\mathrm{id}_f}{\Rightarrow}"] & B \arrow[d, "\mathrm{id}_B", "\bullet" marking] \\
A \arrow[r, "f"'] & B
\end{tikzcd}
\qquad \beta\alpha = \mathrm{id}_f
$$

$$
\begin{tikzcd}
A \arrow[r, "1_A"] \arrow[d, "\mathrm{id}_A"', "\bullet" marking] \arrow[dr, phantom, "\overset{\alpha}{\Rightarrow}"] & A \arrow[d, "v", "\bullet" marking] \\
A \arrow[r, "f"] \arrow[d, "v"', "\bullet" marking] \arrow[dr, phantom, "\overset{\beta}{\Rightarrow}"] & B \arrow[d, "\mathrm{id}_B", "\bullet" marking] \\
B \arrow[r, "1_A"'] & B
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, "1_A"] \arrow[d, "v"', "\bullet" marking] \arrow[dr, phantom, "\overset{1_v}{\Rightarrow}"] & A \arrow[d, "v", "\bullet" marking] \\
B \arrow[r, "1_B"'] & B
\end{tikzcd}
\qquad \beta \bullet \alpha = 1_v
$$

Properties

- Companions, when they exist, are unique up to globular isomorphism

We make a choice of companion f_* and, following Ronnie Brown, denote the binding cells by corner brackets

- We have $(1_A)_* \cong \mathrm{id}_A$ and $(gf)_* \cong g_* f_*$
-

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "v"'] \arrow[dr, phantom, "\overset{\phi}{\Longrightarrow}"] & B \arrow[d, "w"] \\
C \arrow[r, "g"'] & D
\end{tikzcd}
\quad \longmapsto \quad
\begin{tikzcd}
A \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\ulcorner"] & A \arrow[d, "f_*"] \\
A \arrow[r, "f"] \arrow[d, "v"'] \arrow[dr, phantom, "\overset{\phi}{\Longrightarrow}"] & B \arrow[d, "w"] \\
C \arrow[r, "g"'] \arrow[d, "g_*"'] \arrow[dr, phantom, "\lrcorner"] & D \arrow[d, equal] \\
D \arrow[r, equal] & D
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
A \arrow[r, equal] \arrow[d, "v"'] & A \arrow[d, "f_*"] \\
C \arrow[r, phantom, "\overset{\psi}{\Longrightarrow}"] \arrow[d, "g_*"'] & B \arrow[d, "w"] \\
D \arrow[r, equal] & D
\end{tikzcd}
$$

gives a bijection between ϕ's and ψ's

Conjoints

There is a dual notion of *conjoint* f^*

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\bullet" marking, "\mathrm{id}_A"'] \arrow[dr, phantom, "\overset{\psi}{\Longrightarrow}"] & B \arrow[r, "1_B"] \arrow[d, "\bullet" marking, "f^*"] \arrow[dr, phantom, "\overset{\chi}{\Longrightarrow}"] & B \arrow[d, "\bullet" marking, "\mathrm{id}_B"] \\
A \arrow[r, "1_A"'] & A \arrow[r, "f"'] & B
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\bullet" marking, "\mathrm{id}_A"'] \arrow[dr, phantom, "\overset{\mathrm{id}_f}{\Longrightarrow}"] & B \arrow[d, "\bullet" marking, "\mathrm{id}_B"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}
\qquad \chi\psi = \mathrm{id}_f
$$

$$
\begin{tikzcd}
B \arrow[r, "1_B"] \arrow[d, "\bullet" marking, "f^*"'] \arrow[dr, phantom, "\overset{\chi}{\Longrightarrow}"] & B \arrow[d, "\bullet" marking, "\mathrm{id}_B"] \\
A \arrow[r, "f"] \arrow[d, "\bullet" marking, "\mathrm{id}_A"'] \arrow[dr, phantom, "\overset{\psi}{\Longrightarrow}"] & B \arrow[d, "\bullet" marking, "f^*"] \\
A \arrow[r, "1_A"'] & A
\end{tikzcd}
=
\begin{tikzcd}
B \arrow[r, "1_B"] \arrow[d, "\bullet" marking, "f^*"'] \arrow[dr, phantom, "\overset{1_{f^*}}{\Longrightarrow}"] & B \arrow[d, "\bullet" marking, "f^*"] \\
A \arrow[r, "1_A"'] & A
\end{tikzcd}
\qquad \psi \bullet \chi = 1_{f^*}
$$

Examples

- In $\mathbb{R}\mathrm{ing}$, $f : R \longrightarrow S$
  f_* is S considered as an S-R bimodule
  f^* is S considered as an R-S bimodule

- In $\mathbb{C}\mathrm{at}$, $F : \mathbf{A} \longrightarrow \mathbf{B}$
  $F_* = \mathbf{B}(F-, =)$ and $F^* = \mathbf{B}(-, F =)$

