Paths in Double Categories

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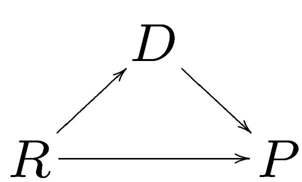
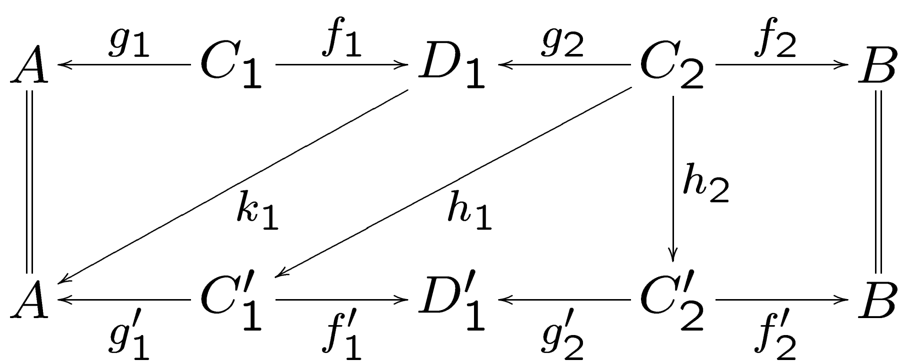


Motivation: understanding the Π_2 construction

- a more informative two dimensional version of Π_1, where 2-cells are equivalence classes of diagrams, called fences, of the form:



- What is the equivalence relation for 2-cells in Π_2 of a 2-category?

- Π_2 can be viewed as a composition of two constructions which we will call **Path** and **Span**

Overview

- Review: Path and Path_* for Categories

- $\mathbb{P}\text{ath}$ for Double Categories
  - The Construction
  - The Morphism $\Xi\colon \mathbb{A} \to \mathbb{P}\text{ath}\,\mathbb{A}$

- Universal Properties and Oplax Double Categories

- $\mathbb{P}\text{ath}_*$ for Double Categories
  - The Construction
    (With Equivalence Relation)
  - The Morphism $\Xi_*\colon \mathbb{A} \to \mathbb{P}\text{ath}_*\,\mathbb{A}$

- Universal Properties and Oplax Normal Double Categories

- The Equivalence Relation on $\Pi_2\mathcal{A}$

Path for Categories

Cat: category of small categories
Gph: category of directed multi-graphs

$$\mathbf{Cat} \underset{U}{\overset{F}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} \mathbf{Gph}$$

• U is monadic, E-M algebras are small categories
• F is comonadic, graphs are categories with a costructure

Comonad: $(FU, \varepsilon, F\eta U) = (\mathsf{Path}, E, D)$

Coalgebras: categories with a comultiplication (coaction) $\Phi \colon \mathbf{A} \to \mathsf{Path}(\mathbf{A})$, factorizing each arrow into 'primes'.

These primes form the edges of a graph.

$$\mathbf{Gph} \simeq \mathbf{Cat}_{\mathsf{Path}}$$

Note:

- **Path**: **Cat** $\to$ **Cat** is not a 2-functor.

- Any category has at most one coalgebra structure on it.

- Kleisli category: category of small categories with 'functors' which don't preserve identities nor composition.

$\mathbf{Path}_*$ for Categories

Cat: category of small categories
RGph: category of *reflexive* directed multi-graphs

$$\mathbf{Cat} \underset{U}{\overset{F}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} \mathbf{RGph}$$

U is monadic, E-M algebras are small categories
F is comonadic, a reflexive graph is a category with a costructure

$\mathbf{Path}_*$ differs from **Path** in treatment of identities.

Comonad: $(FU, \varepsilon, F\eta U) = (\mathsf{Path}_*, E_*, D_*)$

Coalgebras: categories with a comultiplication (coaction) $\Phi\colon \mathbf{A} \to \mathsf{Path}\,(\mathbf{A})$, factorizing each arrow into non-identity 'primes'.

$$\mathbf{Rgph} \simeq \mathbf{Cat}_{\mathsf{Path}_*}. \tag{1}$$

Note:

- $\mathbf{Path}_*\colon \mathbf{Cat} \to \mathbf{Cat}$ is not a 2-functor.
- Any category has at most one coalgebra structure on it.
- Kleisli category: small categories with 'functors' which don't preserve composition.

