Normalisation for the fundamental crossed complex of a simplicial set
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Joint work with Rafael Sivera

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Part of a book on Nonabelian algebraic topology by RB, Higgins, Sivera in preparation.

Intended to give a full account of the work on crossed complexes by RB and PJH *et al* since 1971.

Simp is the category of simplicial sets.
Crs is the category of crossed complexes, an extension of the notion of chain complexes with operators.
A crossed complex has the structure of a family of groupoids over C_0

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & C_n & \xrightarrow{\delta_n} & C_{n-1} & \longrightarrow \cdots \longrightarrow & C_2 & \xrightarrow{\delta_2} & C_1 \\
 & \downarrow t & & \downarrow t & & \downarrow t & & s \downarrow\downarrow t \\
 & C_0 & & C_0 & & C_0 & & C_0
\end{array}$$

which consists of a crossed module (of groupoids) in dimensions $\leqslant 2$, and many other rules.

Normalisation Theorem

There are functors

$$\Pi : \mathsf{Simp} \to \mathsf{Crs}, \quad \Pi^{\mathrm{norm}} : \mathsf{Simp} \to \mathsf{Crs}$$

such that for a simplicial set K
ΠK is free on the simplices of K,
$\Pi^{\mathrm{norm}} K$ is free on the non degenerate simplices of K,
and there are morphisms

$$^{p}\Pi_{\lambda} K \underset{s}{\longleftarrow} \Pi^{\mathrm{norm}} K$$

making $\Pi^{\mathrm{norm}} K$ a natural strong deformation retract of ΠK.

Corollary 1

The natural map $\|K\| \to |K|$ is a homotopy equivalence.

Corollary 2

In using crossed complexes for nonabelian cohomology, you may use the normalised or unnormalised cocycles.

This theorem is completely analogous to a classical theorem of Eilenberg-Mac Lane for simplicial abelian groups, but is much more tricky to prove.
These crossed complexes usefully strengthen chain complexes with a group of groupoid of operators.

What is the problem?

1. Crossed complexes have different types of structures in dimensions 0,1,2,$\geqslant 3$.
2. The face and degeneracy maps $\partial_i, \varepsilon_j$ are defined on the free generators but are not 'morphisms' as in the abelian groups situation.
3. For homotopies $(f, h) : f^0 \simeq f$, of morphisms of crossed complexes, h_1 is an f_1-derivation, so care is needed in working out $h_1 \delta_2$.
4. We need to discuss normal subcrossed complexes, and free subcrossed complexes.

Why crossed complexes?

1. Bring the fundamental group (and groupoid) and its actions back into the center of algebraic topology, and applications.
2. Set up and apply acyclic models for crossed complexes.
3. Give the full setting for the Homotopy Addition Lemma.
4. Utilise crossed complexes as a linear approximation to homotopy theory.
5. Crossed modules give a complete algebraic model of homotopy 2-types.
6. Crossed complexes as a strengthening of the concept of chain complex with a group of operators.
7. See later section on crossed complexes.

Boundary of a simplex: Abelian theory

Classic formula for the boundary of a simplex:
If x has dimension n, $\partial_i x$ is its ith face, then

$$d_n x = \sum_{i=0}^{n} (-1)^i \partial_i x,$$

This replaces older formulae involving orientation. It works in simplicial abelian groups, and it is easy to prove

$$d_{n-1} d_n = 0.$$

So you get homology groups

$$H_n = \text{Ker } d_n / \text{Im } d_{n+1}.$$

Homotopy addition lemma

This also gives 'the boundary of a simplex', but it also takes account of:

- a set of base points (the vertices of the simplex);
- operators of dimension 1 on dimensions $\geqslant 2$;
- nonabelian structures in dimensions 1 and 2.

In dimension 1, we have end points:

$$0 \xrightarrow{x} 1 \qquad \text{(HAL1-diagram)}$$

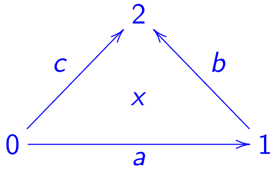
$$0 = \partial_1 x, \qquad 1 = \partial_0 x.$$

Homotopy addition lemma

In dimension 2 we have a groupoid rule:

$$\delta_2 x = -\partial_1 x + \partial_2 x + \partial_0 x, \qquad \text{(HAL2)}$$

which is represented by the diagram



(HAL2-diagram)

and the easy to understand formula (HAL2) says that

$$\delta_2 x = -c + a + b$$

Homotopy addition lemma

In dimension 3 we have the nonabelian rule:

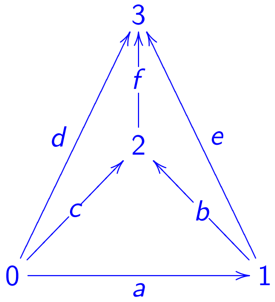
$$\delta_3 x = (\partial_3 x)^{u_3} - \partial_0 x - \partial_2 x + \partial_1 x. \qquad \text{(HAL3)}$$

Note that in HAL3 we have an exponent u_3: this is given by $f = \partial_0^2 x$.The necessity for this is that our convention is that each n-simplex x has as base point its last vertex $\partial_0^n x$. Thus the base point of the above 3-simplex x is 3, while the base point of $\partial_0 x$ is 2. The exponent f relocates $\partial_0 x$ to have base point at 3, and so obtains a well defined formula.

