and $K = \lambda xy.x$. Then $(\Lambda, \cdot, S, K)$ is a combinatory algebra. Also note that, by Corollary 4.5, this algebra is non-trivial, i.e., it has more than one element.

Similar examples are obtained by replacing $=_\beta$ by $=_{\beta\eta}$, and/or replacing Λ by the set Λ_0 of closed terms.

Example 5.5. We construct a combinatory algebra of SK-terms as follows. Let V be a given set of variables. The set $\mathfrak{C}$ of *terms* of combinatory logic is given by the grammar:

$$A, B \quad ::= \quad x \mid \mathbf{S} \mid \mathbf{K} \mid AB,$$

where x ranges over the elements of V.

On $\mathfrak{C}$, we define combinatory equivalence $=_c$ as the smallest equivalence relation satisfying $\mathbf{S}ABC =_c (AC)(BC)$, $\mathbf{K}AB =_c A$, and the rules (*cong*$_1$) and (*cong*$_2$) (see page 2.5). Then the set $\mathfrak{C}/{=_c}$ is a combinatory algebra (called the *free* combinatory algebra generated by V, or the *term algebra*). You will prove in Exercise 19 that it is non-trivial.

Note that all of the above examples of combinatory algebras are either trivial or syntactic. It is not easy to find a true "mathematical" example of a combinatory algebra. We will see how to find such models later.

Exercise 19. On the set $\mathfrak{C}$ of combinatory terms, define a notion of *single-step reduction* by the following laws:

$$\begin{array}{l} \mathbf{S}ABC \to_c (AC)(BC), \\ \mathbf{K}AB \to_c A, \end{array}$$

together with the usual rules (*cong*$_1$) and (*cong*$_2$) (see page 2.5). As in lambda calculus, we call a term a *normal form* if it cannot be reduced. Prove that the reduction $\to_c$ satisfies the Church-Rosser property. (Hint: similarly to the lambda calculus, first define a suitable parallel one-step reduction $\triangleright$ whose reflexive transitive closure is that of $\to_c$. Then show that it satisfies the diamond property.)

Corollary 5.6. *It immediately follows from the Church-Rosser Theorem for combinatory logic (Exercise 19) that two normal forms are $=_c$-equivalent if and only if they are equal.*

5.4 The failure of soundness for combinatory algebras

A combinatory algebra is almost a model of the lambda calculus. Indeed, given a combinatory algebra $\mathbf{A}$, we can interpret any lambda term as follows. To each term M with free variables among $x_1, \ldots, x_n$, we recursively associate a polynomial $[\![M]\!] \in \mathbf{A}\{x_1, \ldots, x_n\}$:

$$\begin{array}{l} [\![x]\!] := x, \\ [\![NP]\!] := [\![N]\!][\![P]\!], \\ [\![\lambda x.M]\!] := \lambda^* x.[\![M]\!]. \end{array}$$

Notice that this definition is almost the identity function, except that we have replaced the ordinary lambda abstractor of lambda calculus by the derived lambda abstractor of combinatory logic. The result is a polynomial in $\mathbf{A}\{x_1, \ldots, x_n\}$. In the particular case where M is a closed term, we can regard $[\![M]\!]$ as an element of $\mathbf{A}$.

To be able to say that $\mathbf{A}$ is a "model" of the lambda calculus, we would like the following property to be true:

$$M =_\beta N \Rightarrow [\![M]\!] = [\![N]\!] \text{ holds in } \mathbf{A}.$$

This property is called *soundness* of the interpretation. Unfortunately, it is in general false for combinatory algebras, as the following example shows.

Example 5.7. Let $M = \lambda x.x$ and $N = \lambda x.(\lambda y.y)x$. Then clearly $M =_\beta N$. On the other hand,

$$\begin{array}{l} [\![M]\!] = \lambda^* x.x = i, \\ [\![N]\!] = \lambda^* x.(\lambda^* y.y)x = \lambda^* x.ix = s(ki)i. \end{array}$$

It follows from Exercise 19 and Corollary 5.6 that the equation $i = s(ki)i$ does not hold in the combinatory algebra $\mathfrak{C}/{=_c}$. In other words, the interpretation is not sound.