$\mathbb{P}$ath for Double Categories

Let $\mathbb{A}$ be a double category. Then $\mathbb{P}\mathrm{ath}\,\mathbb{A}$ is defined as:

- same objects and vertical arrows as in $\mathbb{A}$
- A horizontal arrow in $\mathbb{P}\mathrm{ath}\,\mathbb{A}$ is a path

$$A_0 \xrightarrow{f_1} A_1 \xrightarrow{f_2} A_2 \longrightarrow \cdots \longrightarrow A_{m-1} \xrightarrow{f_m} A_m$$

- Double cells are triples $(\varphi, \langle v_i \rangle, \langle \alpha_i \rangle)$:

∗ an order preserving indexing function

$$\varphi\colon \{0, 1, \ldots, m\} \to \{0, 1, \ldots, n\}$$

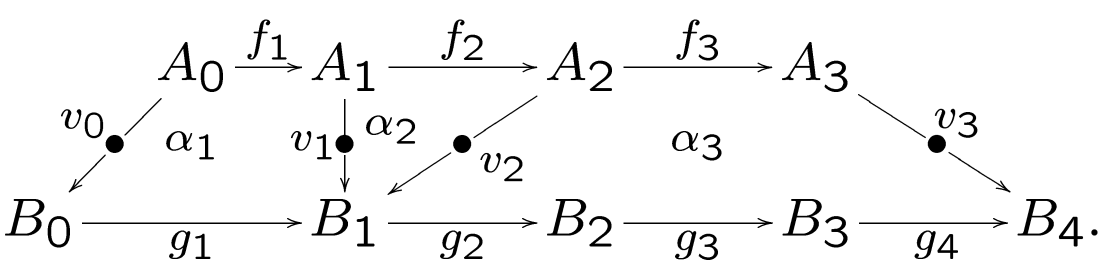
with $\varphi(0) = 0$ and $\varphi(m) = n$;

∗ vertical arrows $v_i\colon A_i \longrightarrow\!\!\bullet\!\!\longrightarrow B_{\varphi(i)}$;

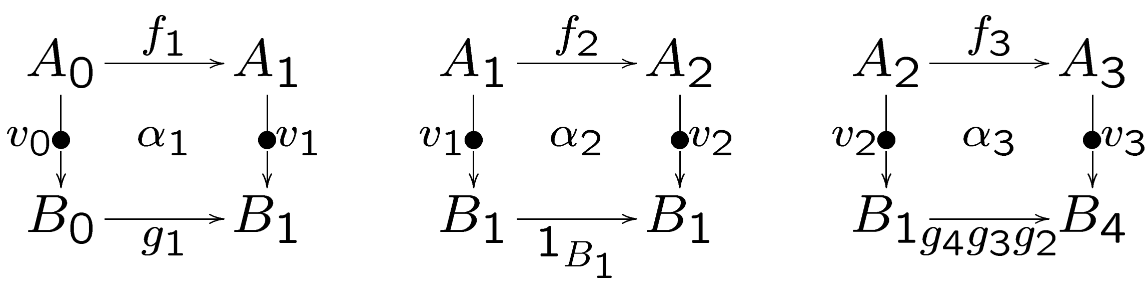
∗ double cells

$$\begin{array}{ccc} A_{i-1} & \xrightarrow{\quad f_i \quad} & A_i \\ {\scriptstyle v_{i-1}}\big\downarrow\bullet & \alpha_i & \bullet\big\downarrow{\scriptstyle v_i} \\ B_{\varphi(i)} & \xrightarrow[\;g^{\varphi(i-1)}_{\varphi(i)}\;]{} & B_{\varphi(i)} \end{array}$$

A typical cell:



where



are cells in $\mathbb{A}$.

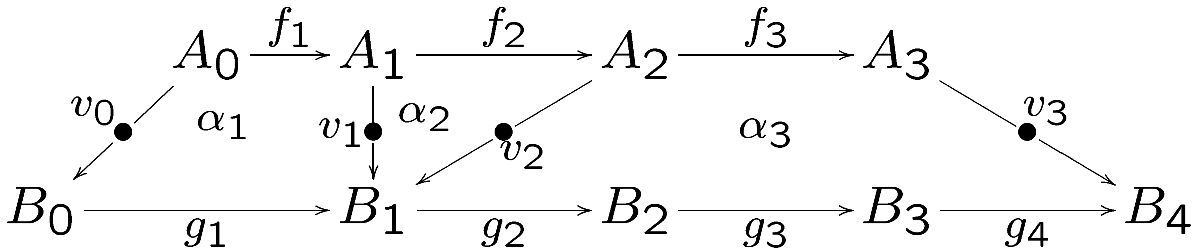
Notation $g_4^2 = g_4 g_3 g_2$.

Composition

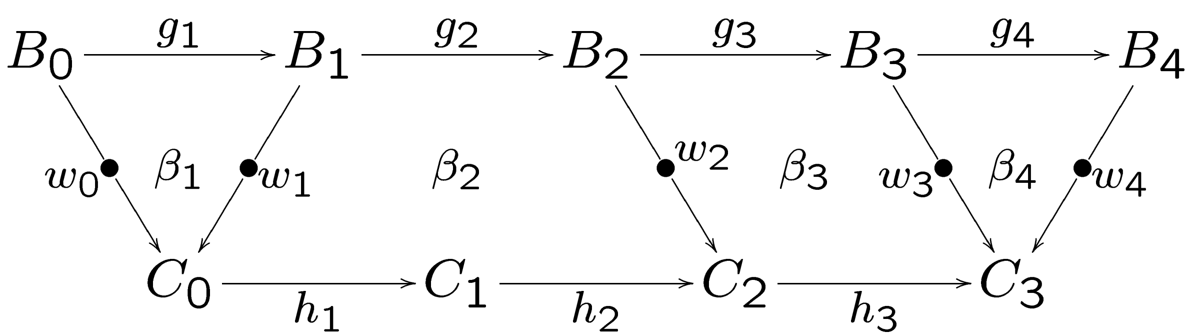
- Horizontal composition of boths arrows and cells is by concatenation.