Homotopy addition lemma

Understanding of this is helped by considering the diagram

$x :=$



(HAL3-diagram)

We have the groupoid formula

$$-f+(-c+a+b)+f-(-e+b+f)-(-d+a+e)+(-d+c+f)=0.$$

This is a translation of the rule $\delta_2\delta_3=0$, provided we assume

$$\delta_2(y^f)=-f+\delta_2 y+f,$$

which is the first rule for a crossed module.

Homotopy addition lemma

In dimension $n \geqslant 4$ we have the abelian rule, but still with operators:

$$\delta_n x = (\partial_n x)^{u_n} + \sum_{i=0}^{n-1} (-1)^{n-i} \partial_i x, \qquad (\text{HAL} \geqslant 4)$$

where $u_n x = \partial_0^{n-1} x$. These, or analogous, formulae underly much nonabelian cohomology theory.

The rule $\delta_{n-1}\delta_n = 0$ is straightforward to verify for $n > 4$, through working in abelian groups,but for $n = 4$ we require the second crossed module rule, that for x, y of dimension 2

$$-y + x + y = x^{\delta_2 y}.$$

Homotopy addition lemma

In the right context, the HAL is an inductive cone formula.

$$\delta_n x = (\partial_n x)^{u_n} - \text{non-exponent formula in dimension } n-1.$$

The context for these formulae is the notion of crossed complex.

Crossed complexes

Crossed complexes in the reduced (single vertex) case are due to Blakers (1948), Whitehead (1949); using relative homotopy groups they defined

$$\Pi : \mathrm{FTop}_{red} \to \mathrm{Crs}_{red}$$

where FTop is the category of filtered spaces

$$X_* : X_0 \subseteq X_1 \subseteq \cdots \subseteq X_n \subseteq \cdots \subseteq X_\infty = X$$

Philip Higgins and RB generalised to the groupoid (non reduced) case and used a deep relation with cubical methods to set up a rich algebraic theory of these so that they can be used for calculation and expression.
The HAL is then proved purely algebraically, and one also proves crossed complexes model the topology.
That is the hard bit!

1. The functor $\Pi : \mathsf{FTop} \to \mathsf{Crs}$ from the category of filtered spaces to crossed complexes preserves certain colimits.
2. The category Crs is monoidal closed with an exponential law of the form

$$\mathsf{Crs}(A \otimes B, C) \cong \mathsf{Crs}(A, \mathsf{CRS}(B, C)). \qquad \text{(exponential law)}$$

3. The category Crs has a *unit interval object* written $\mathcal{I}$, which is essentially just the indiscrete groupoid on two objects $0, 1$, and so has only element $\iota : 0 \to 1$ in dimension. For a crossed complex A, this gives rise to a *cylinder object*

$$\mathsf{Cyl}(A) = (A \rightrightarrows \mathcal{I} \otimes A)$$

and so a homotopy theory for crossed complexes.

5. There is for filtered spaces X_*, Y_* a natural transformation

$$\eta : \Pi X_* \otimes \Pi Y_* \to \Pi(X_* \otimes Y_*),$$

which is an isomorphism if X_*, Y_* are the skeletal filtrations of CW-complexes.

4. There is a *classifying space functor* $B : \mathrm{Crs} \to \mathrm{Top}$ and for a CW-complex X with its skeletal filtration, and crossed complex C a homotopy classification theorem

$$[X, BC] \cong [\Pi X_*, C].$$

6. (Whitehead Theorem) If $f : X_* \to Y_*$ is cellular map of CW-filtrations, then $f : X \to Y$ is a homotopy equivalence if and only if $\Pi f : \Pi X_* \to \Pi Y_*$ is a weak equivalence.

Simplicial sets

The crossed complex model of the simplex, is defined inductively as

$$a\Delta^n = \mathrm{Cone}(a\Delta^{n-1})$$

in the category Crs where $\mathrm{Cone}(A)$ is given by the pushout

$$\begin{array}{ccc} \{1\} \otimes A & \longrightarrow & \{v\} \\ \downarrow & & \downarrow \\ \mathcal{I} \otimes A & \longrightarrow & \mathrm{Cone}(A) \end{array}$$

This gives the formulae above.

A simplicial set K is a colimit (or coend) of its simplices Δ^n and so we can define ΠK as the colimit (or coend) of the $a\Delta^n$ over the simplicial face category.
We can similarly define the $\Pi^{\mathrm{norm}} K$ as the coend over the full simplicial category.

Theorem

$$\Pi K \cong \Pi \|K\|, \qquad \Pi^{\mathrm{norm}} K \cong \Pi|K|$$

where $\|K\|$, $|K|$ have the skeletal filtrations.

Conclusion The algebra of crossed complexes is a useful model of the geometry.
To paraphrase JHC Whitehead: Crossed Complexes have better realisation properties than chain complexes with operators.
They can also be used for calculation, and in relation to Čech theory, stacks and gerbes.