These are copies of the slides for my talk

Higher dimensional diagrams via computads

at CT 2006, White Point, June 25 – July 1,

The last four slides (pp 19-22) were not shown at the talk (for lack of time ...).

More importantly: on bottom p. 7 and on p 9, now you find corrected pictures of "atoms"; very regrettably, those pictures were shown in wrong variants in the talk.

July 5 / 2006

Higher Dimensional Diagrams (HDD's) via Computads

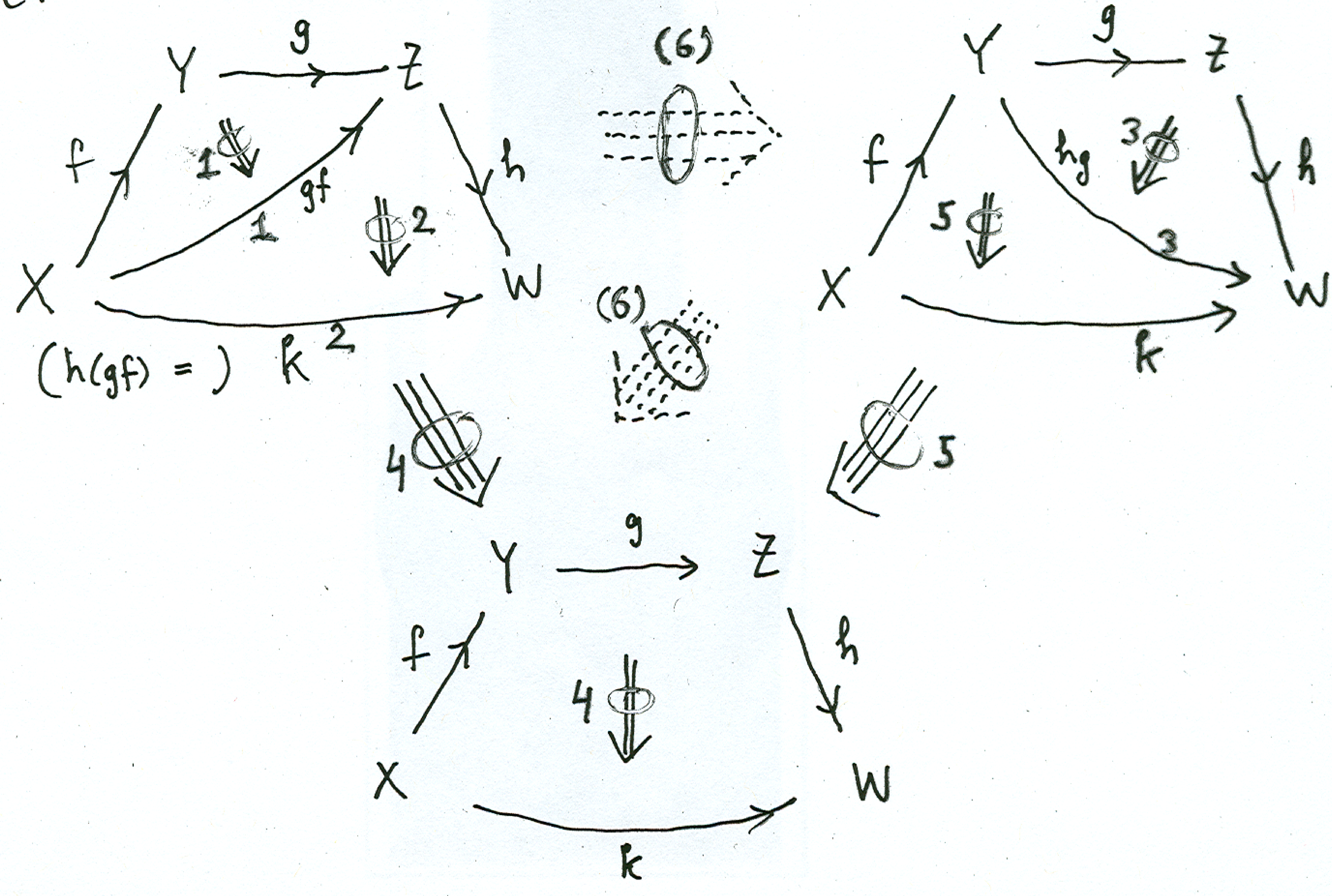
(M. Makkai)

Example: 'Proof' of the associative law for composition: $f(gh) \cong (fg)h \; (\cong fgh)$

Given:

$$\begin{tikzcd} & Y \arrow[r, "g"] & Z \arrow[dr, "h"] & \\ X \arrow[ur, "f"] & & & W \end{tikzcd}$$

Derive:



SLOGANS:

- $\left.\begin{array}{l}\text{Some}\\ \text{Many}\\ \text{All}\end{array}\right\}$ HDD's are needed for:

weak n-categories

- HDD = computad (see below)

Sample theorem:

The category $\mathbf{Comp}_3^+$ of positive 3-computads is a presheaf category

N.B.
- The category $\mathbf{Comp}_3$ of all 3-computads is not a presheaf category (M. Zawadowski – M.M.)

- The category $\mathbf{Comp}_2$ of all 2-computads is a presheaf category (Steve Schanuel)

Adjoining indeterminate cells and computads

$\mathbb{X}$: ω-category

(Notation : $\mathbb{X}_n$: set of n-cells ($n = 0, 1, 2, \dots$)

convention : $\mathbb{X}_{-1} = \{*\}$

$$a \in \mathbb{X}_0 \Longrightarrow da = ca = *$$

(d : domain ; c : codomain

convention : $* \parallel *$

$$a \parallel b \Longleftrightarrow da = db \ \& \ ca = cb$$

$$\|\mathbb{X}\| \overset{\text{def}}{=} \coprod_{n \in \mathbb{N} \cup \{-1\}} \mathbb{X}_n \quad)$$

Definition:

A set of indeterminates (attached to $\mathbb{X}$)

$$\underline{U} = (U, d, c) : \underset{\text{set}}{U} \underset{c}{\overset{d}{\rightrightarrows}} \|\mathbb{X}\|$$

functions

such that:

$$u \in U \Rightarrow du \parallel cu \quad (\text{in } \mathbb{X})$$

To define: $\mathbb{X}[\underline{U}]$: the result of freely adjoining all $u \in \underline{U}$ to $\mathbb{X}$.

