Recursive Coalgebras

J. Adámek & S. Milius

Result: Induction $\Longrightarrow$ Recursion

$\not\Longleftarrow$ ..... in general

$\Longleftarrow$ ..... under mild assumptions

Assumption: $H : \mathrm{Set} \longrightarrow \mathrm{Set}$, ($H$ preserves monos)

Recursion

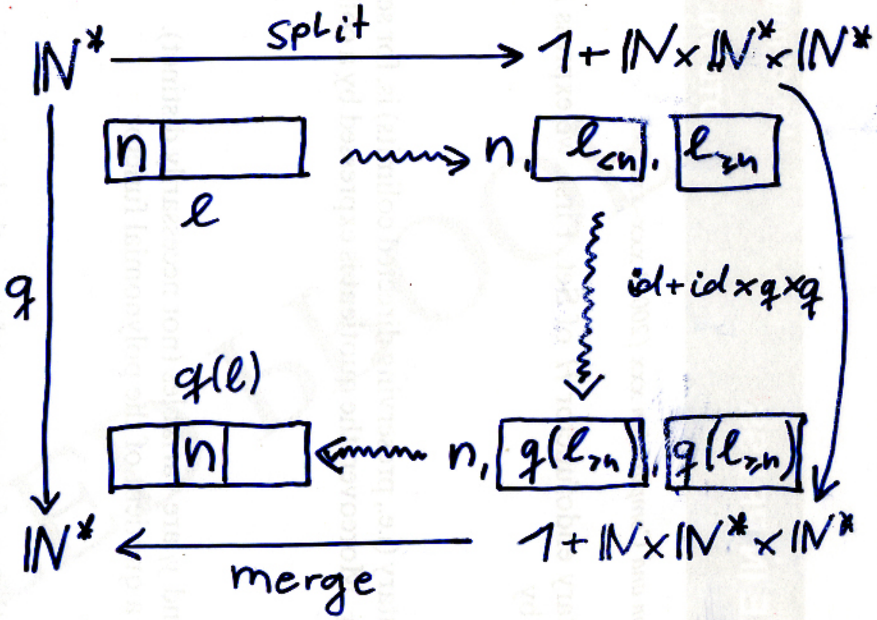
Examples: (i) Initial Algebras

$$\begin{tikzcd}
I \arrow[d, dashed, "\exists!"'] \arrow[r, bend right, "\varphi^{-1}"'] & HI \arrow[l, "\varphi"'] \arrow[d] \\
X & HX \arrow[l, "\forall e"]
\end{tikzcd}$$

- natural numbers : $HX = 1 + X$
- lists over D : $HX = 1 + D \times X$
- trees over D : $HX = 1 + D \times X \times X$

etc.

(ii) Quicksort



Definition: $\alpha : A \longrightarrow HA$ recursive coalgebra $:\Longleftrightarrow$

$$\begin{tikzcd}
A \arrow[r, "\alpha"] \arrow[d, "\exists! e^{\dagger}"'] & HA \arrow[d, "He^{\dagger}"] \\
X & HX \arrow[l, "\forall e"]
\end{tikzcd}$$

- G. Osius (1974)
- P. Taylor (1999)
- A. Eppendahl (1999)
- V. Capretta, T. Uustalu, V. Vene (2003)

Parametric Recursion

Definition: $\alpha : A \longrightarrow HA$ parametrically recursive $:\Longleftrightarrow$

$$
\begin{tikzcd}
A \arrow[r, "{\langle \alpha, id \rangle}"] \arrow[d, "\exists! e^\dagger"'] & HA \times A \arrow[d, "He^\dagger \times A"] \\
X & HX \times A \arrow[l, "\forall e"]
\end{tikzcd}
$$

Examples:

(i) Factorial function: $HX = 1 + X$

$$\mathbb{N} \xrightarrow{pred} 1 + \mathbb{N}$$

$$
\begin{tikzcd}
\mathbb{N} \arrow[r] \arrow[d, "f"'] & (1+\mathbb{N}) \times \mathbb{N} = \mathbb{N} + \mathbb{N} \times \mathbb{N} \arrow[d, "id + f \times id"] \\
\mathbb{N} & \mathbb{N} + \mathbb{N} \times \mathbb{N} \arrow[l, "e"']
\end{tikzcd}
$$

$n \longmapsto 0 \ / \ (n-1, n)$

$1 \longleftarrow\!\shortmid\ 0$

$x \cdot n \longleftarrow\!\shortmid\ (x, n)$

(ii) Fibonacci sequence: $HX = 1 + X^2$

$$\mathbb{N} \longrightarrow 1 + \mathbb{N} \times \mathbb{N}$$

$0, 1 \longmapsto *$

$n \longmapsto (n-2, n-1)$

$$
\begin{tikzcd}
\mathbb{N} \arrow[r] \arrow[d] & (1+\mathbb{N} \times \mathbb{N}) \times \mathbb{N} = \mathbb{N} + \mathbb{N}^3 \arrow[d, "id + f^2 \times id"] \\
\mathbb{N} & \mathbb{N} + \mathbb{N}^3 \arrow[l]
\end{tikzcd}
$$

$n \longmapsto 0 \ / \ 1 \ / \ (n-2, n-1, n)$

$a_0 + a_1 \longleftarrow\!\shortmid\ 0, 1$

$x + n \longleftarrow\!\shortmid\ (x, y, n)$

Theorem: initial Algebra $\equiv$ terminal parametrically recursive coalgebra

Induction (Well-founded Coalgebras)