Let us analyze the failure of the soundness property further. Recall that β-equivalence is the smallest equivalence relation on lambda terms satisfying the six rules in Table 2.

If we define a relation $\sim$ on lambda terms by

$$M \sim N \qquad \Longleftrightarrow \qquad [\![M]\!] = [\![N]\!] \text{ holds in } \mathbf{A},$$

then we may ask which of the six rules of Table 2 the relation $\sim$ satisfies. Clearly, not all six rules can be satisfied, or else we would have $M =_\beta N \Rightarrow M \sim N \Rightarrow [\![M]\!] = [\![N]\!]$, i.e., the model would be sound.

Clearly, $\sim$ is an equivalence relation, and therefore satisfies (*refl*), (*symm*), and (*trans*). Also, (*cong*) is satisfied, because whenever p, q, p', q' are polynomials

(*refl*)	$\dfrac{}{M = M}$	(*cong*)	$\dfrac{M = M' \qquad N = N'}{MN = M'N'}$
(*symm*)	$\dfrac{M = N}{N = M}$	(ξ)	$\dfrac{M = M'}{\lambda x.M = \lambda x.M'}$
(*trans*)	$\dfrac{M = N \qquad N = P}{M = P}$	(β)	$\dfrac{}{(\lambda x.M)N = M[N/x]}$

Table 2: The rules for β-equivalence

such that $p = p'$ and $q = q'$ holds in $\mathbf{A}$, then clearly $pq = p'q'$ holds in $\mathbf{A}$ as well. Finally, we know from the first Remark of Section 5.3 that the rule (β) is satisfied.

So the rule that fails is the (ξ) rule. Indeed, Example 5.7 illustrates this. Note that $x \sim (\lambda y.y)x$ (from the proof of Theorem 5.1), but $\lambda x.x \not\sim \lambda x.(\lambda y.y)x$, and therefore the ($\xi$) rule is violated.

5.5 Lambda algebras

A lambda algebra is, by definition, a combinatory algebra that is a sound model of lambda calculus, and in which s and k have their expected meanings.

Definition (Lambda algebra). A *lambda algebra* is a combinatory algebra $\mathbf{A}$ satisfying the following properties:

$$\begin{array}{ll} (\forall M, N \in \Lambda)\ M =_\beta N \ \Rightarrow\ [\![M]\!] = [\![N]\!] & (\textit{soundness}), \\ s = \lambda^* x.\lambda^* y.\lambda^* z.(xz)(yz) & (\textit{s-derived}), \\ k = \lambda^* x.\lambda^* y.x & (\textit{k-derived}). \end{array}$$

The purpose of the remainder of this section is to give an axiomatic description of lambda algebras.

Lemma 5.8. *Recall that Λ_0 is the set of closed lambda terms, i.e., lambda terms without free variables. Soundness is equivalent to the following:*

$$(\forall M, N \in \Lambda_0)\ M =_\beta N \ \Rightarrow\ [\![M]\!] = [\![N]\!] \quad (\textit{closed soundness})$$

Proof. Clearly soundness implies closed soundness. For the converse, assume closed soundness and let $M, N \in \Lambda$ with $M =_\beta N$. Let $FV(M) \cup FV(N) = \{x_1, \ldots, x_n\}$. Then

$$\begin{array}{lll} M =_\beta N & & \\ \Rightarrow & \lambda x_1 \ldots x_n.M =_\beta \lambda x_1 \ldots x_n.N & \text{by } (\xi) \\ \Rightarrow & [\![\lambda x_1 \ldots x_n.M]\!] = [\![\lambda x_1 \ldots x_n.N]\!] & \text{by closed soundness} \\ \Rightarrow & \lambda^* x_1 \ldots x_n.[\![M]\!] = \lambda^* x_1 \ldots x_n.[\![N]\!] & \text{by def. of } [\![-]\!] \\ \Rightarrow & (\lambda^* x_1 \ldots x_n.[\![M]\!])x_1 \ldots x_n & \\ & \quad = (\lambda^* x_1 \ldots x_n.[\![N]\!])x_1 \ldots x_n & \\ \Rightarrow & [\![M]\!] = [\![N]\!] & \text{by proof of Thm 5.1} \end{array}$$