- In $\mathbb{S}\mathrm{pan}(\mathbf{A})$, $f : A \longrightarrow B$

$$f_* \ \text{is}\ \begin{tikzcd} & A \\ A \arrow[ur, "1_A"] \arrow[dr, "f"'] & \\ & B \end{tikzcd} \qquad \text{and} \quad f^* \ \text{is}\ \begin{tikzcd} & B \\ A \arrow[ur, "f"] \arrow[dr, "1_A"'] & \\ & A \end{tikzcd}$$

- In $\mathbf{V}\mathbb{S}\mathrm{et}$, $f_* = [\Delta_{a,fa}]$ and $f^* = [\Delta_{fa,a}]$, $\Delta_{cd} = \begin{cases} I & \text{if } c = d \\ 0 & \text{o.w.} \end{cases}$

What strong means

• The strong universal property is equivalent to the globular one plus the stability property

$$y \mathbin{\backslash\!\!\!\bullet} (z \bullet f_*) \cong (y \mathbin{\backslash\!\!\!\bullet} z) \bullet f_*$$

• If every horizontal arrow has a conjoint, then the strong universal property is equivalent to the globular one

• So all of our examples have the strong universal property, i.e. they are vertically biclosed double categories

Left duals

- Suppose $\mathbb{A}$ left closed
- For $v : A \xrightarrow{\;\bullet\;} B$ we can define its *left Isbell dual*

${}^{\bullet}v = v \mathbin{\backslash\!\!\!\bullet}\, \mathrm{id}_B : B \xrightarrow{\;\bullet\;} A$

We have

$$
{}^{\bullet}\,\mathrm{id}_B \cong \mathrm{id}_B
$$

$$
{}^{\bullet}v \bullet {}^{\bullet}w \longrightarrow {}^{\bullet}(w \bullet v)
$$

So perhaps we get a lax normal

$$
\mathbb{A}^{co} \longrightarrow \mathbb{A}
$$

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "v"', "\bullet" marking] \arrow[dr, phantom, "\overset{\alpha}{\Longrightarrow}"] & C \arrow[d, "w", "\bullet" marking] \\
B \arrow[r, "g"'] & D
\end{tikzcd}
\quad \overset{?}{\rightsquigarrow} \quad
\begin{tikzcd}
B \arrow[r, "g"] \arrow[d, "{}^{\bullet}v"', "\bullet" marking] \arrow[dr, phantom, "\overset{{}^{\bullet}\alpha}{\Longrightarrow}"] & D \arrow[d, "{}^{\bullet}w", "\bullet" marking] \\
A \arrow[r, "f"'] & C
\end{tikzcd}
$$

Retrocells

From now on we assume $\mathbb{A}$ has companions

A *retrocell*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "v"'] \arrow[dr, phantom, "\overset{\alpha}{\Leftarrow}"] & C \arrow[d, "w"] \\ B \arrow[r, "g"'] & D \end{tikzcd} \quad \text{is a cell} \quad \begin{tikzcd} A \arrow[r, equal] \arrow[d, "f_*"'] & A \arrow[d, "v"] \\ C \arrow[r, phantom, "\overset{\alpha}{\Rightarrow}"] \arrow[d, "w"'] & B \arrow[d, "g_*"] \\ D \arrow[r, equal] & D \end{tikzcd} \quad \text{in } \mathbb{A}$$

Quintets

- **Example:** In $\mathbb{Q}(\mathcal{A})$, a cell is a quintet

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "h"'] \arrow[dr, phantom, "\swarrow"] & B \arrow[d, "k"] \\
C \arrow[r, "g"'] & D
\end{tikzcd}
$$

and a retrocell is a coquintet

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "h"'] \arrow[dr, phantom, "\nearrow"] & B \arrow[d, "k"] \\
C \arrow[r, "g"'] & C
\end{tikzcd}
$$

Mates

Proposition

(1) If v and w as below have right adjoints v' and w' in $\mathcal{V}ert\mathbb{A}$, then retrocells α are in bijection with standard cells β:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "v"'] \arrow[dr, phantom, "\Leftarrow\alpha"] & B \arrow[d, "w"] \\ C \arrow[r, "g"'] & D \end{tikzcd} \longleftrightarrow \begin{tikzcd} C \arrow[r, "g"] \arrow[d, "v'"'] \arrow[dr, phantom, "\overset{\beta}{\Rightarrow}"] & D \arrow[d, "w'"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

(2) If f and g have right adjoints h and k in $\mathcal{H}or\mathbb{A}$, then retrocells α are in bijection with standard cells γ:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "v"'] \arrow[dr, phantom, "\Leftarrow\alpha"] & B \arrow[d, "w"] \\ C \arrow[r, "g"'] & D \end{tikzcd} \longleftrightarrow \begin{tikzcd} B \arrow[r, "h"] \arrow[d, "w"'] \arrow[dr, phantom, "\overset{\gamma}{\Rightarrow}"] & A \arrow[d, "v"] \\ D \arrow[r, "k"'] & C \end{tikzcd}$$