Let: $\mathcal{F}$: category : (of "frames")

objects : $(\mathbb{X}; \underline{U})$

arrows : $(\mathbb{X}; \underline{U}) \xrightarrow{(\Gamma, \Lambda)} (\mathbb{Y}, \underline{V})$

$$\mathbb{X} \xrightarrow{\Gamma} \mathbb{Y}$$
$$\underline{U} \xrightarrow{\Lambda} \underline{V}$$

subject to:

$$\begin{tikzcd}
U \arrow[rr, "\Lambda"] \arrow[d, "{\langle d, c\rangle}"'] \arrow[drr, phantom, "\circ"] & & V \arrow[d, "{\langle d, c\rangle}"] \\
\|\mathbb{X}\| \times \|\mathbb{X}\| \arrow[rr] & & \|\mathbb{Y}\| \times \|\mathbb{Y}\|
\end{tikzcd}$$

'tautological frame': U those of $\mathbb{Y}$

$$\underset{\omega\text{-cat}}{\tau(\mathbb{Y})} \overset{\mathrm{DEF}}{=} (\mathbb{Y}; (\|\mathbb{Y}\|, d, c))$$

$\exists$: $E \dashv \tau$

$$\begin{tikzcd}
\mathcal{F} \arrow[rr, bend right, "E"'] \arrow[rr, phantom, "\top"] & & \underline{\omega\,\mathrm{Cat}} \arrow[ll, bend right, "\tau"']
\end{tikzcd}$$

(ordinary) category of (strict) ω-cat's & (strict) ω-functors

$$(\mathbb{X}; \underline{U}) \longmapsto \mathbb{X}[\underline{U}]$$

0- computad: a set

<u>Recursively</u>: strict!

$(n+1)$-computad: an $(n+1)$-category of the form $\mathbb{X}[\underline{\mathcal{U}}]$, with

$\mathbb{X}$: n-computad, $\underline{\mathcal{U}} = (\mathcal{U}, d, c)$ a set of $(n+1)$-indeterminates ($u \in \mathcal{U} \Rightarrow du, cu \in \mathbb{X}_n$).

Computad ($= \omega$-computad): ω-category $\mathbb{X}$ such that $\forall n \in \mathbb{N}$, $\underbrace{\mathbb{X}\restriction n}_{n\text{-truncation}}$ is an n-computad.

<u>N.B.</u> no need for 'tracking' indet's: they are exactly the <u>indecomposables</u>:

y is indecomposable iff either $\dim(y) = 0$

or: $y \neq 1_b \ \forall b$

& $\forall b, e \in \mathbb{Y}_{\underset{\dim(y)}{n}}$, $k < n$, $y = e \circ_k b$

$\Rightarrow$ $b = 1_a^{(n)}$ or $e = 1_a^{(n)}$ for some $a \in \mathbb{Y}_k$

($m = \dim(a)$: $1_a^{(m)} = a$;

$1_a^{(p+1)} = 1_{1_a^{(p)}}$.)

<u>Comp</u> : the category of computads;

objects : (small) computads

arrows : ω-cat morphisms

taking indet's (indecomposables)

to indet's.

Non-full inclusion

$$\underline{\mathrm{Comp}} \longrightarrow \underline{\omega\,\mathrm{Cat}}$$

full on isomorphisms.

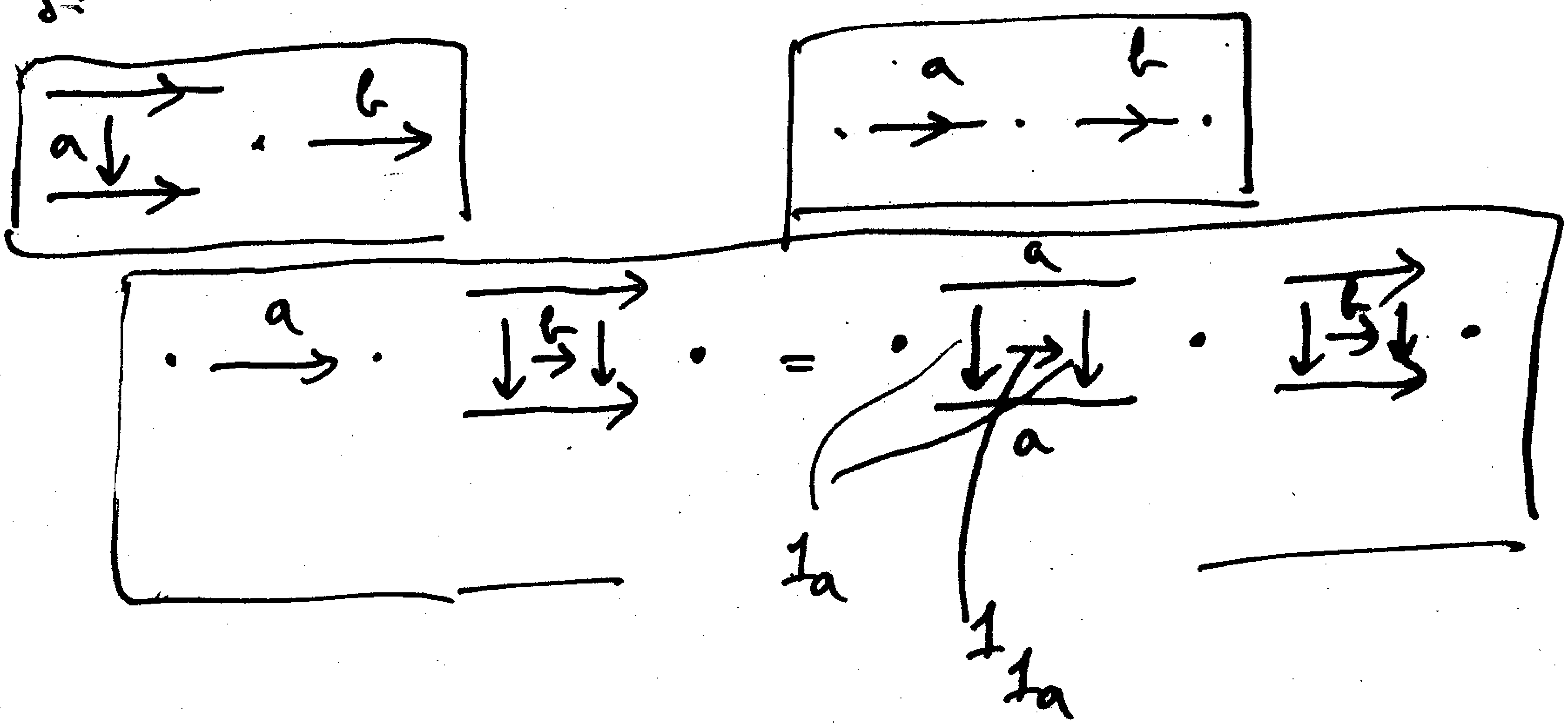
Dot-compositions:

$$a \cdot b \overset{\text{def}}{=} b \circ_k a = 1_b^{(p)} \circ_k 1_a^{(p)}$$

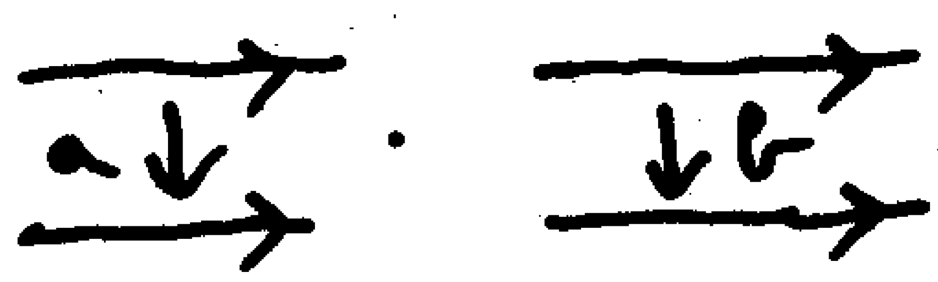
for: $k = k(a,b) \overset{\text{def}}{=} \min(\underbrace{m}_{\dim(a)}, \underbrace{n}_{\dim(b)}) - 1$

$$p \overset{\text{def}}{=} \max(m, n)$$

Ex's:

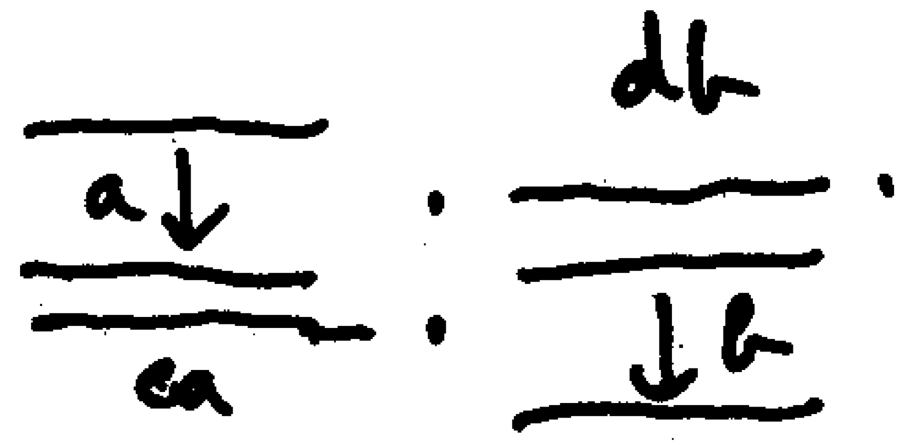


Non example



||

(but:



$(a \cdot db) \cdot (ca \cdot b)$)

The 'commutative' law

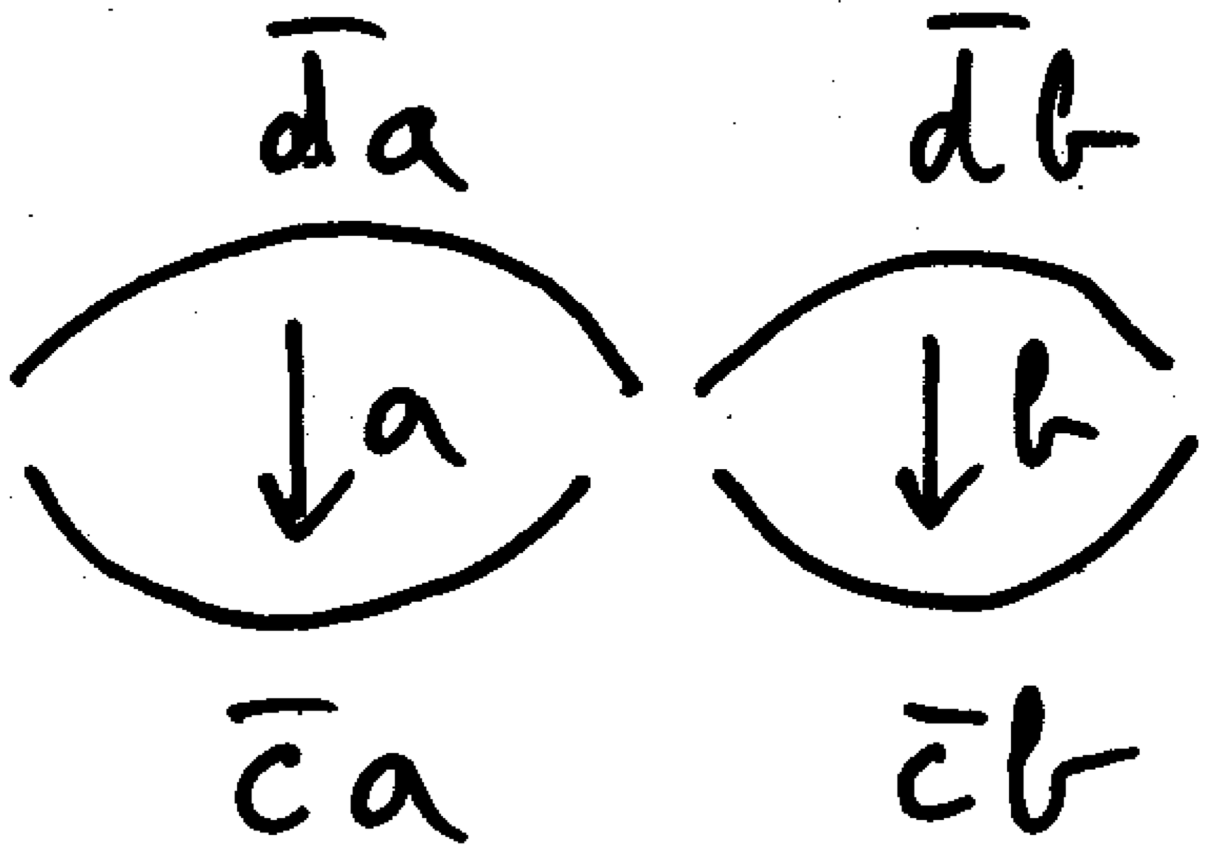
For: $m, n \geq 2$ $\quad (m = \dim(a),\ n = \dim(b))$

$k = k(a,b)$, $\bar{d} = d^{(k)}$, $\bar{c} = c^{(k)}$,

<u>assume:</u> $c\bar{c}a = c\bar{d}a$ ⊜ $d\bar{d}b = d\bar{c}b$

Then:

$$\boxed{(a\cdot \bar{d}b)\cdot(\bar{c}a\cdot b) = ((\bar{d}a\cdot b)\cdot(a\cdot\bar{c}b))}$$