This proves soundness. □

Definition (Translations between combinatory logic and lambda calculus). Let $A \in \mathfrak{C}$ be a combinatory term (see Example 5.5). We define its translation to lambda calculus in the obvious way: the translation A_λ is given recursively by:

$$\begin{array}{lcl} \mathbf{S}_\lambda & = & \lambda xyz.(xz)(yz), \\ \mathbf{K}_\lambda & = & \lambda xy.x, \\ x_\lambda & = & x, \\ (AB)_\lambda & = & A_\lambda B_\lambda. \end{array}$$

Conversely, given a lambda term $M \in \Lambda$, we recursively define its translation M_c to combinatory logic like this:

$$\begin{array}{lcl} x_c & = & x, \\ (MN)_c & = & M_c N_c, \\ (\lambda x.M)_c & = & \lambda^* x.(M_c). \end{array}$$

Lemma 5.9. *For all lambda terms M, $(M_c)_\lambda =_\beta M$.*

Lemma 5.10. *Let $\mathbf{A}$ be a combinatory algebra satisfying $s = \lambda^* x.\lambda^* y.\lambda^* z.(xz)(yz)$ and $k = \lambda^* x.\lambda^* y.x$. Then for all combinatory terms A, $(A_\lambda)_c = A$ holds in $\mathbf{A}$.*

Exercise 20. Prove Lemmas 5.9 and 5.10.

RLet $\mathfrak{C}_0$ be the set of *closed* combinatory terms. The following is our first useful characterization of lambda calculus.

Lemma 5.11. *Let $\mathbf{A}$ be a combinatory algebra. Then $\mathbf{A}$ is a lambda algebra if and only if it satisfies the following property:*

$$(\forall A, B \in \mathfrak{C}_0)\ A_\lambda =_\beta B_\lambda \ \Rightarrow\ A = B \text{ holds in } \mathbf{A}. \quad (\textit{alt-soundness})$$

Proof. First, assume that $\mathbf{A}$ satisfies (*alt-soundness*). To prove (*closed soundness*), let M, N be lambda terms with $M =_\beta N$. Then $(M_c)_\lambda =_\beta M =_\beta N =_\beta (N_c)_\lambda$, hence by (*alt-soundness*), $M_c = N_c$ holds in $\mathbf{A}$. But this is the definition of $[\![M]\!] = [\![N]\!]$.

To prove (*k-derived*), note that

$$\begin{array}{rcll} k_\lambda & = & (\lambda x.\lambda y.x) & \text{by definition of } (-)_\lambda \\ & = & ((\lambda x.\lambda y.x)_c)_\lambda & \text{by Lemma 5.9} \\ & = & (\lambda^* x.\lambda^* y.x)_\lambda & \text{by definition of } (-)_c. \end{array}$$

Hence, by (*alt-soundness*), it follows that $k = (\lambda^* x.\lambda^* y.x)$ holds in $\mathbf{A}$. Similarly for (*s-derived*).

Conversely, assume that $\mathbf{A}$ is a lambda algebra. Let $A, B \in \mathfrak{C}_0$ and assume $A_\lambda =_\beta B_\lambda$. By soundness, $[\![A_\lambda]\!] = [\![B_\lambda]\!]$. By definition of the interpretation, $(A_\lambda)_c = (B_\lambda)_c$ holds in $\mathbf{A}$. But by (*s-derived*), (*k-derived*), and Lemma 5.10, $A = (A_\lambda)_c = (B_\lambda)_c = B$ holds in $\mathbf{A}$, proving (*alt-soundness*). $\square$

Definition (Homomorphism). Let $(\mathbf{A}, \cdot_\mathbf{A}, s_\mathbf{A}, k_\mathbf{A})$, $(\mathbf{B}, \cdot_\mathbf{B}, s_\mathbf{B}, k_\mathbf{B})$ be combinatory algebras. A *homomorphism* of combinatory algebras is a function $\varphi : \mathbf{A} \to \mathbf{B}$ such that $\varphi(s_\mathbf{A}) = s_\mathbf{B}$, $\varphi(k_\mathbf{A}) = k_\mathbf{B}$, and for all $a, b \in \mathbf{A}$, $\varphi(a \cdot_\mathbf{A} b) = \varphi(a) \cdot_\mathbf{B} \varphi(b)$.