Composition

Retrocells can be composed horizontally

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "v"'] \arrow[dr, phantom, "\overset{\alpha}{\Leftarrow}"] & B \arrow[r, "g"] \arrow[d, "w"'] \arrow[dr, phantom, "\overset{\beta}{\Leftarrow}"] & C \arrow[d, "x"] \\
A' \arrow[r, "f'"'] & B' \arrow[r, "g'"'] & C'
\end{tikzcd}
\quad \longleftrightarrow \quad
\begin{tikzcd}
A \arrow[r, equal] \arrow[d, "f_*"'] \arrow[dr, phantom, "="] & A \arrow[r, equal] \arrow[d, "f_*"] & A \arrow[d, "v"] \\
B \arrow[r, equal] \arrow[d, "g_*"'] & B \arrow[r, phantom, "\overset{\alpha}{\Rightarrow}"] \arrow[d, "w"] & A' \arrow[d, "f'_*"] \\
C \arrow[r, phantom, "\overset{\beta}{\Rightarrow}"] \arrow[d, "x"'] & B' \arrow[r, equal] \arrow[d, "g'_*"] \arrow[dr, phantom, "="] & B' \arrow[d, "g'_*"] \\
C' \arrow[r, equal] & C' \arrow[r, equal] & C'
\end{tikzcd}
$$

Composition

and vertically

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "v"'] \arrow[dr, phantom, "\overset{\alpha}{\Leftarrow}"] & B \arrow[d, "w"] \\
A' \arrow[r, "f'"'] \arrow[d, "v'"'] \arrow[dr, phantom, "\overset{\alpha'}{\Leftarrow}"] & B' \arrow[d, "w'"] \\
A'' \arrow[r, "f''"'] & B''
\end{tikzcd}
\quad \longleftrightarrow \quad
\begin{tikzcd}
A \arrow[r, equal] \arrow[d, "f_*"'] & A \arrow[r, equal] \arrow[d, "v"] \arrow[dr, phantom, "="] & A \arrow[d, "v"] \\
B \arrow[r, phantom, "\overset{\alpha}{\Rightarrow}"] \arrow[d, "w"'] & A' \arrow[r, equal] \arrow[d, "f'_*"] & A' \arrow[d, "v'"] \\
B' \arrow[r, equal] \arrow[d, "w'"'] \arrow[dr, phantom, "="] & B' \arrow[r, phantom, "\overset{\alpha'}{\Rightarrow}"] \arrow[d, "w'"] & A'' \arrow[d, "f''_*"] \\
B'' \arrow[r, equal] & B'' \arrow[r, equal] & B''
\end{tikzcd}
$$

Theorem

This gives a double category $\mathbb{A}^{ret}$. $\mathbb{A}^{ret}$ has companions and $(\mathbb{A}^{ret})^{ret} \cong \mathbb{A}$

Functoriality of duals

Theorem

If $\mathbb{A}$ has companions and left duals, the left dual is a lax normal double functor which is the identity on objects and horizontal arrows

$$^{\bullet}(\ ) : \mathbb{A}^{ret\ co} \longrightarrow \mathbb{A}$$

- The proof uses strong universality

Functoriality of $\mathbin{\backslash\!\!\!\bullet}$

A cell α and a retrocell β as in

$$
\begin{tikzcd}
B \arrow[r, "g"] \arrow[d, "y"', "\bullet" marking] & B' \arrow[d, "y'", "\bullet" marking] \\
C \arrow[r, "h"'] \arrow[ur, phantom, "\overset{\beta}{\Leftarrow}"] & C'
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "z"', "\bullet" marking] & A' \arrow[d, "z'", "\bullet" marking] \\
C \arrow[r, "h"'] \arrow[ur, phantom, "\overset{\alpha}{\Rightarrow}"] & C'
\end{tikzcd}
\qquad \text{produce a cell} \qquad
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "y \mathbin{\backslash\!\!\!\bullet} z"', "\bullet" marking] & A' \arrow[d, "y' \mathbin{\backslash\!\!\!\bullet} z'", "\bullet" marking] \\
B \arrow[r, "g"'] \arrow[ur, phantom, "\overset{\beta \mathbin{\backslash\!\!\!\bullet} \alpha}{\Longrightarrow}"] & B'
\end{tikzcd}
$$

given by

$$
\begin{tikzcd}
A \arrow[r, equal] \arrow[d, "y \backslash\!\!\bullet z"'] \arrow[dr, phantom, "="] & A \arrow[r, equal] \arrow[d, "y \backslash\!\!\bullet z"] & A \arrow[r, "f"] \arrow[dd, "z"] & A' \arrow[dd, "z'"] \\
B \arrow[r, equal] \arrow[d, "g_*"'] & B \arrow[d, "y"] \arrow[r, phantom, "\overset{\epsilon}{\Rightarrow}"] & {} \arrow[r, phantom, "\overset{\alpha}{\Rightarrow}"] & {} \\
B' \arrow[r, phantom, "\overset{\beta}{\Rightarrow}"] \arrow[d, "y'"'] & C \arrow[r, equal] \arrow[d, "h_*"] \arrow[dr, phantom, "="] & C \arrow[r, "h"] \arrow[d, "h_*"] \arrow[dr, phantom, "\lrcorner"] & C' \arrow[d, "\mathrm{id}_{C'}"] \\
C' \arrow[r, equal] & C' \arrow[r, equal] & C' \arrow[r, equal] & C'
\end{tikzcd}
$$