- Vertical composition of arrows is as in $\mathbb{A}$.

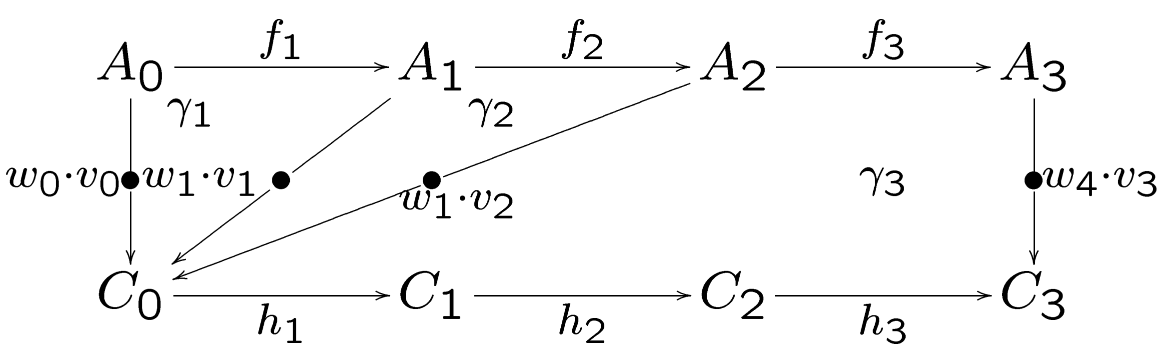
- Vertical composition of



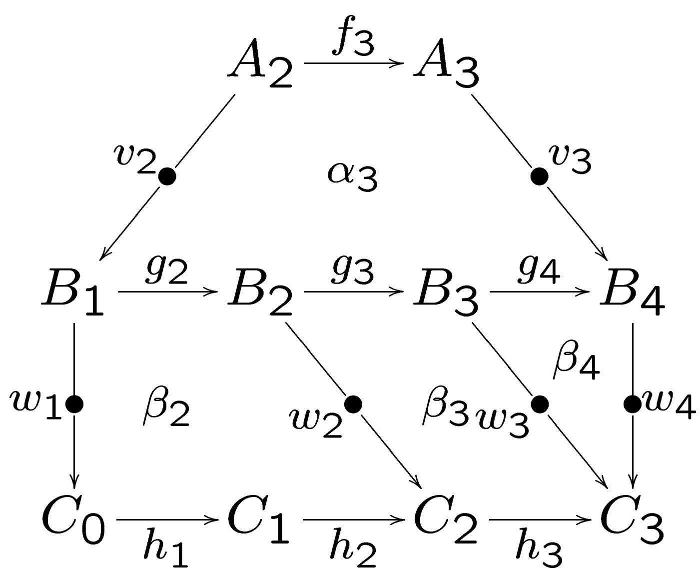
and



is



where γ_3 is the pasting of

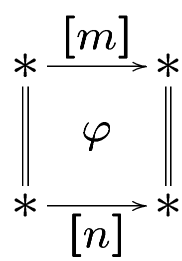


.

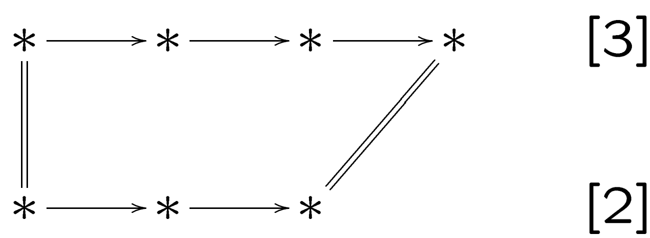
Example: $\mathbb{P}\text{ath}\,\mathbb{1}$, where $\mathbb{1}$ is the terminal double category, with one object $*$ and one vertical arrow, 1_*.

• Horizontal arrows correspond to non-empty ordinals $[m] = \{0, 1, \ldots, m\}$ (a path of m indentity arrows).