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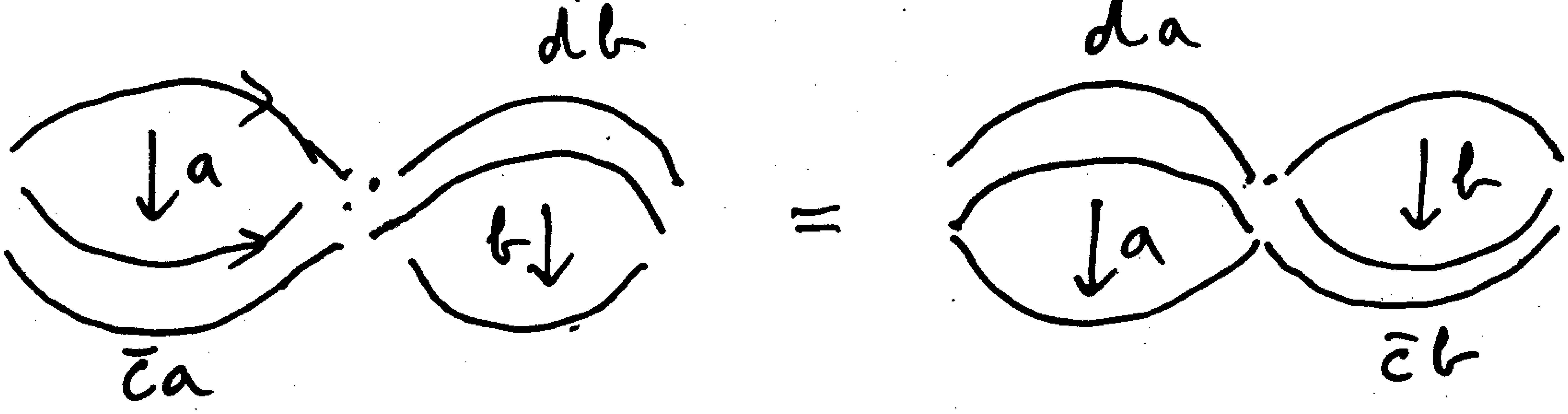


$\boxed{\text{Constructing } \mathbb{X}[\underline{\mathcal{U}}]}$

$\mathbb{X}$: n-category

$\underline{\mathcal{U}} = (\mathcal{U}, d, c)$: set of $(n+1)$-indets (attached to $\mathbb{X}$)

$\boxed{\text{Fact 1}}$ Every $(n+1)$-cell of (the $(n+1)$-cat) $\mathbb{X}[\underline{\mathcal{U}}]$ is of the form

$$\varphi_1 \cdot \varphi_2 \cdots \varphi_N \qquad (\underset{\substack{| \\ (!\,\ldots)}}{N = 0}, 1, 2, \ldots)$$

where each φ_i is an atom

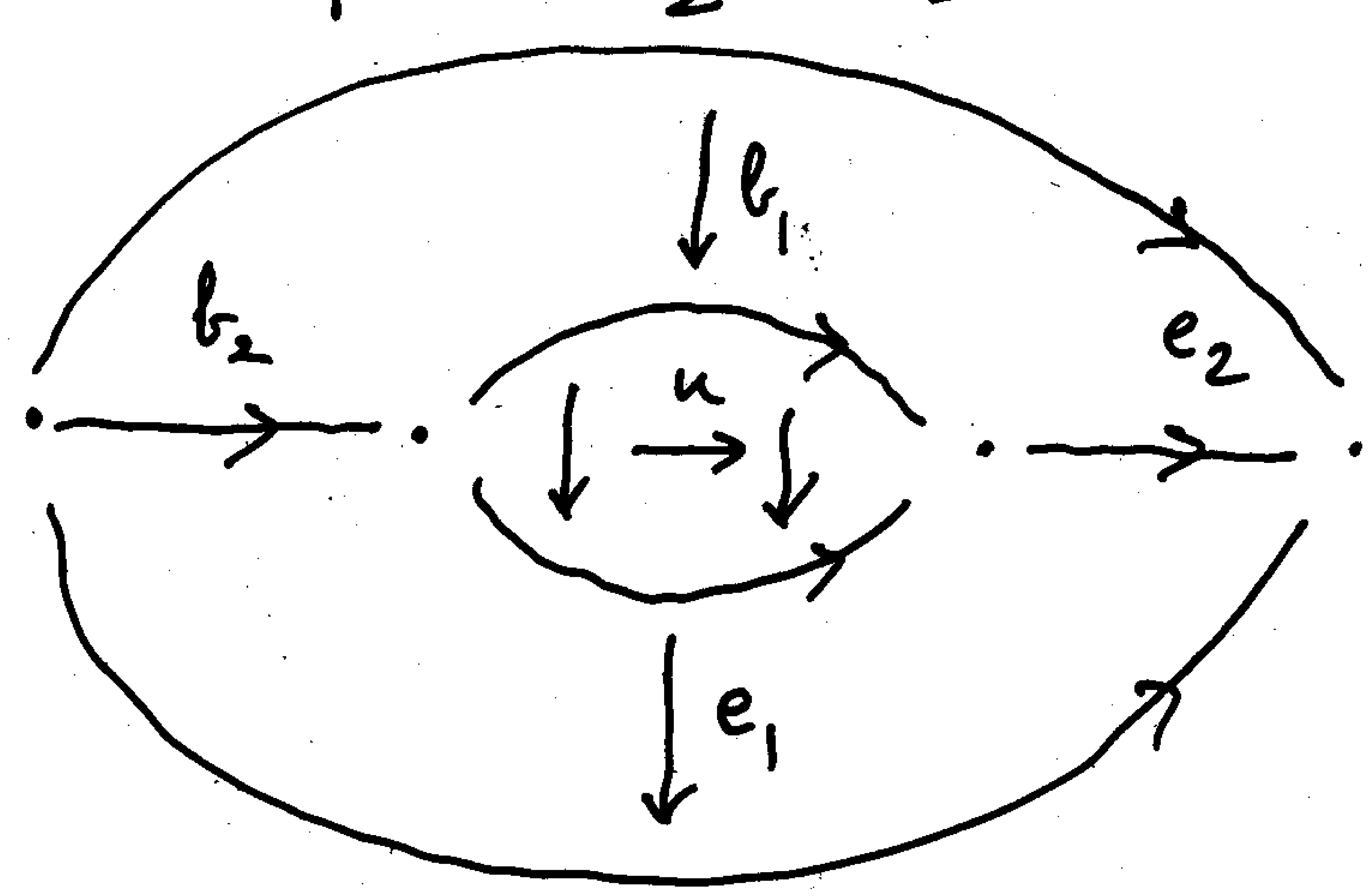
i.e. an $(n+1)$-cell of the form

$$b_n \cdot (b_{n-1} \cdot (\cdots (b_1 \cdot u \cdot e_1) \cdots) \cdot e_{n-1}) \cdot e_n$$

where $u \in \mathcal{U}$; $b_i \in \mathbb{X}_{n-i+1}$, $e_i \in \mathbb{X}_{n-i+1}$

$(i = 1, \ldots, n)$

For $n=2$: $\varphi = b_2 \cdot (b_1 \cdot u \cdot e_1) \cdot e_2$:



Let: $\quad U \longrightarrow \bar{U}$ ..... new set of n-indet's

$$u \longmapsto \bar{u}$$

$$d\bar{u} \stackrel{\text{def}}{=} ddu = cdu$$

$$c\bar{u} \stackrel{\text{def}}{=} cdu = ccu$$

$\underline{U} = (U, d, c) \rightsquigarrow \underline{\bar{U}} = (\bar{U}, d, c)$

attached to $\mathbb{X}$

$\boxed{\text{Consider}}$: $\quad \mathbb{X}[\underline{\bar{U}}]$ ..... n-category

n-cat $\quad$ n-indet's

: $\boxed{\text{Known}}$

[N.B. When $\mathbb{X} = \mathbb{Y}[\underline{V}]$

$(n-1)$-cat $\quad$ n-indets

then $\mathbb{X}[\underline{\bar{U}}] = \mathbb{Y}[\underbrace{\underline{V} \sqcup \underline{\bar{U}}}_{n\text{-indets}}]$