Theorem

$\backslash\!\!\bullet$ *is functorial in both variables, covariant in the numerator and retrovariant in the denominator*

Commuter cells

• In M. Grandis, R. Paré, "Kan extensions in double categories" (TAC 2008), we introduced *commutative cells* to express the universal property of comma double categories

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "v"'] \arrow[rd, phantom, "\overset{\alpha}{\Rightarrow}"] & B \arrow[d, "w"] \\ C \arrow[r, "g"'] & D \end{tikzcd}$$

is a *commuter cell* if

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, equal] \arrow[rd, phantom, "\ulcorner"] & A \arrow[d, "f_*"] \\ A \arrow[r, "f"] \arrow[d, "v"'] \arrow[rd, phantom, "\overset{\alpha}{\Rightarrow}"] & B \arrow[d, "w"] \\ C \arrow[r, "g"'] \arrow[d, "g_*"'] \arrow[rd, phantom, "\lrcorner"] & D \arrow[d, equal] \\ D \arrow[r, equal] & D \end{tikzcd}$$

is horizontally invertible

• The inverse would be a retrocell (although, how is it inverse to α?)

Lax functors

- If $F : \mathbb{A} \longrightarrow \mathbb{B}$ is a double functor, we get $F^{ret} : \mathbb{A}^{ret} \longrightarrow \mathbb{B}^{ret}$

- If $F : \mathbb{A} \longrightarrow \mathbb{B}$ is just lax, it doesn't extend to $\mathbb{A}^{ret}$; it should properly respect companions

- If F is lax normal, then F preserves companions and also composites of the form $A \xrightarrow{f_*} B \xrightarrow{v} C$

$$\phi(v, f_*) : F(v) \bullet F(f_*) \longrightarrow F(v \bullet f_*) \quad \text{iso}$$

Dawson, Paré, Pronk, "The Span Construction" (TAC 2010)

Paranormal

Definition

F is *paranormal* if it is normal and also preserves compositions of the form $g_* \bullet v$

$$\phi(g_*, v) : F(g_*) \bullet F(v) \longrightarrow F(g_* \bullet v) \quad \text{iso}$$

Theorem

If F is lax paranormal, then it extends to $F^{ret} : \mathbb{A}^{ret} \longrightarrow \mathbb{B}^{ret}$, oplax paranormal

Retrocells of spans

- In $\mathbb{S}\mathrm{pan}(\mathbf{A})$

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "S"', "\bullet" marking] & B \arrow[d, "T", "\bullet" marking] \\
C \arrow[r, "g"'] & D
\arrow[from=1-2, to=2-1, phantom, "\overset{\phi}{\Leftarrow}"]
\end{tikzcd}
\quad \text{is} \quad
\begin{tikzcd}
A \arrow[rrrrr, equal] \arrow[dr, equal] & & & & & A \\
& A \arrow[dl, "f"'] & & & & \\
B & PB & P \arrow[ul] \arrow[dl] \arrow[r, "\phi"] & S \arrow[uurr, "\sigma_0"'] \arrow[dr, "\sigma_1"] & & \\
& T \arrow[ul, "\tau_0"] \arrow[dl, "\tau_1"] & & & C \arrow[dr, "g"] & \\
D \arrow[rrrrr, equal] & & & & & D
\end{tikzcd}
$$

- In $\mathbb{S}\mathrm{et} = \mathbb{S}\mathrm{pan}(\mathbf{Set})$

Denote an element of T by $t\colon b \xrightarrow{\bullet} d$ if $\tau_0(t) = b$ and $\tau_1(t) = d$, and similarly for an element of S, $s : a \xrightarrow{\bullet} c$

Then

$$\phi : (a, fa \xrightarrow[\bullet]{t} d) \longmapsto (a \xrightarrow[\bullet]{\phi t} c_t), \quad g(c_t) = d$$

Category objects

- A category object in $\mathbf{A}$ is a vertical monad in $\mathbb{S}\mathrm{pan}(\mathbf{A})$
- An internal functor $F : \mathbb{A} \longrightarrow \mathbb{B}$ is a cell

$$\begin{tikzcd}
A_0 \arrow[r, "F_0"] \arrow[d, "A_1"', "\bullet" marking] \arrow[dr, phantom, "\overset{F_1}{\Longrightarrow}"] & B_0 \arrow[d, "B_1", "\bullet" marking] \\
A_0 \arrow[r, "F_0"'] & B_0
\end{tikzcd}$$

respecting composition and identities

- A retrocell ϕ is an object function F_0 together with a lifting operation

$$\begin{tikzcd}
\mathbb{A} \arrow[d, dash, dashed] & A \arrow[r, dashed, "\phi b"] \arrow[d, mapsto] & A_b \arrow[d, mapsto] \\
\mathbb{B} & F_0 A \arrow[r, "b"'] & B
\end{tikzcd}$$