• Cells



such as



correspond to order preserving functions

$$\varphi\colon [m] \to [n]$$

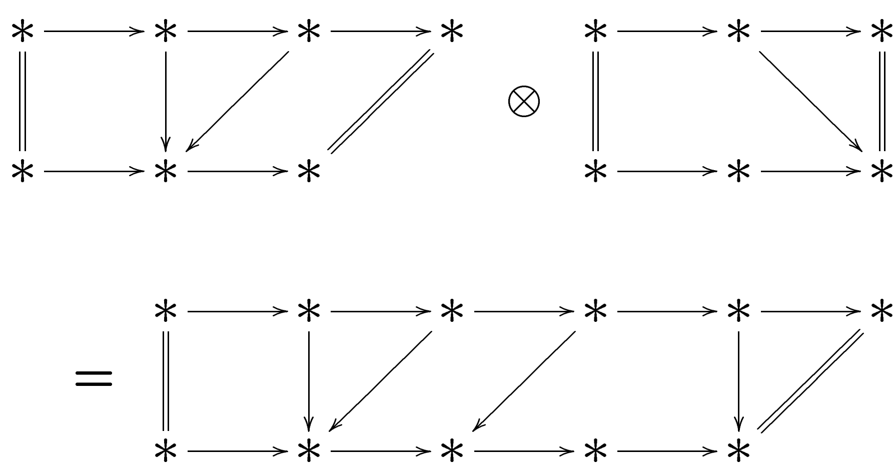
such that $\varphi(0) = 0$ and $\varphi(m) = n$.

Horizontal composition:

$$[m'] \otimes [m] = [m' + m],$$

$$(\varphi' \otimes \varphi)(i) = \begin{cases} \varphi'(i) & \text{if } i \leq m' \\ m' + \varphi(i) & \text{otherwise.} \end{cases}$$

For example,



Note: $\mathbb{P}\text{ath}\, \mathbb{1} \simeq \mathbb{\Delta}^{\text{op}}$

Oplax Morphisms of Double Categories

Definition [Grandis-Paré] $F\colon \mathbb{A} \to \mathbb{B}$, mapping

$$
\begin{array}{ccc}
A_1 & \xrightarrow{f_1} & A_2 \\
{\scriptstyle v_1}\downarrow\bullet & \alpha & \bullet\downarrow{\scriptstyle v_2} \\
A_3 & \xrightarrow[f_3]{} & A_4
\end{array}
\quad \mapsto \quad
\begin{array}{ccc}
FA_1 & \xrightarrow{Ff_1} & FA_2 \\
{\scriptstyle Fv_1}\downarrow\bullet & F\alpha & \bullet\downarrow{\scriptstyle Fv_2} \\
FA_3 & \xrightarrow[Ff_3]{} & FA_4,
\end{array}
$$

is an oplax morphism if

- it preserves the vertical structure;
- for every object A of $\mathbb{A}$, there is a cell

$$
\begin{array}{ccc}
FA & \xrightarrow{F1_A} & FA \\
\Vert & \varphi_A & \Vert \\
FA & \xrightarrow[1_{FA}]{} & FA,
\end{array}
$$

and for every $A \xrightarrow{f} A' \xrightarrow{f'} A''$, a cell

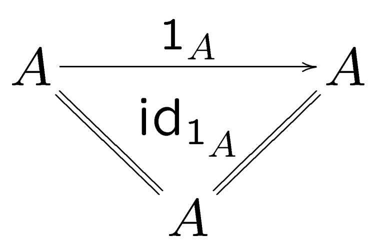
$$
\begin{array}{ccccc}
FA & & \xrightarrow{F(f'f)} & & FA'' \\
\Vert & & \varphi_{f',f} & & \Vert \\
FA & \xrightarrow[Ff]{} & FA' & \xrightarrow[Ff']{} & FA'',
\end{array}
$$

satisfying coherence conditions.

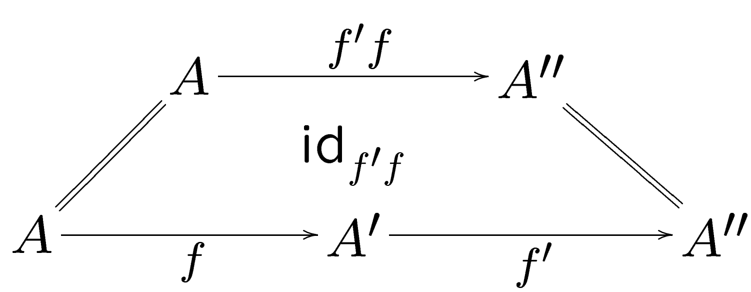
The Morphism $\Xi \colon \mathbb{A} \to \mathbb{P}\mathrm{ath}\mathbb{A}$

The morphism $\Xi \colon \mathbb{A} \to \mathbb{P}\mathrm{ath}\mathbb{A}$ is defined as follows:

- it is the identity on objects and vertical arrows;
- it takes a cell, or a horizontal arrow, to itself, considered as a path of such of length 1;
- for an object A in $\mathbb{A}$, let ξ_A be the cell



- for a path $A \xrightarrow{f} A' \xrightarrow{f'} A''$, let $\xi_{f',f}$ be the cell



Vertical Transformations

Let $F, G\colon \mathbb{A} \rightrightarrows \mathbb{B}$ be oplax morphisms. A *vertical transformation* $t\colon F \to G$ assigns

- to each object A in $\mathbb{A}$ a vertical arrow

$$tA\colon FA \mathrel{-\!\!\bullet\!\!\rightarrow} GA\,;$$

- to each horizontal arrow $f\colon A \longrightarrow A'$ a cell

$$\begin{array}{ccc} FA & \xrightarrow{Ff} & FA' \\ {\scriptstyle tA}\,\bullet\!\downarrow & tf & \downarrow\!\bullet\,{\scriptstyle tA'} \\ GA & \xrightarrow[Gf]{} & GA' \end{array}$$

satisfying functoriality conditions in the horizontal direction and naturality conditions in the vertical direction.

