Math 4680/5680, Topics in Logic and Computation, Winter 2017
Handout 2: Natural Deduction for Quantifiers

Problems.

Problem 1 Prove the following in natural deduction:

(a) $Q \to \forall x\, P(x) \equiv \forall x\, (Q \to P(x))$ — assume that x does not occur in Q.

(b) $\neg\exists x\, P(x) \equiv \forall y\, \neg P(y)$.

(c) $\forall x\, P(x) \wedge \forall x\, Q(x) \equiv \forall x\, (P(x) \wedge Q(x))$.

(d) $\forall x\, P(x) \vee \forall x\, Q(x) \vdash \forall x\, (P(x) \vee Q(x))$.

(e) $\exists x\, \forall y\, P(x,y) \vdash \forall y\, \exists x\, P(x,y)$.

(f) $\exists x\, \forall y\, P(x,y) \vdash \exists z\, P(z,z)$.

(g) $\exists x\, P(x) \vee \exists x\, Q(x) \equiv \exists x(P(x) \vee Q(x))$.

(h) $\exists x(P(x) \wedge Q(x)) \vdash \exists x\, P(x) \wedge \exists x\, Q(x)$.

(i) $\exists x\, P(x,x) \vdash \exists y\, \exists z\, P(y,z)$.

(j) $\forall x\, (A(x) \to B(x)) \vdash \exists x\, \neg B(x) \to \exists x\, \neg A(x)$.

(k) $\neg\exists x\, (A(x) \wedge B(x)) \equiv \forall x\, (A(x) \to \neg B(x))$.

(l) $\exists x\, \forall y\, P(x,y,x) \vdash \exists x\, \forall y\, \exists z\, P(x,y,z)$.

(m) $\vdash \forall x(P(x) \to \exists y\, P(y))$.

(n) $\vdash \forall x(\forall y\, P(y) \to P(x))$.

(o) $\forall x\, P(x) \vdash \exists x\, P(x)$.

(p) $\forall x(A(x) \to B(x)), \forall y(B(y) \to C(y)) \vdash \forall z(A(z) \to C(z))$.

(q) $\exists x\, A(x), \forall x\, (A(x) \to B(x)) \vdash \exists x(A(x) \wedge B(x))$.

(r) $\forall x\, A(x), \exists x\, (A(x) \to B(x)) \vdash \exists x(A(x) \wedge B(x))$.

(s) $\neg\exists x(A(x) \vee B(x)) \equiv \forall x \neg A(x) \wedge \forall x \neg B(x)$.

(t) $\exists x\, P(x) \to \forall y\, Q(y) \equiv \forall x\, \forall y(P(x) \to Q(y))$.

Problem 2 Prove the following by natural deduction. Note: each of these problems requires the $\neg\neg$-elimination rule.

(u) $Q \to \exists x\, P(x) \equiv \exists x\, (Q \to P(x))$ — assume that x does not occur in Q.

(v) $\neg\forall x\, P(x) \equiv \exists y\, \neg P(y)$.

(w) $\exists x\, (A(x) \wedge B(x)) \equiv \neg\forall x\, (A(x) \to \neg B(x))$.

(x) $\vdash \exists x(\exists y\, P(y) \to P(x))$.

(y) $\neg\forall x(A(x) \wedge B(x)) \equiv \exists x \neg A(x) \vee \exists x \neg B(x)$.

(z) $\forall x\, P(x) \to \exists y\, Q(y) \equiv \exists x\, \exists y(P(x) \to Q(y))$.

Problem 3 Prove the equivalences in Problem 1 (a), (b), (c), (g), (k), (s), (t) and Problem 2 (u), (v), (w), (y), (z) by the laws of statement algebra.

Problem 4 In Problem 1 (d), (e), (f), (h), (i), (j), (l), (o), (p), (q), (r), prove that the converse direction does not hold by giving a counterexample.