Any given homomorhism $\varphi : \mathbf{A} \to \mathbf{B}$ can be extended to polynomials in the obvious way: we define $\hat{\varphi} : \mathbf{A}\{x_1, \ldots, x_n\} \to \mathbf{B}\{x_1, \ldots, x_n\}$ by

$$\begin{array}{ll} \hat{\varphi}(a) = \varphi(a) & \text{for } a \in \mathbf{A}, \\ \hat{\varphi}(x) = x & \text{if } x \in \{x_1, \ldots, x_n\}, \\ \hat{\varphi}(pq) = \hat{\varphi}(p)\hat{\varphi}(q). & \end{array}$$

Example 5.12. If $\varphi(a) = a'$ and $\varphi(b) = b'$, then $\hat{\varphi}((ax)(by)) = (a'x)(b'y)$.

The following is the main technical concept needed in the characterization of lambda algebras. We say that an equation *holds absolutely* if it holds in $\mathbf{A}$ and in any homomorphic image of $\mathbf{A}$.

Definition (Absolute equation). Let $p, q \in \mathbf{A}\{x_1, \ldots, x_n\}$ be two polynomials with coefficients in $\mathbf{A}$. We say that the equation $p = q$ *holds absolutely* in $\mathbf{A}$ if for all combinatory algebras $\mathbf{B}$ and all homomorphisms $\varphi : \mathbf{A} \to \mathbf{B}$, $\hat{\varphi}(p) = \hat{\varphi}(q)$ holds in $\mathbf{B}$. If an equation holds absolutely, we write $p =_{\text{abs}} q$.

We can now state the main theorem characterizing lambda algebras. Let $\mathbf{1} = s(ki)$.

$$\begin{array}{lrcl} (a) & \mathbf{1}k & =_{\text{abs}} & k, \\ (b) & \mathbf{1}s & =_{\text{abs}} & s, \\ (c) & \mathbf{1}(kx) & =_{\text{abs}} & kx, \\ (d) & \mathbf{1}(sx) & =_{\text{abs}} & sx, \\ (e) & \mathbf{1}(sxy) & =_{\text{abs}} & sxy, \\ (f) & s(s(kk)x)y & =_{\text{abs}} & \mathbf{1}x, \\ (g) & s(s(s(ks)x)y)z & =_{\text{abs}} & s(sxz)(syz), \\ (h) & k(xy) & =_{\text{abs}} & s(kx)(ky), \\ (i) & s(kx)i & =_{\text{abs}} & \mathbf{1}x. \end{array}$$

Table 3: An axiomatization of lambda algebras. Here $\mathbf{1} = s(ki)$.

Theorem 5.13. *Let $\mathbf{A}$ be a combinatory algebra. Then the following are equivalent:*

1. *$\mathbf{A}$ is a lambda algebra,*
2. *$\mathbf{A}$ satisfies (alt-soundness),*
3. *for all $A, B \in \mathfrak{C}$ such that $A_\lambda =_\beta B_\lambda$, the equation $A = B$ holds absolutely in $\mathbf{A}$,*
4. *$\mathbf{A}$ absolutely satisfies the nine axioms in Table 3,*
5. *$\mathbf{A}$ satisfies (s-derived) and (k-derived), and for all $p, q \in \mathbf{A}\{y_1, \ldots, y_n\}$, if $px =_{\text{abs}} qx$ then $\mathbf{1}p =_{\text{abs}} \mathbf{1}q$,*
6. *$\mathbf{A}$ satisfies (s-derived) and (k-derived), and for all $p, q \in \mathbf{A}\{x, y_1, \ldots, y_n\}$, if $p =_{\text{abs}} q$ then $\lambda^* x.p =_{\text{abs}} \lambda^* y.q$.*

The proof proceeds via $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4 \Rightarrow 5 \Rightarrow 6 \Rightarrow 1$.