Double Categories, 2-Categories, and Bicategories

$\mathbf{Doub}_{\mathrm{Opl}}$ is the *2-category* of double categories, oplax morphisms and vertical transformations.

There is no 2-category consisting of bicategories, oplax morphisms and any of the obvious choices of 2-cells (lax, oplax, or pseudo).

The correct 2-cells for a 2-category **Bicat** are the specializations of the vertical transoformations in $\mathbf{Doub}_{\mathrm{Opl}}$, *i.e.*, cells for which the one-dimensional components are all identities.

The Universal Oplax Morphism

Let $\mathbf{Doub}(\mathbb{A}, \mathbb{B})$ denote the category of double functors from $\mathbb{A}$ to $\mathbb{B}$ with vertical transformations, and $\mathbf{Oplax}(\mathbb{A}, \mathbb{B})$ the category of oplax morphisms from $\mathbb{A}$ to $\mathbb{B}$ with vertical transformations.

Theorem For any double category $\mathbb{A}$,

$$\Xi \colon \mathbb{A} \to \mathbb{P}\mathrm{ath}\, \mathbb{A}$$

is the *universal oplax morphism* in the sense that composition with Ξ,

$$\Xi^* \colon \mathbf{Doub}(\mathbb{P}\mathrm{ath}\, \mathbb{A}, \mathbb{B}) \to \mathbf{Oplax}(\mathbb{A}, \mathbb{B})$$

is an isomorphism of categories.

The Comonad $\mathbb{P}\mathrm{ath}$ on **Doub**

Proposition The inclusion

$$\mathbf{Doub} \hookrightarrow \mathbf{Doub}_{\mathrm{Opl}}$$

of the 2-category of double categories, double functors, and vertical transformations into the 2-category of double categories, oplax morphisms, and vertical transformations, has a 2-left adjoint

$$\mathbb{P}\mathrm{ath} \colon \mathbf{Doub}_{\mathrm{Opl}} \to \mathbf{Doub}$$

with unit Ξ.

$\mathbf{Doub}_{\mathrm{Opl}}$ is the Kleisli 2-category for the 2-comonad $\mathbb{P}\mathrm{ath}$ induced on **Doub**.

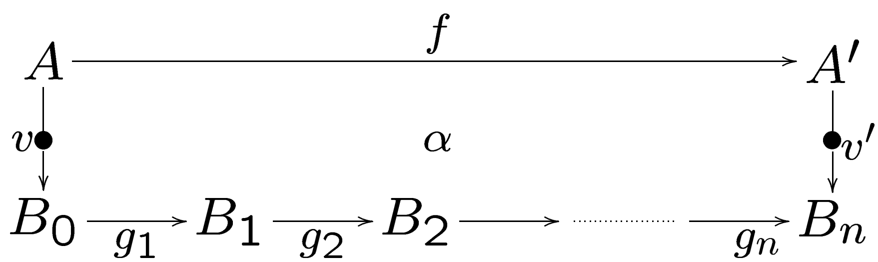
Note: The fact that $\mathbb{P}\mathrm{ath}$ is a 2-functor is a double category phenomenon.
We must consider 2-categories as double categories to get the two dimensional structure of **Doub** and $\mathbb{P}\mathrm{ath}$, because we need the vertical transformations to make things work.

Theorem The 2-comonad $\mathbb{P}\mathrm{ath}$ on **Doub** is an oplax idempotent comonad.

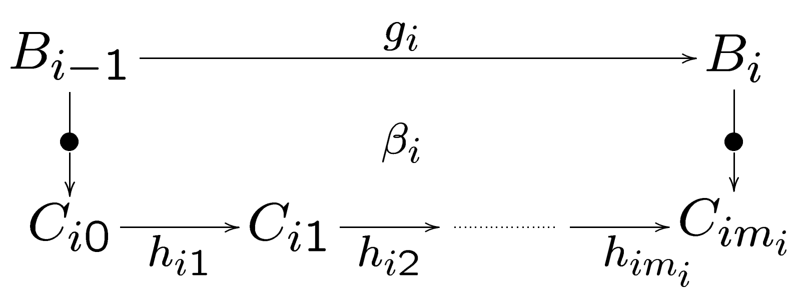
Consequence: There is at most one coalgebra structure $\Phi\colon \mathbb{A} \to \mathbb{P}\mathrm{ath}\,\mathbb{A}$, namely a (vertical) left adjoint to $E\colon \mathbb{P}\mathrm{ath}\,\mathbb{A} \to \mathbb{A}$.