- If ϕ respects composition and identities, then this is exactly a *cofunctor* $\mathbb{B} \longrightarrow \mathbb{A}$ in the sense of Aguiar

Vertical monads in $\mathbb{A}$

A *vertical monad* (A, t, η, μ) is an object A, a vertical arrow $t \colon A \longrightarrow\!\!\bullet\!\!\rightarrow A$ and cells η, μ such that

$$
\begin{tikzcd}
A \arrow[r, equal] \arrow[d, "t"'] \arrow[dr, phantom, "="] & A \arrow[r, equal] \arrow[d, "t"] \arrow[dddr, phantom, "\overset{\mu}{\Rightarrow}"] & A \arrow[ddd, "t"] \\
A \arrow[r, equal] \arrow[d, "t"'] & A \arrow[dd, "t"] & \\
A \arrow[d, "t"'] \arrow[r, phantom, "\overset{\mu}{\Rightarrow}"] & {} & \\
A \arrow[r, equal] & A \arrow[r, equal] & A
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, equal] \arrow[d, "t"'] & A \arrow[r, equal] \arrow[dd, "t"] \arrow[dddr, phantom, "\overset{\mu}{\Rightarrow}"] & A \arrow[ddd, "t"] \\
A \arrow[d, "t"'] \arrow[r, phantom, "\overset{\mu}{\Rightarrow}"] & {} & \\
A \arrow[r, equal] \arrow[d, "t"'] \arrow[dr, phantom, "="] & A \arrow[d, "t"] & \\
A \arrow[r, equal] & A \arrow[r, equal] & A
\end{tikzcd}
$$

$$
\begin{tikzcd}
A \arrow[r, equal] \arrow[d, "\mathrm{id}_A"'] \arrow[dr, phantom, "\overset{\eta}{\Rightarrow}"] & A \arrow[r, equal] \arrow[d, "t"] \arrow[ddr, phantom, "\overset{\mu}{\Rightarrow}"] & A \arrow[dd, "t"] \\
A \arrow[r, equal] \arrow[d, "t"'] \arrow[dr, phantom, "="] & A \arrow[d, "t"] & \\
A \arrow[r, equal] & A \arrow[r, equal] & A
\end{tikzcd}
\cdot = \cdot
\begin{tikzcd}
A \arrow[r, equal] \arrow[d, "t"'] \arrow[dr, phantom, "\overset{1_t}{\Rightarrow}"] & A \arrow[d, "t"] \\
A \arrow[r, equal] & A
\end{tikzcd}
\cdot = \cdot
\begin{tikzcd}
A \arrow[r, equal] \arrow[d, "t"'] \arrow[dr, phantom, "="] & A \arrow[r, equal] \arrow[d, "t"] \arrow[ddr, phantom, "\overset{\mu}{\Rightarrow}"] & A \arrow[dd, "t"] \\
A \arrow[r, equal] \arrow[d, "\mathrm{id}_A"'] \arrow[dr, phantom, "\overset{\eta}{\Rightarrow}"] & A \arrow[d, "t"] & \\
A \arrow[r, equal] & A \arrow[r, equal] & A
\end{tikzcd}
$$

Monad morphism

- A *morphism of monads* $(f,\phi)\colon (A,t,\eta,\mu) \longrightarrow (B,s,\kappa,\nu)$ is a cell ϕ such that

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "t"'] \arrow[dr, phantom, "\overset{\phi}{\Rightarrow}"] & B \arrow[r, equal] \arrow[d, "s"] & B \arrow[dd, "s"] \\
A \arrow[r, "f"'] \arrow[d, "t"'] \arrow[dr, phantom, "\overset{\phi}{\Rightarrow}"] & B \arrow[d, "s"] \arrow[r, phantom, "\overset{\nu}{\Rightarrow}"] & {} \\
A \arrow[r, "f"'] & B \arrow[r, equal] & B
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
A \arrow[r, equal] \arrow[d, "t"'] & A \arrow[r, "f"] \arrow[dd, "t"] & B \arrow[dd, "s"] \\
A \arrow[d, "t"'] \arrow[r, phantom, "\overset{\mu}{\Rightarrow}"] & {} \arrow[r, phantom, "\overset{\phi}{\Rightarrow}"] & {} \\
A \arrow[r, equal] & A \arrow[r, "f"'] & B
\end{tikzcd}
$$

$$
\begin{tikzcd}
A \arrow[r, equal] \arrow[d, "\mathrm{id}_A"'] \arrow[dr, phantom, "\overset{\eta}{\Rightarrow}"] & A \arrow[r, "f"] \arrow[d, "t"] \arrow[dr, phantom, "\overset{\phi}{\Rightarrow}"] & B \arrow[d, "s"] \\
A \arrow[r, equal] & A \arrow[r, "f"'] & B
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "id_A"'] \arrow[dr, phantom, "\overset{\mathrm{id}_f}{\Rightarrow}"] & B \arrow[r, equal] \arrow[d, "\mathrm{id}_B"] \arrow[dr, phantom, "\overset{\kappa}{\Rightarrow}"] & B \arrow[d, "s"] \\
A \arrow[r, "f"'] & B \arrow[r, equal] & B
\end{tikzcd}
$$