$(n-1)$-cat

for computads : induction]

The <u>barred atoms</u> of $\mathbb{X}[\underline{\bar{u}}]$:

n-cells of the form

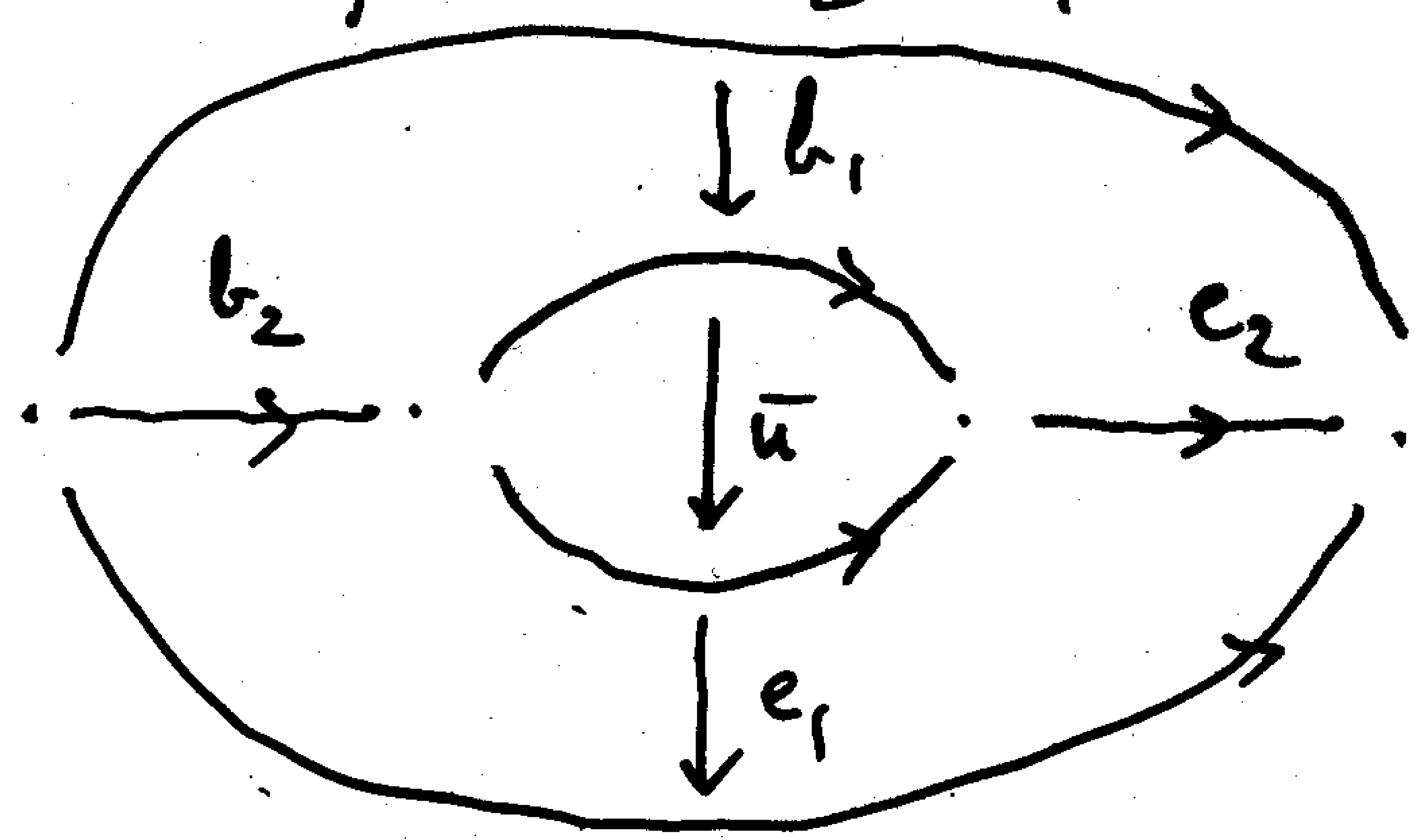
$$b_n \cdot (b_{n-1} \cdot (\cdots (b_1 \cdot \bar{u} \cdot e_1) \cdots) \cdot e_{n-1}) \cdot b_n$$

$u \in \mathcal{U}$

$$b_i,\ e_i \in \mathbb{X}_{n-i+1}$$

[N.B.. A barred atom is not necessarily an atom of $(\mathbb{X}[\underline{\bar{u}}] =) \mathbb{Y}[\underline{V} \cup \underline{\bar{u}}]$]

Picture: $n=2$: $\varphi[\bar{u}] = b_2 \cdot (b_1 \cdot u \cdot e_1) \cdot e_2$:



$\boxed{\text{Fact 2}}$ The map Atoms of $\mathbb{X}[\underline{u}] \longrightarrow$ Barred A's of $\mathbb{X}[\underline{\bar{u}}]$

$$\varphi[u] \longmapsto \varphi[\bar{u}]$$

$$b_n(b_{n-1}(\cdots(b_1 u e_1)\cdots)e_{n-1})e_n$$

$$\longmapsto b_n(b_{n-1}(\cdots(b_1 \bar{u} e_1)\cdots)e_{n-1})e_n$$

is a bijection (well def'd ...)

$\therefore$ An atom of $\mathbb{X}[\underline{u}]$ is 'the same' as a barred atom of $\mathbb{X}[\bar{\underline{u}}]$, hence, "inductively understood"

$\boxed{\text{Fact 3}}$

$$d(\varphi[u]) = \varphi[du]$$
$$c(\varphi[u]) = \varphi[cu]$$

A **molecule**: by definition:

a tuple $\langle \varphi_1, \dots, \varphi_N \rangle$ of $(n+1)$-atoms (of $\mathbb{X}[U]$) such that $\varphi_1 \cdots \varphi_N$ is well-defined

Define: $\sim$

$$(\vec{\varphi} =)\langle \varphi_1, \dots, \varphi_N \rangle \sim \langle \psi_1, \dots, \psi_M \rangle (= \vec{\psi})$$

$$\overset{}{\underset{\text{def}}{\Longleftrightarrow}}$$

$$\varphi_1 \cdots \varphi_N = \psi_1 \cdots \psi_M \quad \text{in } \mathbb{X}[U]$$

