Pseudo-Exponentiability

in

Homotopy Slices of **Top**

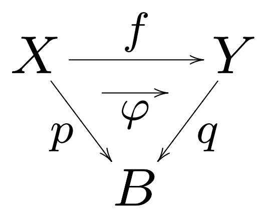
and

Pseudo-Slices of **Cat**

Top$//B$ *homotopy slice of* **Top**

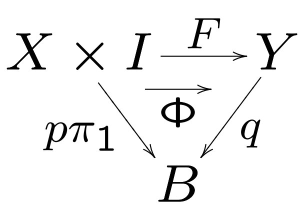
objects $p: X \longrightarrow B$

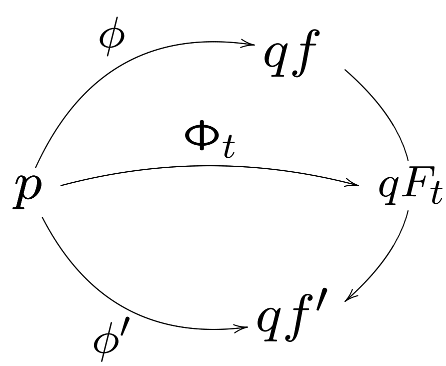
morphisms



Get a bicategory *(composition is not associative)*

What are the 2-cells? Equivalence classes of

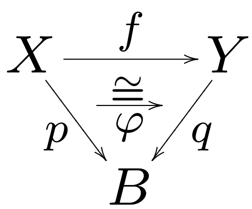


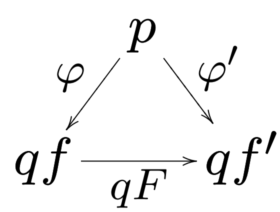


To obtain **Top**$//B$, we'll use a variation of the construction of the lax slice **Cat**$\nearrow B$ as the Kleisli 2-category of a 2-monad on **Cat**$/B$ (Street SLN 420).

$\mathbf{Cat}//B$ *pseudo-slice of* $\mathbf{Cat}$ *(2-category)*

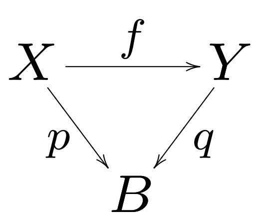
objects $p\colon X \longrightarrow B$

morphisms 

2-cells $F\colon f \longrightarrow f'$ s.t. 

$\mathbf{Cat}/B$ *2-slice of* $\mathbf{Cat}$ *(2-category)*

objects $p\colon X \longrightarrow B$

morphisms 

2-cells $F\colon f \longrightarrow f'$ s.t. $qF = id_p$

NOTATION

Consider B^I in $\mathbf{Cat}$, where I is the category

$$0 \xrightarrow{\cong} 1$$

The *composition* functor is denoted by

$$B^I \times_B B^I \xrightarrow{\ \circ\ } B^I$$

and the *identity-valued* functor by

$$B \xrightarrow{\ \iota.\ } B^I$$

where

$$\begin{array}{ccc} B^I \times_B X & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle p} \\ B^I & \xrightarrow[ev_1]{} & B \end{array} \qquad\qquad \begin{array}{ccc} X \times_B B^I & \longrightarrow & B^I \\ \downarrow & & \downarrow{\scriptstyle ev_0} \\ X & \xrightarrow[p]{} & B \end{array}$$

Note.

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & {\scriptstyle p}\searrow \ \underset{\varphi}{\overset{\cong}{\Rightarrow}} \ \swarrow{\scriptstyle q} & \\ & B & \end{array} \qquad \longleftrightarrow \qquad \begin{array}{ccc} X & \xrightarrow{\langle \widehat{\varphi}, f \rangle} & B^I \times_B Y \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle ev_0 \pi_1} & \\ & B & \end{array}$$

Define $T\colon \mathbf{Cat}/B \longrightarrow \mathbf{Cat}/B$ by

$$T(X \xrightarrow{p} B) = B^I \times_B X \xrightarrow{ev_0 \pi_1} B$$

$$\eta_X\colon X \xrightarrow{\langle \iota_p, id_X \rangle} B^I \times_B X$$

$$\mu_X\colon B^I \times_B B^I \times_B X \xrightarrow{\circ \times id_X} B^I \times_B X$$