Definition: (B. Jacobs 2002)

Given $\alpha : A \longrightarrow HA$ define $\text{nexttime} : \mathrm{Sub}(A) \circlearrowleft$

$$\begin{tikzcd}
 & \text{nexttime}(M) \arrow[r] \arrow[d, hook] \arrow[dr, phantom, very near start, "\lrcorner"] & HM \arrow[d, hook, "Hm"] \\
M \arrow[r, hook, "m"'] & A \arrow[r, "\alpha"'] & HA
\end{tikzcd}$$

Examples:

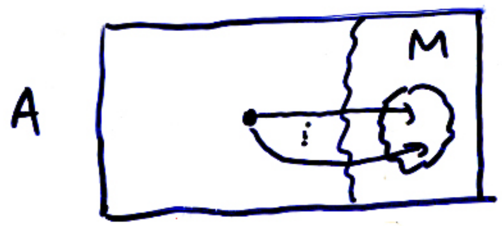
(i) $HX = 1 + \mathbb{N} \times X \times X$

$A \xrightarrow{\alpha} 1 + \mathbb{N} \times A \times A$

$\text{nexttime}(M) = \{ a \mid a \text{ halting or } \alpha(a) \in \mathbb{N} \times M^2 \}$

(ii) $H = \mathcal{P}$

$A \xrightarrow{\alpha} \mathcal{P}A$



$\text{nexttime}(M) = \{ a \mid \alpha(a) \subseteq M \}$

Definition: (P. Taylor) $A \xrightarrow{\alpha} HA$ well-founded $:\Longleftrightarrow$ $\dfrac{\text{nexttime}(M) \subseteq M}{M = A}$ $(M \subseteq$

Well-founded Coalgebras

Definition: Well-founded core of any $\alpha : A \longrightarrow HA$:

$$A^* = \mu M.\ \mathrm{nexttime}(M)$$

Facts:

- A well-founded $\iff$ $A = A^*$
- A^* is a well-founded coreflection of the coalgebra A

Examples:

(i) $HX = 1 + \mathbb{N} \times X \times X$

$$A \xrightarrow{\alpha} 1 + \mathbb{N} \times A \times A$$

automaton with binary input and output from $\mathbb{N}$

$\vdots$

A^* = all states allways reaching a halting state in finitely many steps

$\vdots$

(A, α) well-founded iff all states have this "halting property"

(ii) $H = \mathcal{P}$

$$A \xrightarrow{\alpha} \mathcal{P}A$$

a graph

$\vdots$

A^* = all nodes not on an infinite path

$\vdots$

(A, α) well-founded iff no node lies on an infinite path

General Categories

Example: Discrete-time dynamical systems given by $HX = 1 + X$ on Top:

$A \xrightarrow{\alpha} 1 + A$

A well-founded

iff every state has the "halting property"

BUT: nexttime : Sub(A) $\circlearrowleft$

has proper subobjects as fixed points

e.g. id : $A_0 \longrightarrow A$

discrete space

$\hookrightarrow$ monos $\rightsquigarrow$ strong monos

Assumptions:
- $\mathcal{C}$ well-powered, complete, cocomplete and has universally chain-complete strong monos
- $H : \mathcal{C} \longrightarrow \mathcal{C}$ preserving strong monos and inverse images of strong subob

Theorem: If $\mathcal{C}$ also satisfies:
- every object is generated
- every non-empty mono splits

then for every $\alpha : A \longrightarrow HA$ t.f.a.e

(i) A well-founded (ii) A parametrically recursive

(iii) A recursive (iv) A has a homomorphism into I

Theorem: (P. Johnstone)

If $\mathcal{C}$ has a strong-subobject classifier, then for every $A \xrightarrow{\alpha} HA$ properties (i) – (iii) are equivalent.

Induction $\Longrightarrow$ Recursion

Theorem: For a coalgebra $\alpha : A \longrightarrow HA$ we have

well-founded	$\Longrightarrow$	parametrically recursive	$\Longrightarrow$	recursive	$\Longrightarrow$	a homomorphism into I
	$\overset{?}{\Longleftarrow}$		$\not\Longleftarrow$		$\not\Longleftarrow$	

Open Problems:

- Are parametrically recursive coalgebras well-founded?
- Does every coalgebra have a recursive coreflection?

Theorem: For an endofunctor $H : \mathrm{Set} \to \mathrm{Set}$ preserving inverse images the 4 properties above are equivalent.

Application: Recursion $\Longrightarrow$ Partial Recursion

$H: \mathsf{Set} \longrightarrow \mathsf{Set}$ preserving inverse images ..... $H_p: \mathsf{Set}_p \circlearrowleft$ lifting of H

$A \xrightarrow{\alpha} HA$ recursive for H $\Longrightarrow$ A well-founded for H

$\Longrightarrow$ A well-founded for H_p

$\Longrightarrow$ A recursive for H_p

$$
\begin{tikzcd}
A \arrow[r, "\alpha"] \arrow[d, dashed, "\exists! e^t"'] & H_pA \arrow[d, "H_p e^t"] \\
X & H_pX \arrow[l, "\forall e"]
\end{tikzcd}
$$

Open Problem: How about relations in lieu of partial maps?