Note: This also explains why the coalgebra structure for categories is unique.

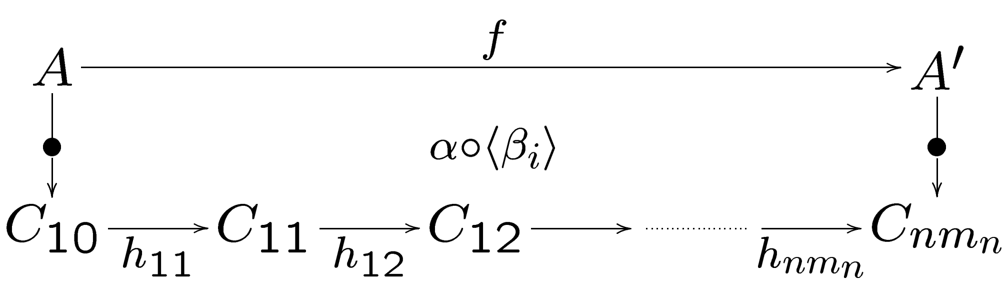
An oplax double category has objects, horizontal and vertical arrows, and multicells. The vertical arrows form a category whereas the horizontal arrows don't. A multicell has domains and codomains as illustrated



Such an α can be composed with n cells



to give a cell



There are identity multicells

$$\begin{array}{ccc} A & \xrightarrow{f} & A' \\ \| & \mathsf{id}_f & \| \\ A & \xrightarrow[f]{} & A' \end{array}$$

and the composition is associative.

Remarks

- A lax double category is a $\mathbb{T}$-multicategory in $\mathbf{Set}^{\bullet \rightrightarrows \bullet}$, where $\mathbb{T}$ is the free category monad.

- A lax double category in which all vertical arrows are identities is what Hermida calls a multicategory with several objects.

- A lax double category with a single object is the same as a multicategory as defined by Leinster.

- A. Burroni (1971): T-catégories
- C. Hermida (2000) multicategories
- C. Hermida, M. Makkai, J. Power (2000) multitopic categories
- J. Koslowski (2005) S-T-Spans
- J. Lambek, multicategories
- T. Leinster (2002) **fc**-multicategories

Definition A *morphism of (op)lax double categories* $F: \mathbb{A} \to \mathbb{B}$ takes arrows to arrows of the same type and cells to cells respecting domains and codomains, and preserving vertical composition of arrows and cells, and identities.

For any double category $\mathbb{A}$, there is an oplax double category $\mathbb{O}\mathrm{plax}\,\mathbb{A}$, defined as follows:

- its objects and vertical arrows are as in $\mathbb{A}$;
- its multicells

$$
\begin{array}{ccccccc}
A & & & \xrightarrow{\quad f \quad} & & & A' \\
{\scriptstyle v}\bullet\downarrow & & & \alpha & & & \downarrow\bullet{\scriptstyle v'} \\
B_0 & \xrightarrow[g_1]{} & B_1 & \xrightarrow[g_2]{} & \cdots & \xrightarrow[g_n]{} & B_n
\end{array}
$$

correspond to double cells

$$
\begin{array}{ccc}
A & \xrightarrow{\quad f \quad} & A' \\
{\scriptstyle v}\bullet\downarrow & \alpha & \downarrow\bullet{\scriptstyle v'} \\
B_0 & \xrightarrow[(g_n g_{n-1})\cdots g_2)g_1]{} & B_n
\end{array}
$$

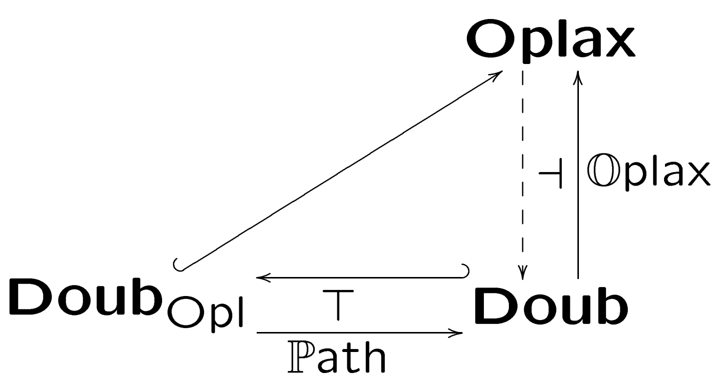
in $\mathbb{A}$.

Theorem [Hermida] Let $\mathbb{A}$ and $\mathbb{B}$ be double categories. A morphism of oplax double categories $\mathbb{O}\mathrm{plax}\,\mathbb{A} \to \mathbb{O}\mathrm{plax}\,\mathbb{B}$ is the same as an oplax morphism of double categories $\mathbb{A} \to \mathbb{B}$.