$\mathbf{T} = (T, \eta, \mu)$ is a *2-monad* on $\mathcal{K} = \mathbf{Cat}/B$

$$\begin{array}{ccccc} T & \xrightarrow{T\eta} & T^2 & \xleftarrow{\eta T} & T \\ & {\scriptstyle id_T} \searrow & \downarrow{\scriptstyle \mu} & \swarrow {\scriptstyle id_T} & \\ & & T & & \end{array} \qquad \begin{array}{ccc} T^3 & \xrightarrow{T\mu} & T^2 \\ {\scriptstyle \mu T}\downarrow & & \downarrow{\scriptstyle \mu} \\ T^2 & \xrightarrow[\mu]{} & T \end{array}$$

$\mathbf{Cat}//B$ is the *Kleisli 2-category* $\mathcal{K}_{\mathbf{T}}$ of $\mathbf{T}$

$$|\mathcal{K}_{\mathbf{T}}| = |\mathcal{K}| \qquad \mathcal{K}_{\mathbf{T}}(X, Y) = \mathcal{K}(X, TY)$$

with $id_X = \eta_X$ and composition induced by μ

$\mathbf{Top}/B$ *2-slice of* $\mathbf{Top}$ *(2-category)*

objects $p\colon X \longrightarrow B$

morphisms
$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & B & \end{array}$$

2-cells equivalence classes* of
$$\begin{array}{ccc} X \times I & \xrightarrow{F} & Y \\ & {\scriptstyle p\pi_1}\searrow \quad \swarrow{\scriptstyle q} & \\ & B & \end{array}$$

s.t. $F|_{X\times 0} = f,\ F|_{X\times 1} = f'$

* $F \sim F'$ if there exists
$$\begin{array}{ccc} X \times I^2 & \xrightarrow{h} & Y \\ & {\scriptstyle p\pi_1}\searrow \quad \swarrow{\scriptstyle q} & \\ & B & \end{array}$$

$h|_{X\times I\times 0} = F,\ h|_{X\times I\times 1} = F',\ h|_{X\times 0\times I} = f,\ h|_{X\times 1\times I} = f'$

Define $T\colon \mathbf{Top}/B \longrightarrow \mathbf{Top}/B$ by

$$T(X \xrightarrow{p} B) = B^I \times_B X \xrightarrow{ev_0\pi_1} B$$

$$\eta_X\colon X \xrightarrow{\langle \iota_p, id_X \rangle} B^I \times_B X$$

$$\mu_X\colon B^I \times_B B^I \times_B X \xrightarrow{\circ \times id_X} B^I \times_B X$$

$\mathbf{T} = (T, \eta, \mu)$ is a *pseudo-monad* on $\mathbf{Top}/B$

Note. T is a 2-functor and η, μ are 2-natural, but

$$\begin{array}{ccccc} T & \xrightarrow{T\eta} & T^2 & \xleftarrow{\eta T} & T \\ & \underset{id_T}{\searrow} \cong & \downarrow{\scriptstyle \mu} & \cong \underset{id_T}{\swarrow} & \\ & & T & & \end{array} \qquad\qquad \begin{array}{ccc} T^3 & \xrightarrow{T\mu} & T^2 \\ {\scriptstyle \mu T}\downarrow & \cong & \downarrow{\scriptstyle \mu} \\ T^2 & \xrightarrow[\mu]{} & T \end{array}$$

only commute up to invertible modifications

Define $\mathbf{Top}//B$ to be the *Kleisli bicategory*

Remark. Given $q: Y \to B$ exponentiable in the *category* $\mathbf{Top}/B$, the natural bijections

$$\theta_{p,r}: \mathbf{Top}/B(X \times_B Y, Z) \to \mathbf{Top}/B(X, Z^Y)$$

are 2-natural isos of categories, or equivalently, the adjunction $- \times_B Y \dashv (\ )^Y$ is a 2-adjunction.

When is q pseudo-exponentiable in $\mathbf{Top}//B$?