Fact: $\vec{\varphi} \sim \vec{\psi} \implies N = M$

$\boxed{\text{Explaining } \sim :}$

Define: $\sim_0$:

molecules

$$\langle \varphi_1, \dots, \varphi_N \rangle \sim_0 \langle \psi_1, \dots, \psi_N \rangle$$

$$\overset{\text{def}}{\Longleftrightarrow}$$

there is $i \in \{1, \dots, N-1\}$ such that

$\varphi_k = \psi_k$ for $k \in \{1, \dots, N\} - \{i, i+1\}$

and

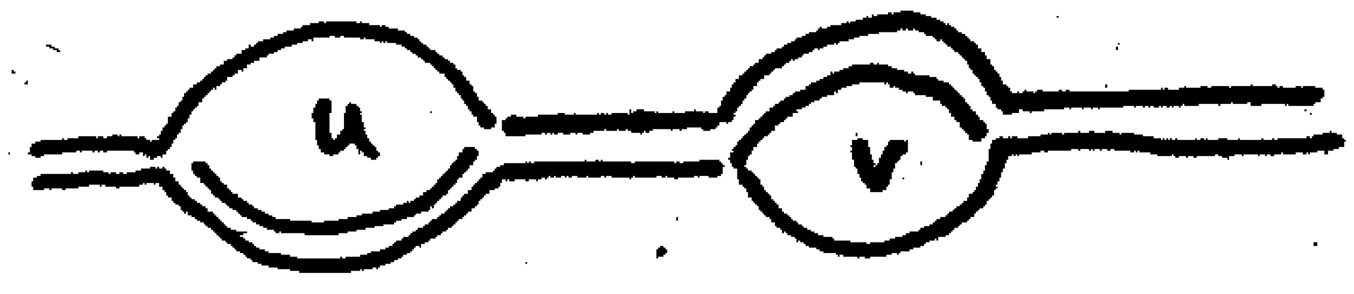
$$\left.\text{“}\varphi_i \cdot \varphi_{i+1} = \psi_i \cdot \psi_{i+1} \text{ because of the commutative law”}\right\} (*)$$

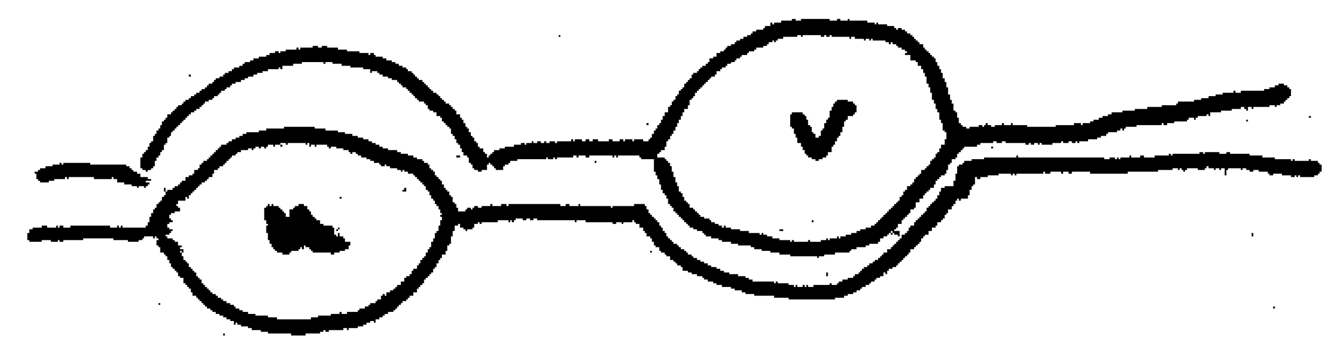
More precisely:

$L(\rho, \sigma, \varphi, \psi) \overset{\text{def}}{\Longleftrightarrow}$ there exist atoms α, β such that

atoms

$\rho = \alpha \cdot d\beta$	ρ
$\sigma = c\alpha \cdot \beta$	σ
$\varphi = d\alpha \cdot \beta$	φ
$\psi = \alpha \cdot c\beta$	ψ





(*) above is, precisely,

"$L(\varphi_i, \varphi_{i+1}, \psi_i, \psi_{i+1})$

or

$L(\psi_i, \psi_{i+1}, \varphi_i, \varphi_{i+1})$"

$\boxed{\textbf{Fact 4}}$ $\sim$ is the transitive/reflexive closure of $\sim_0$.

$\boxed{\textbf{Fact 5}}$ Assume: $\mathbb{X}$ is an n-computad. The $\sim$-equivalence class of any molecule $\langle \varphi_1, \dots, \varphi_N \rangle$ is finite.

$\boxed{\text{Theorem}}$ The word-problem for computads is recursively solvable

The case $n = 1$, $n+1 = 2$: 2-pasting diagrams

(also the subject of:

A. J. Power, A 2-categorical pasting theorem. J. Algebra <u>129</u> (1990), 439–445)

Let : $\mathbb{X}$: 1-computad

arrows of $\mathbb{X}$: composable chains

1-pd's: $\bullet \xrightarrow{f_1} \bullet \xrightarrow{f_2} \bullet \cdots \bullet \xrightarrow{f_\ell} \bullet$

of 1-indeterminates ($\ell = 0$ is allowed ...)

Fix: $N \in \mathbb{N} - \{0\}$

N distinct 2-indet's. $u_1, \dots, u_N$

with $u_i \longmapsto (du_i \,\|\, cu_i)$

1-pd's : non-identity cells

'Molecule' (for now):

$A = \langle \varphi_1, \dots, \varphi_N \rangle$ such that each u_i ("positivity")

occurs in exactly one φ_k.

Write $\varphi[i]$ for this φ_k.

Define: $i < _A j \underset{\text{def}}{\Longleftrightarrow} k < l$

for $\varphi[i] = \varphi_k$
$\varphi[j] = \varphi_l$

a total order
of $[N] = \{1, \dots, N\}$.

For atoms ρ, σ, write:

$\rho \Rightarrow \sigma$ ("ρ & σ are left switchable")

$\underset{\text{def}}{\Longleftrightarrow} \exists: \varphi, \psi$ s.t. $L(\rho, \sigma, \varphi, \psi)$

and:

$\varphi \Leftarrow \psi$

$\underset{\text{def}}{\Longleftrightarrow} \exists: \rho, \sigma$ s.t. $L(\rho, \sigma, \varphi, \psi)$

Fact 6 ($n=1$; the positive case only)

$\rho \Rightarrow \sigma$, $\rho \Leftarrow \sigma$ cannot both hold
(false for the non-positive context)

& if $L(\rho, \sigma, \varphi, \psi)$,
φ & ψ are uniquely determined
(and vice versa)

The tree of variants of a molecule

<u>Define</u> recursively the labelled tree

$$T[A]$$

given molecule
(in terms of $u_1, \dots, u_N$)

At the root r, the label is : A

Suppose we've arrived at node : t — level : p
with label A_t, a molecule.

Define the level-$(p+1)$ successors $\hat{t}$ of t,
<u>if</u> <u>any</u>, with their labels. $A_{\hat{t}}$.
(there may be none).