2-Cells

- For monad morphisms $(f,\phi),(g,\psi)\colon (A,t,\eta,\mu)\longrightarrow(B,s,\kappa,\nu)$ a 2-cell $\alpha\colon (f,\phi)\Rightarrow(s,\psi)$ is a cell α such that

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "t"'] \arrow[dr, phantom, "\overset{\alpha}{\Rightarrow}"] & B \arrow[r, equal] \arrow[d, "s"] \arrow[ddr, phantom, "\overset{\nu}{\Rightarrow}"] & B \arrow[dd, "s"] \\
A \arrow[r, "g"'] \arrow[d, "t"'] \arrow[dr, phantom, "\overset{\psi}{\Rightarrow}"] & B \arrow[d, "s"] & \\
A \arrow[r, "g"'] & B \arrow[r, equal] & B
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, equal] \arrow[d, "t"'] \arrow[ddr, phantom, "\overset{\mu}{\Rightarrow}"] & A \arrow[r, "f"] \arrow[dd, "t"] \arrow[ddr, phantom, "\overset{\alpha}{\Rightarrow}"] & B \arrow[dd, "s"] \\
A \arrow[d, "t"'] & & \\
A \arrow[r, equal] & A \arrow[r, "g"'] & B
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "t"'] \arrow[dr, phantom, "\overset{\phi}{\Rightarrow}"] & B \arrow[r, equal] \arrow[d, "s"] \arrow[ddr, phantom, "\overset{\nu}{\Rightarrow}"] & B \arrow[dd, "s"] \\
A \arrow[r, "f"'] \arrow[d, "t"'] \arrow[dr, phantom, "\overset{\alpha}{\Rightarrow}"] & B \arrow[d, "s"] & \\
A \arrow[r, "g"'] & B \arrow[r, equal] & B
\end{tikzcd}
$$

- Note: Can also formulate in terms of a cell
$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\mathrm{id}_A"'] \arrow[dr, phantom, "\overset{\beta}{\Rightarrow}"] & B \arrow[d, "s"] \\
A \arrow[r, "g"'] & B
\end{tikzcd}
$$

In $\mathbb{Span}(\mathbf{A})$

- Vertical monads are internal categories
- Monad morphisms are internal functors
- 2-cells are internal natural transformations

Vertical comonads

- Vertical comonads, comonad morphisms, and 2-cells are defined to be monads, etc. in $\mathbb{A}^{op}$ (horizontal dual)
- Because the η and μ for a monad are globular, they can be considered as retrocells in the opposite direction
- So a monad (A, t, η, μ) in $\mathbb{A}$ is a comonad in $\mathbb{A}^{ret}$

$$\begin{tikzcd}
A \arrow[r, equal] \arrow[dd, "t"'] & A \arrow[d, "t"] \\
 & A \arrow[d, "t"] \arrow[l, phantom, "\overset{\mu}{\Leftarrow}"] \\
A \arrow[r, equal] & A
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
A \arrow[r, equal] \arrow[d, "t"'] \arrow[dr, phantom, "\overset{\eta}{\Leftarrow}"] & A \arrow[d, "\mathrm{id}_A"] \\
A \arrow[r, equal] & A
\end{tikzcd}$$

- Internal categories are *comonads* in $\mathbb{S}\mathrm{pan}(\mathbf{A})^{ret}$!

Cofunctors

- But morphisms of comonads and 2-cells are not globular so we get something different: in $\mathbb{S}\mathrm{pan}(\mathbf{A})^{ret}$ they're cofunctors
- A morphism is a retrocell ϕ such that

$$
\begin{tikzcd}
A \arrow[r, equal] \arrow[dd, "t"'] & A \arrow[r, "f"] \arrow[d, "t"] \arrow[dr, phantom, "\overset{\phi}{\Leftarrow}"] & B \arrow[d, "s"] \\
\arrow[r, phantom, "\overset{\mu}{\Leftarrow}"] & A \arrow[r, "f"] \arrow[d, "t"] \arrow[dr, phantom, "\overset{\phi}{\Leftarrow}"] & B \arrow[d, "s"] \\
A \arrow[r, equal] & A \arrow[r, "f"'] & B
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, "f"] \arrow[dd, "t"'] & B \arrow[r, equal] \arrow[dd, "s"] & B \arrow[d, "s"] \\
\arrow[r, phantom, "\overset{\phi}{\Leftarrow}"] & \arrow[r, phantom, "\overset{\nu}{\Leftarrow}"] & B \arrow[d, "s"] \\
A \arrow[r, "f"'] & B \arrow[r, equal] & B
\end{tikzcd}
$$