Proposition [Hermida] An oplax double category is of the form $\mathbb{O}\mathrm{plax}\,\mathbb{A}$ if and only if all composites are strongly representable.

Definition Oplax is the 2-category, whose objects are oplax double categories, arrows are morphisms of such, and 2-cells are vertical transformations.

Proposition $\mathbb{O}$plax is a locally full and faithful 2-functor **Doub** $\to$ **Oplax**.



Theorem $\mathbb{P}$ath is the object part of a 2-functor **Oplax** $\to$ **Doub** which is a 2-left adjoint to the inclusion $\mathbb{O}$plax: **Doub** $\to$ **Oplax**.

Theorem $\mathbb{P}\mathrm{ath} : \mathbf{Oplax} \to \mathbf{Doub}$ is 2-comonadic.

Remarks:

- **Oplax** is both complete and cocomplete, but $\mathbf{Double}_{\mathrm{Opl}}$ is neither.

- Oplax double categories are the right structure on which to define oplax morphisms.

- **Oplax** is the 2-dimensional analog of **Gph**.

- It is a property of a double category to be $\mathbb{P}\mathrm{ath}$ of an oplax one, not extra structure.

$\mathbb{P}\mathsf{ath}_*$ for Double Categories

Definition An oplax morphism of double categories $F\colon \mathbb{A} \to \mathbb{B}$ is called *normal* if for every A the given cell

$$\begin{array}{ccc} FA & \xrightarrow{F1_A} & FA \\ \| & \varphi_A & \| \\ FA & \xrightarrow[1_{FA}]{} & FA \end{array}$$

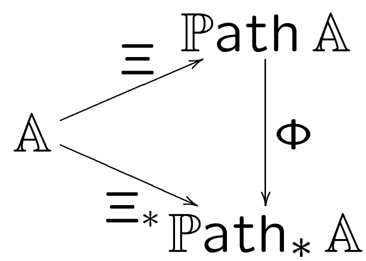
is vertically invertible.

Construct the universal oplax normal morphism

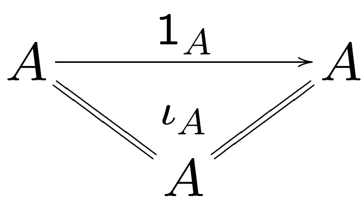
$$\mathbb{A} \xrightarrow{\Xi_*} \mathbb{P}\mathsf{ath}_*\,\mathbb{A}$$

for any double category $\mathbb{A}$.