Recall. An object Y is *pseudo-exponentiable* in a bicategory $\mathcal{K}$ if $- \times Y: \mathcal{K} \to \mathcal{K}$ has a right pseudo-adjoint, i.e., there are equivalences

$$\theta_{X,Z}: \mathcal{K}(X \times Y, Z) \to \mathcal{K}(X, Z^Y)$$

pseudo-natural in X and Z.

Lemma 1. If $\mathbf{T}$ is a pseudo-monad on a bicategory $\mathcal{K}$ with binary pseudo-products and

$$\rho: T(X \times TY) \longrightarrow TX \times TY$$

is an equivalence in $\mathcal{K}$, then $X \times TY$ is a pseudo-product of X and Y in $\mathcal{K}_{\mathbf{T}}$.

Examples. $X \times_B B^I \times_B Y$ is a pseudo-product in $\mathbf{Cat}//B$ and $\mathbf{Top}//B$, since

$$\rho: B^I \times_B (X \times_B B^I \times_B Y) \longrightarrow (B^I \times_B X) \times_B (B^I \times_B Y)$$

is given in both cases by

$$\big(b \overset{\alpha}{\to} px, x, px \overset{\beta}{\to} qy, y\big) \mapsto \big((b \overset{\alpha}{\to} px, x), (b \overset{\alpha}{\to} px \overset{\beta}{\to} qy, y)\big)$$

Suppose TY is pseudo-exponentiable in $\mathcal{K}$, and consider:

$$\mathcal{K}_{\mathbf{T}}(X{\times}TY{,}Z){=}\mathcal{K}(X{\times}TY{,}TZ)\longrightarrow\mathcal{K}(T(X{\times}TY){,}T^2Z)$$

$$\xrightarrow{\mathcal{K}(id,\mu)}\mathcal{K}(T(X{\times}TY){,}TZ){\simeq}\mathcal{K}(TX{\times}TY{,}TZ){\simeq}\mathcal{K}(TX{,}TZ^{TY})$$

$$\xrightarrow{\mathcal{K}(id,\eta)}\mathcal{K}(TX{,}T(TZ^{TY}))\xleftarrow{\mathcal{K}(id,\mu)}\mathcal{K}(TX{,}T^2(TZ^{TY}))$$

$$\longleftarrow\mathcal{K}(X{,}T(TZ^{TY})){=}\mathcal{K}_{\mathbf{T}}(X{,}TZ^{TY})$$

If these functors are all equivalences, then Y will be pseudo-exponentiable in $\mathcal{K}_{\mathbf{T}}$.

Lemma 2. If $\mathbf{T}$ is a pseudo-monad on $\mathcal{K}$ and $\eta T \cong T\eta$, then

$$\mathcal{K}(X, TY) \longrightarrow \mathcal{K}(TX, T^2Y) \xrightarrow{\mathcal{K}(id,\mu)} \mathcal{K}(TX, TY)$$

is an equivalence, for all X, Y.

Lemma 3. If $\mathbf{T}$ is as in Lemma 2 and TY is pseudo-exponentiable in $\mathcal{K}$, then

$$\eta : TZ^{TY} \longrightarrow T(TZ^{TY})$$

is an equivalence in $\mathcal{K}$, for all Z.

Examples. $\eta T \cong T\eta$ in $\mathbf{Cat}/B$ and $\mathbf{Top}/B$

Theorem. Suppose $\mathcal{K}$ has pseudo-products, $\mathbf{T}$ is a pseudo-monad on $\mathcal{K}$, $\eta T \cong T\eta$, and

$$\rho: T(X \times TY) \to TX \times TY$$

is an equivalence, for all X.

If TY is pseudo-exponentiable in $\mathcal{K}$, then Y is pseudo-exponentiable in $\mathcal{K}_{\mathbf{T}}$.

Proof. By Lemmas 2 and 3, the functors

$$\mathcal{K}_{\mathbf{T}}(X \times TY, Z) = \mathcal{K}(X \times TY, TZ) \longrightarrow \mathcal{K}(T(X \times TY), T^2 Z)$$

$$\xrightarrow{\mathcal{K}(id,\mu)} \mathcal{K}(T(X \times TY), TZ) \simeq \mathcal{K}(TX \times TY, TZ) \simeq \mathcal{K}(TX, TZ^{TY})$$

$$\xrightarrow{\mathcal{K}(id,\eta)} \mathcal{K}(TX, T(TZ^{TY})) \xleftarrow{\mathcal{K}(id,\mu)} \mathcal{K}(TX, T^2(TZ^{TY}))$$

$$\longleftarrow \mathcal{K}(X, T(TZ^{TY})) = \mathcal{K}_{\mathbf{T}}(X, TZ^{TY})$$

are all equivalences of categories.