First, define:

$$i \xrightarrow{A} j \overset{\text{det}}{\iff} \underbrace{i<j}_{\substack{j \text{ is next larger to } i \text{ in } <_{A_t}}} \;\&\; i <_t ! j \;\&\; \varphi[i] \Rightarrow \varphi[j]$$

$$i \xleftarrow{A} j \overset{\text{det}}{\iff} i<j \;\&\; i <_t ! j \;\&\; \varphi[i] \Leftarrow \varphi[j]$$

The successors $\hat{t}$ of t in $T[A]$ are, <u>by definition</u>, in a bijective correspondence with the pairs (i, j) such that

$$i \xrightarrow{A} j \quad \text{or} \quad i \xleftarrow{A} j$$

The label $A_{\hat{t}}$, for $\hat{t}$ corr. to $i \xrightarrow{A} j$:

$$\hat{\varphi}[h] = \varphi[h] \quad \text{for } h \neq i, j$$

and $\hat{\varphi}[i]$, $\hat{\varphi}[j]$ are determined by:

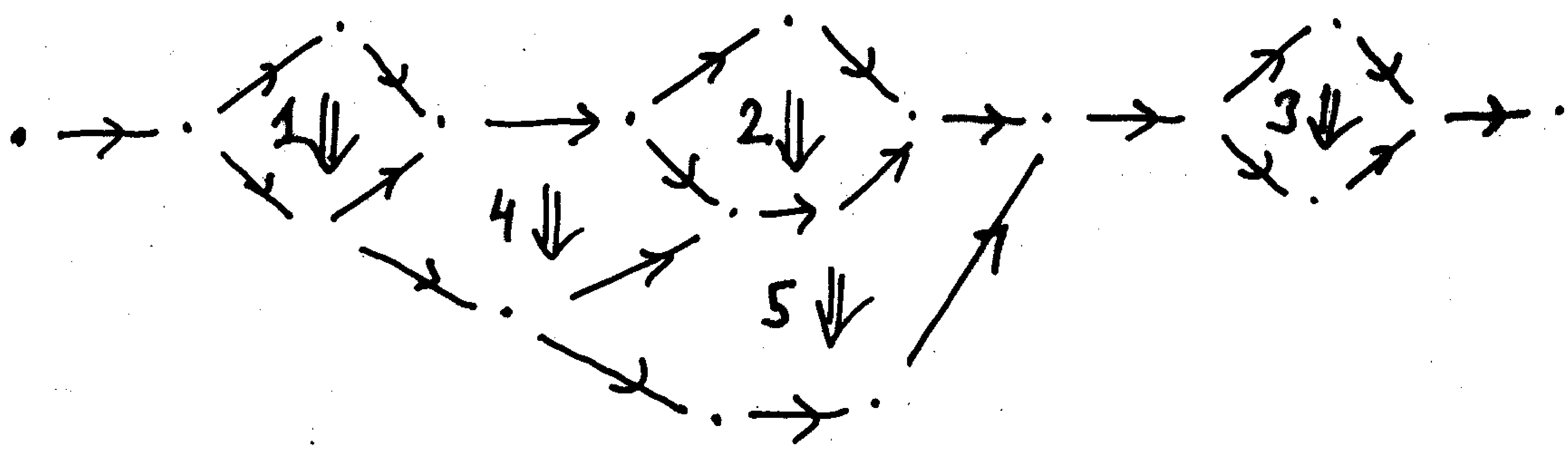
$$L(\varphi[i], \varphi[j], \hat{\varphi}[j], \hat{\varphi}[i])$$

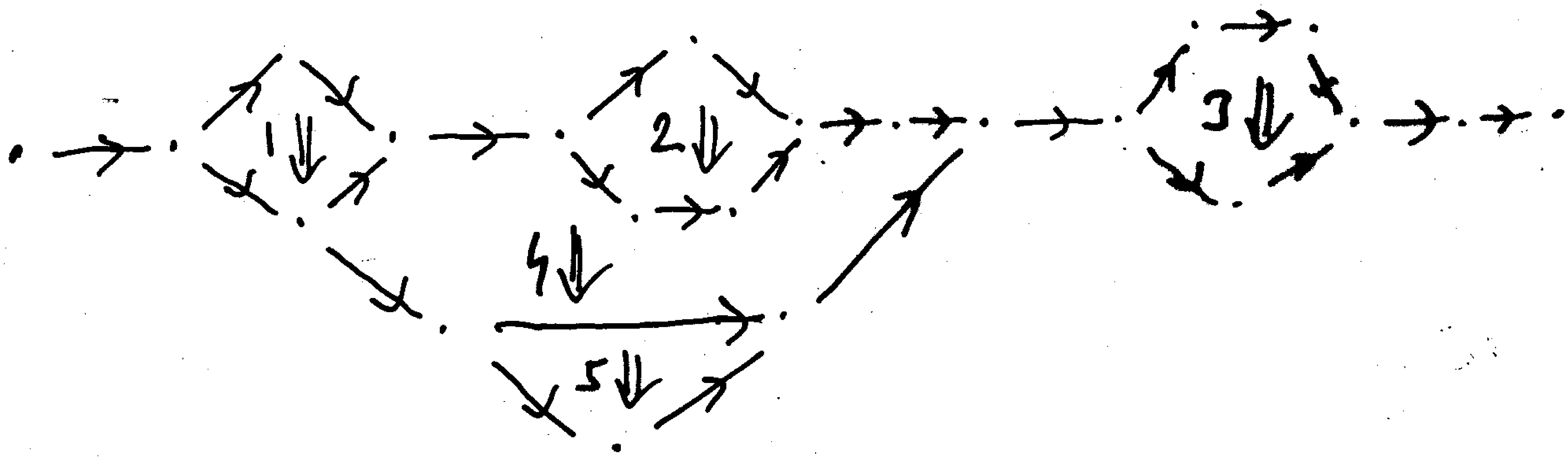
"$\varphi[i]$, $\varphi[j]$ in A_t have been left-switched, to get $A_{\hat{t}}$"