Universality of Ξ gives us a unique Φ such that

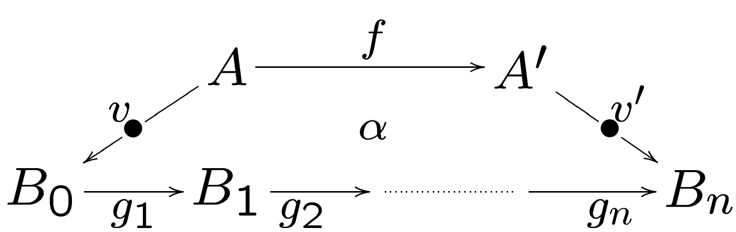


So we must add vertical inverses for

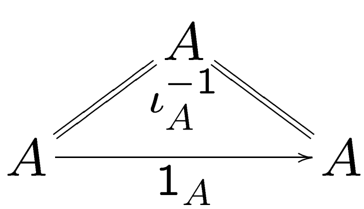


So the new double cells are generated by

- Cells of the form

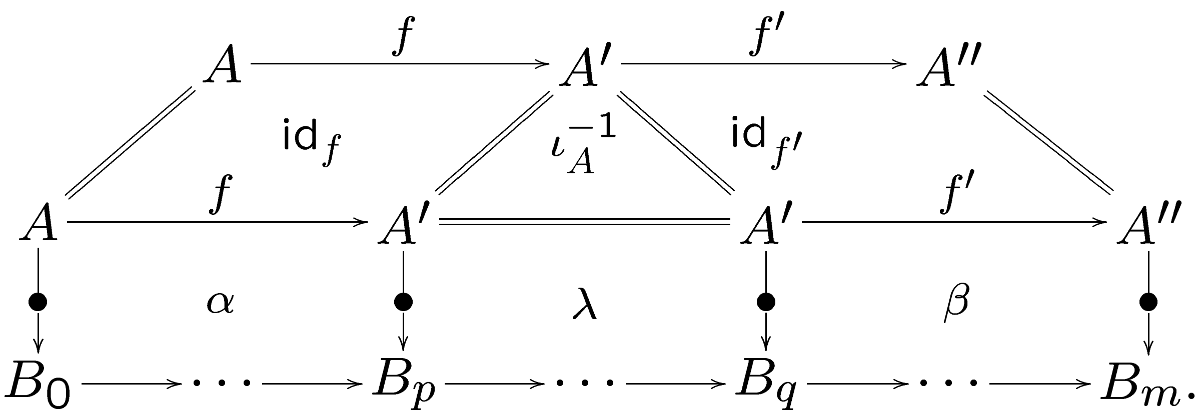


- Cells of the form

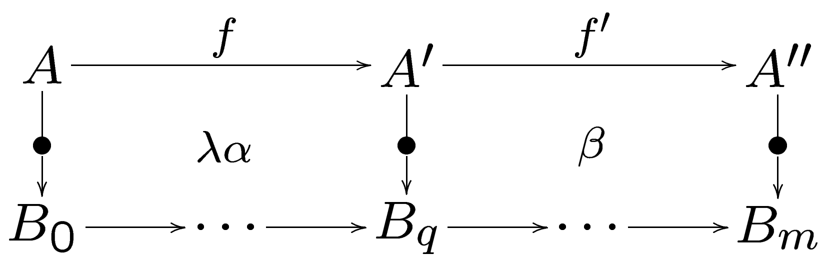


A Flavour of the Equivalence Relation

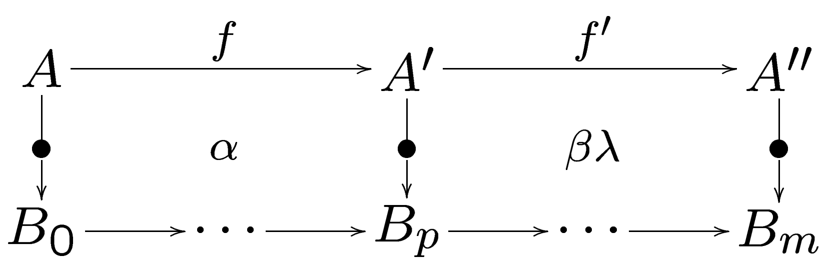
Pasting this diagram



shows that



is equivalent to



Theorem For any double category $\mathbb{A}$,

$$\Xi_*: \mathbb{A} \to \mathbb{P}\mathsf{ath}_* \mathbb{A}$$

is the universal oplax normal morphism, *i.e.*, for any (strict) double category $\mathbb{B}$ composing with Ξ_* induces an *isomorphism* of categories

$$\mathbf{Doub}(\mathbb{P}\mathsf{ath}_*\mathbb{A}, \mathbb{B}) \xrightarrow{\cong} \mathbf{Doub}_{OLN}(\mathbb{A}, \mathbb{B}).$$

Proposition $\mathbf{Doub}_{OLN}$ is the Kleisli 2-category for the comonad $\mathbb{P}\mathsf{ath}_*$ on **Doub**.

Theorem The comonad $\mathbb{P}\mathsf{ath}_*$ on **Doub** is oplax idempotent.

Theorem Let $\mathbb{A}$ be an oplax normal double category. Then

$$\Xi_*\colon \mathbb{A} \to \mathbb{O}\mathrm{plax}_*\mathbb{P}\mathrm{ath}_*\mathbb{A}$$

is the universal oplax normal morphism, *i.e.*, for any strict double category $\mathbb{B}$, composition with Ξ_* induces an *isomorphism* of categories

$$\mathbf{Doub}(\mathbb{P}\mathrm{ath}_*\mathbb{A}, \mathbb{B}) \xrightarrow{\cong} \mathbf{OplaxN}(\mathbb{A}, \mathbb{O}\mathrm{plax}_*\mathbb{B}).$$

Theorem $\mathbb{P}\mathrm{ath}_*\colon \mathbf{OplaxN} \to \mathbf{Doub}$ is comonadic.

The morphism $\mathbb{A} \xrightarrow{\Xi_*} \mathbb{P}\mathsf{ath}_* \mathbb{A}$ is the universal oplax normal morphism (for both double and oplax normal double categories), which induces

$$\mathbf{Doub}(\mathbb{P}\mathsf{ath}_*\mathbb{A}, \mathbb{B}) \xrightarrow{\cong} \mathbf{Doub}_{OLN}(\mathbb{A}, \mathbb{B}),$$

but also

$$\mathbf{Doub}(\mathbb{P}\mathsf{ath}_*\mathbb{A}, \mathbb{B}) \xrightarrow{\cong} \mathbf{OplaxN}(\mathbb{A}, \mathbb{O}\mathsf{plax}_*\mathbb{B}).$$

$\mathbf{Doub}_{OLN}$ is the Kleisli 2-category for the comonad $\mathbb{P}\mathsf{ath}_*$ on **Doub**.

$\mathbb{P}\mathsf{ath}_*$ an oplax idempotent comonad on **Doub**.

$\mathbb{P}\mathsf{ath}_*$: **OplaxN** $\to$ **Doub** is comonadic.

Π_2 of a 2-category

Let $\mathcal{A}$ be a 2-category. Then $\Pi_2\mathcal{A}$ is defined as follows:

- its objects are those of $\mathcal{A}$;
- its arrows are paths of spans in $\mathcal{A}$;
- its 2-cells are equivalence classes of fences