Corollary 1. If $q: Y \longrightarrow B$ is a (Hurewicz) fibration and q is exponentible in $\mathbf{Top}/B$, then q is pseudo-exponentiable in $\mathbf{Top}//B$.

Proof. Show $\eta_Y: Y \to B^I \times_B Y$ is an equivalence in $\mathbf{Top}/B$, with pseudo-inverse given by $\eta'_Y(\beta, y) = H(\beta, y, 0)$, where

$$\begin{array}{ccc} B^I \times_B Y & \xrightarrow{\pi_2} & Y \\ {\scriptstyle \langle id,1\rangle}\big\downarrow & \overset{H}{\nearrow} & \big\downarrow{\scriptstyle q} \\ (B^I \times_B Y) \times I & \xrightarrow[ev \circ \pi_{13}]{} & B \end{array}$$

Then q exp $\Rightarrow$ Tq pseudo-exp in $\mathbf{Top}/B$ $\Rightarrow$ q pseudo-exp in $\mathbf{Top}//B$, by the Theorem.

Examples. Exponentible maps in $\mathbf{Top}/B$

include $q: Y \longrightarrow B$ such that

(1) Y locally compact, B locally Hausdorff

(2) q locally trivial with locally compact fibers

(3) q local homeomorphism

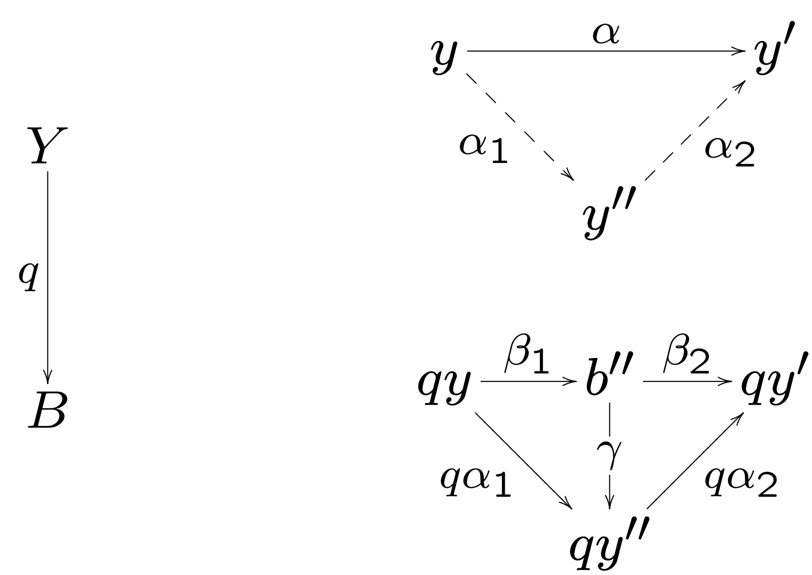
Corollary 2. TFAE for $q: Y \longrightarrow B$ in $\mathbf{Cat}$:

(a) $B^I \times_B Y \xrightarrow{ev_0 \pi_1} B$ is 2-exponentible in $\mathbf{Cat}/B$

(b) q is pseudo-exponentible in $\mathbf{Cat}//B$

(c) q satisfies pseudo-lifting property:

(PLP) Given $y \xrightarrow{\alpha} y'$ in Y and $q\alpha = \beta_2\beta_1$ in B, $\exists \alpha_1, \alpha_2$, and an isomorphism γ s.t.



Proof. (a)⇒(b) by the Theorem

Remarks.

1. The (PLP) with $\gamma = id_{b''}$ is the Giraud-Conduché condition for exponentiability in $\mathbf{Cat}/B$.

2. Corollary 2 (b)⇔(c) is in Johnstone's "Fibrations and partial products in a 2-category", Appl. Categ. Structures 1 (1993), no. 2, 141–179.