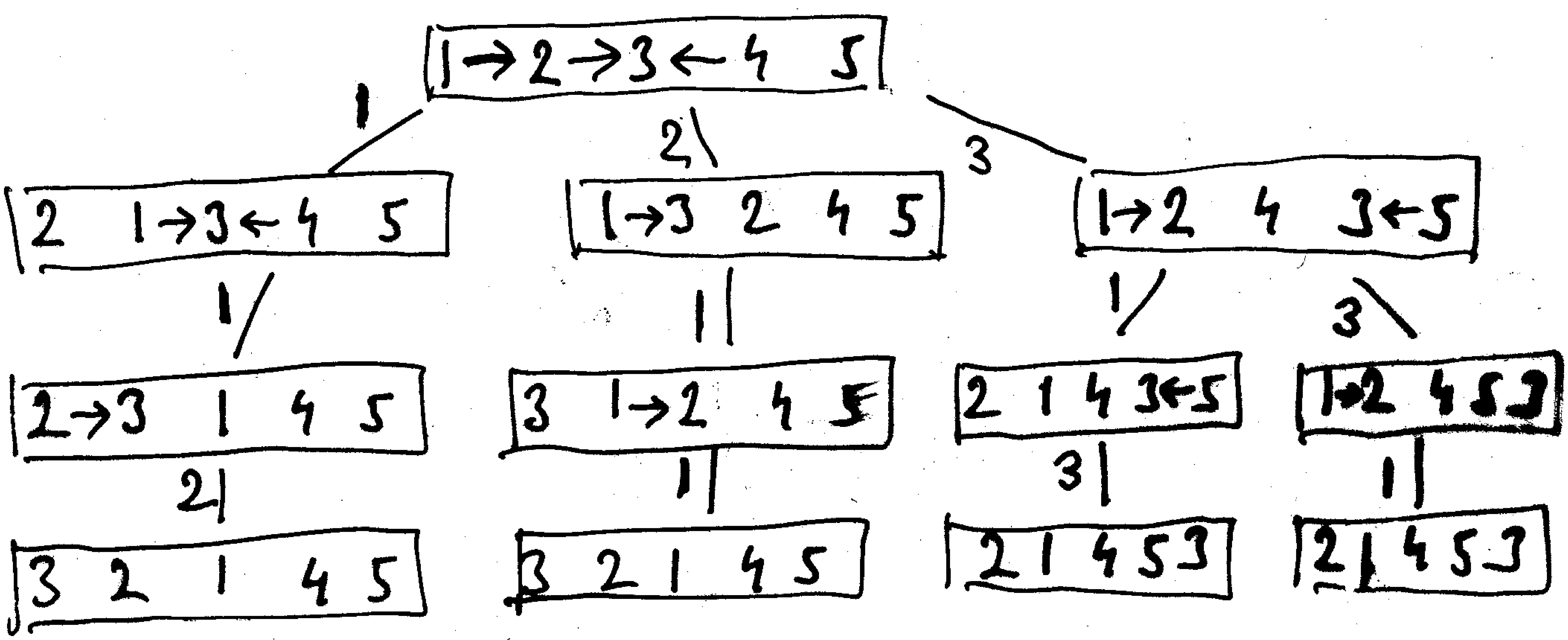
The case $i \xleftarrow{A} j$ is analogous.

(def'n of $T[A]$ is complete)

Ex's:


1⇓
2⇓
3⇓
4⇓
5⇓


1⇓
2⇓
3⇓
4⇓
5⇓


1→2→3←4 5
2 1→3←4 5
1→3 2 4 5
1→2 4 3←5
2→3 1 4 5
3 1→2 4 5
2 1 4 3←5
1→2 4 5 3
3 2 1 4 5
3 2 1 4 5
2 1 4 5 3
2 1 4 5 3

$\boxed{\text{Theorem}}$ ($n=1$; $n+1=2$; positive 2-pd's)

(i) Define: $\lhd \stackrel{\text{def}}{=} \bigcap_{t\in T[\wedge]} <_t$

$$\to \stackrel{\text{def}}{=} \bigcup_{t\in T[\wedge]} \xrightarrow{t} \cup \bigcup_{t\in T} (\xleftarrow{t})^{op}$$

Then: $\lhd$, $\to$ are irreflexive partial orders of $[N]$ which are complementary:

for every $i, j \in [N]$, $i \neq j$:

exactly one of

$i \lhd j$, $j \lhd i$, $i \to j$, $j \to i$

(ii) $<_s = <_t \implies A_s = A_t$

nodes of $T[\wedge]$ — molecules agree atom-by-atom

(iii) Every total order of $[N]$ extending $\lhd$ (see (i)) appears as $<_t$ for at least one t.

(iv) Every molecule B for which $B \sim A$ is $B = A_t$ for at least one t.

To example(s) on p. 18:

Hasse diagram of partial order $\lhd$:

$$\begin{tikzcd}
1 \arrow[d, dash] & 4 \arrow[dl, dash] & \\
2 \arrow[d, dash] & & 3 \\
5 & &
\end{tikzcd}$$

Hasse diagram of partial order $\rightarrow$:

$$\begin{tikzcd}
1 \arrow[r] & 4 \arrow[dr] & \\
2 \arrow[rr] & & 3 \\
5 \arrow[urr] & &
\end{tikzcd}$$

Concrete presheaf categories

Concrete category :

$$\mathbb{A} = (\mathbb{A}, |-|_{\mathbb{A}} : \mathbb{A} \longrightarrow \mathrm{Set})$$

Suppose: $\mathbb{A}, \mathbb{B}$ concrete cat's.

$$\mathbb{A} \simeq \mathbb{B} \quad \underset{def}{\Longleftrightarrow}$$

(equivalent)

$$\exists : \quad \begin{tikzcd} \mathbb{A} \arrow[rr, "\simeq"] \arrow[dr, "|-|_{\mathbb{A}}"'] & \arrow[d, phantom, "\cong"] & \mathbb{B} \arrow[dl, "|-|_{\mathbb{B}}"] \\ & \mathrm{Set} & \end{tikzcd}$$

$\mathbb{A}$ is a concrete presheaf category **if** it is equivalent to the concrete cat

$$\hat{\mathbb{C}} = \left(\mathrm{Set}^{\mathbb{C}^{op}}, |-|_{\hat{\mathbb{C}}}\right) \quad \text{defined by}$$

$$|X|_{\hat{\mathbb{C}}} = \coprod_{U \in ob(\mathbb{C})} X(U)$$

Let X be a finite computad (the set $|X|$ of all indet's in X is finite); and assume there is a unique top-dimensional indet in X; denoted by m_X.

X is **principal** if there is no proper subcomputad of X containing m_X

X is a **computope** if X is principal; and, whenever Y is principal, and $Y \xrightarrow{f} X$ **such** that $f(m_Y) = m_X$, then f is an isomorphism

$\boxed{\text{Theorem}}$ There are enough computopes: for every principal X, there is at least one computope Y, together with a map $Y \xrightarrow{f} X$ such that $f(m_Y) = m_X$.

Proposition (elementary consequence of the non-elementary **Theorem**, p 21)

Suppose $\mathbb{A}$ is a full subcategory of <u>Comp</u>, which is a sieve in <u>Comp</u>,

whenever $B \to A$ and $A \in Ob(\mathbb{A})$,

then $B \in Ob(\mathbb{A})$,.

Then $\mathbb{A}$ is a concrete presheaf category

$\parallel$

$(\mathbb{A}, |{-}|_{\mathbb{A}})$: $|X|_{\mathbb{A}}$ = set of all indets in X

if and only if (a) & (b) :

(a) $\mathbb{A}$ is closed under small colimits in Comp

(b) For every Z in $\mathbb{A}$:

(b1) X computope, $X \overset{f}{\underset{g}{\rightrightarrows}} Z$

$f(m_X) = g(m_X) \implies f = g$

(b2) X, Y computopes, $X \xrightarrow{f} Z \xleftarrow{g} Y$

$f(m_X) = g(m_X) \implies f = g$