$$
\begin{tikzcd}
A \arrow[r, equal] \arrow[d, "t"'] \arrow[dr, phantom, "\overset{\eta}{\Leftarrow}"] & A \arrow[r, "f"] \arrow[d, "\mathrm{id}_A"] \arrow[dr, phantom, "\overset{\mathrm{id}_f}{\Leftarrow}"] & B \arrow[d, "\mathrm{id}_B"] \\
A \arrow[r, equal] & A \arrow[r, "f"'] & B
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "t"'] \arrow[dr, phantom, "\overset{\phi}{\Leftarrow}"] & B \arrow[r, equal] \arrow[d, "s"] \arrow[dr, phantom, "\overset{\kappa}{\Leftarrow}"] & B \arrow[d, "\mathrm{id}_B"] \\
A \arrow[r, "f"'] & B \arrow[r, equal] & B
\end{tikzcd}
$$

2-Cells

For (f, ϕ) and (g, ψ), a 2-cell $\theta\colon (f, \phi) \longrightarrow (g, \psi)$ is a retrocell

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "t"', "\bullet" marking] \arrow[dr, phantom, "\overset{\theta}{\Leftarrow}"] & B \arrow[d, "s", "\bullet" marking] \\
A \arrow[r, "g"'] & B
\end{tikzcd}
$$

such that

$$
\begin{tikzcd}
A \arrow[r, equal] \arrow[dd, "t"', "\bullet" marking] & A \arrow[r, "f"] \arrow[d, "t", "\bullet" marking] \arrow[dr, phantom, "\overset{\theta}{\Leftarrow}"] & B \arrow[d, "s", "\bullet" marking] \\
\arrow[r, phantom, "\overset{\mu}{\Leftarrow}"] & A \arrow[r, "g"] \arrow[d, "t", "\bullet" marking] \arrow[dr, phantom, "\overset{\psi}{\Leftarrow}"] & B \arrow[d, "s", "\bullet" marking] \\
A \arrow[r, equal] & A \arrow[r, "g"'] & B
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, "f"] \arrow[dd, "t"', "\bullet" marking] & B \arrow[r, equal] \arrow[dd, "s"', "\bullet" marking] & B \arrow[d, "s", "\bullet" marking] \\
\arrow[r, phantom, "\overset{\theta}{\Leftarrow}"] & \arrow[r, phantom, "\overset{\nu}{\Leftarrow}"] & B \arrow[d, "s", "\bullet" marking] \\
A \arrow[r, "g"'] & B \arrow[r, equal] & B
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, equal] \arrow[dd, "t"', "\bullet" marking] & A \arrow[r, "f"] \arrow[d, "t", "\bullet" marking] \arrow[dr, phantom, "\overset{\phi}{\Leftarrow}"] & B \arrow[d, "s", "\bullet" marking] \\
\arrow[r, phantom, "\overset{\mu}{\Leftarrow}"] & A \arrow[r, "g"'] \arrow[d, "t", "\bullet" marking] \arrow[dr, phantom, "\overset{\theta}{\Leftarrow}"] & B \arrow[d, "s", "\bullet" marking] \\
A \arrow[r, equal] & A \arrow[r, "g"'] & B
\end{tikzcd}
$$

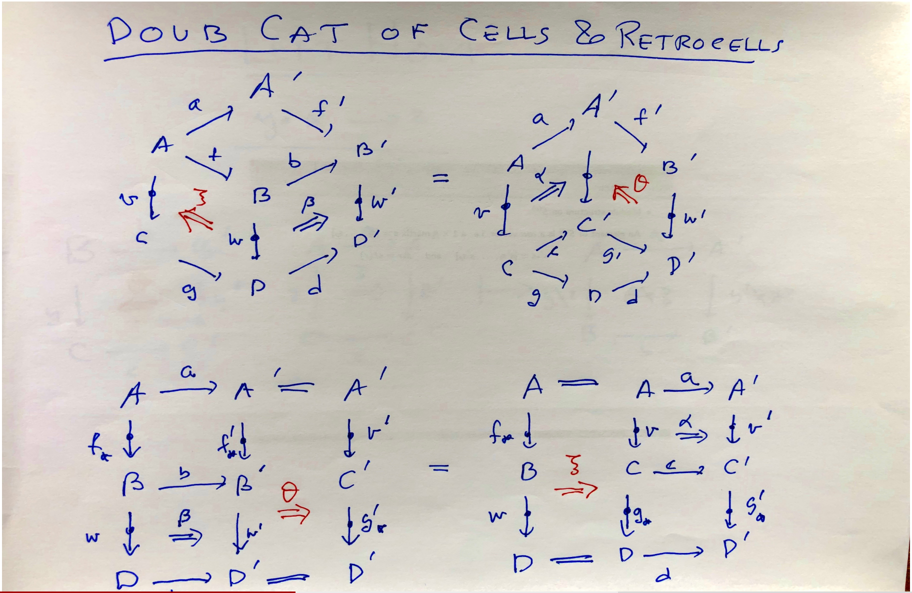
In $\mathbb{Span}(\mathbf{A})^{ret}$

We are given cofunctors $F, G \colon \mathbf{A} \longrightarrow\!\!\!\!\!\!\bullet\;\; \mathbf{B}$
A 2-cell $\theta \colon F \longrightarrow G$ gives a lifting