We have already proven $1 \Rightarrow 2$ in Lemma 5.11.

To prove $2 \Rightarrow 3$, let $FV(A) \cup FV(B) \subseteq \{x_1, \ldots, x_n\}$, and assume $A_\lambda =_\beta B_\lambda$. Then $\lambda x_1 \ldots x_n.(A_\lambda) =_\beta \lambda x_1 \ldots x_n.(B_\lambda)$, hence $(\lambda^* x_1 \ldots x_n.A)_\lambda =_\beta (\lambda^* x_1 \ldots x_n.B)_\lambda$ (why?). Since the latter terms are closed, it follows by (*alt-soundness*) that $\lambda^* x_1 \ldots x_n.A = \lambda^* x_1 \ldots x_n.B$ holds in $\mathbf{A}$. Since closed equations are preserved by homomorphisms, the latter also holds in $\mathbf{B}$ for any homomorphism $\varphi : \mathbf{A} \to \mathbf{B}$. Finally, this implies that $A = B$ holds for any such $\mathbf{B}$, proving that $A = B$ holds absolutely in $\mathbf{A}$.

Exercise 21. Prove the implication $3 \Rightarrow 4$.

The implication $4 \Rightarrow 5$ is the most difficult part of the theorem. We first dispense with the easier part:

Exercise 22. Prove that the axioms from Table 3 imply (*s-derived*) and (*k-derived*).

The last part of $4 \Rightarrow 5$ needs the following lemma:

Lemma 5.14. *Suppose* $\mathbf{A}$ *satisfies the nine axioms from Table 3. Define* $(\mathbf{B}, \bullet, S, K)$ *by:*

$$\begin{aligned}
&\mathbf{B} = \{a \in \mathbf{A} \mid a = \mathbf{1}a\}, \\
&a \bullet b = sab, \\
&S = ks, \\
&K = kk.
\end{aligned}$$

Then $\mathbf{B}$ *is a well-defined combinatory algebra. Moreover, the function* $\varphi : \mathbf{A} \to \mathbf{B}$ *defined by* $\varphi(a) = ka$ *defines a homomorphism.*

Exercise 23. Prove Lemma 5.14.

To prove the implication $4 \Rightarrow 5$, assume $ax = bx$ holds absolutely in $\mathbf{A}$. Then $\hat{\varphi}(ax) = \hat{\varphi}(bx)$ holds in $\mathbf{B}$ by definition of "absolute". But $\hat{\varphi}(ax) = (\varphi a)x = s(ka)x$ and $\hat{\varphi}(bx) = (\varphi b)x = s(kb)x$. Therefore $s(ka)x = s(kb)x$ holds in $\mathbf{A}$. We plug in $x = i$ to get $s(ka)i = s(kb)i$. By axiom (i), $\mathbf{1}a = \mathbf{1}b$.

To prove $5 \Rightarrow 6$, assume $p =_{\text{abs}} q$. Then $(\lambda^* x.p)x =_{\text{abs}} p =_{\text{abs}} q =_{\text{abs}} (\lambda^* x.q)x$ by the proof of Theorem 5.1. Then by 5., $(\lambda^* x.p) =_{\text{abs}} (\lambda^* x.q)$.

Finally, to prove $6 \Rightarrow 1$, note that if 6 holds, then the absolute interpretation satisfies the ξ-rule, and therefore satisfies all the axioms of lambda calculus.

Exercise 24. Prove $6 \Rightarrow 1$.

Remark. The axioms in Table 3 are required to hold *absolutely*. They can be replaced by local axioms by prefacing each axiom with $\lambda^* xyz$. Note that this makes the axioms much longer.