$$
\begin{tikzcd}
A \arrow[r, dashed, "\theta(A,b)"] \arrow[d, dashed, no head, "F"'] & \Theta(A,b) \arrow[r, dashed, "\psi(\Theta(A,b),b')"] \arrow[d, dashed, no head, "G"] & \Psi(\Theta(A,b),b') \arrow[d, dashed, no head, "G"] \\
FA \arrow[r, "b"'] & B = G\Theta(A,b) \arrow[r, "b'"'] & B'
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, dashed, "\theta(A,b'b)"] \arrow[d, dashed, no head, "F"'] & \Theta(A,b'b) \arrow[d, dashed, no head, "G"] \\
FA \arrow[r, "b'b"'] & B'
\end{tikzcd}
$$

$$
=
\begin{tikzcd}
A \arrow[r, dashed, "\phi(A,b)"] \arrow[d, dashed, no head, "F"'] & \Phi(A,b) \arrow[r, dashed, "\theta(\Phi(A,b),b')"] \arrow[d, dashed, no head, "F"] & \Theta(\Phi(A,b),b') \arrow[d, dashed, no head, "G"] \\
FA \arrow[r, "b"'] & B = F(\Phi(A,b)) \arrow[r, "b'"'] & B'
\end{tikzcd}
$$

Cells and retrocells together



Functors and cofunctors together

DOUB CAT OF FUNCTORS & COFUNCTORS

$$
\begin{tikzcd}
\underline{A} \arrow[r, "F"] \arrow[d, "K"'] \arrow[dr, phantom, "\alpha"] & \underline{B} \arrow[d, "L"] \\
\underline{C} \arrow[r, "G"'] & \underline{D}
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, dashed, "k(A,c)"] \arrow[d, dotted, no head] & k(A,c) \arrow[d, dotted, no head] \\
KA \arrow[r, "c"'] & C
\end{tikzcd}
\qquad
\begin{tikzcd}
B \arrow[r, dashed, "\lambda(B,d)"] \arrow[d, dotted, no head] & \ell(B,d) \arrow[d, dotted, no head] \\
LB \arrow[r, "d"'] & D
\end{tikzcd}
$$

(1) $LFA = GKA$

(2)

$$
\begin{tikzcd}
A \arrow[r, "k(A,c)"] & k(A,c) \arrow[r, squiggly, "\text{“}F\text{”}"] & FA \arrow[r, "Fk(A,c)"] \arrow[d, equal] & Fk(A,c) \arrow[d, equal] \\
 & & FA \arrow[r, "\lambda(FA,\,Gc)"] & \ell(FA,\,Gc) \\
KA \arrow[r, "c"'] \arrow[uu, squiggly, "\text{“}K\text{”}"] & C \arrow[r, squiggly, "\text{“}G\text{”}"'] & LFA = GKA \arrow[r, "Gc"'] \arrow[u, squiggly, "\text{“}L\text{”}"'] & GC
\end{tikzcd}
$$

Better?

BETTER DOUB CAT OF FUNCTORS & COFUNCTORS ?

$$\begin{tikzcd} \underline{A} \arrow[r, "F"] \arrow[d, "K"'] \arrow[dr, phantom, "\alpha"] & \underline{B} \arrow[d, "L"] \\ \underline{C} \arrow[r, "G"'] & \underline{D} \end{tikzcd}
\qquad
\begin{tikzcd} A \arrow[r, dashed, "k(A{,}c)"] \arrow[d, dotted, no head] & k(A{,}c) \arrow[d, dotted, no head] \\ KA \arrow[r, "c"'] & C \end{tikzcd}
\qquad
\begin{tikzcd} B \arrow[r, dashed, "\lambda(B{,}d)"] \arrow[d, dotted, no head] & \ell(B{,}d) \arrow[d, dotted, no head] \\ LB \arrow[r, "d"'] & D \end{tikzcd}$$

(1) $LFA \xrightarrow{\alpha A} GKA$

(2)

$$\begin{tikzcd} A \arrow[r, "k(A{,}c)"] & k(A{,}c) \arrow[r, squiggly, "\text{“}F\text{”}"] & FA \arrow[r, "Fk(A{,}c)"] \arrow[d, equal] & Fk(A{,}c) \arrow[d, "\bar{\alpha}(A{,}c)"] \\ & & FA \arrow[r, "\lambda(FA{,}\, Gc\cdot\alpha A)"] & \ell(FA{,}\, Gc\cdot\alpha A) \\ KA \arrow[uu, squiggly, "\text{“}K\text{”}"] \arrow[r, "c"'] & C \arrow[r, squiggly, "\text{“}G\text{”}"'] & LFA \arrow[r, "\alpha A"'] & GKA \arrow[u, squiggly, "\text{“}L\text{”}"'] \arrow[r, "Gc"'] & GC \end{tikzcd}$$