Cohomology without projectives

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From abelian cats with enough projectives ...

$$A \rightarrowtail^{k} B \xrightarrow{h} C \qquad\qquad \mathcal{A} \left\{ \begin{array}{l} \textbf{abelian} \\ \textbf{w. enough} \\ \textbf{projectives} \end{array} \right.$$

From abelian cats with enough projectives ...

$$A \overset{k}{\rightarrowtail} B \overset{h}{\twoheadrightarrow} C \qquad\qquad \mathcal{A}\left\{\begin{array}{l}\text{abelian}\\ \text{w. enough}\\ \text{projectives}\end{array}\right.$$

$$\begin{tikzcd}
\mathrm{Hom}(X,A) \arrow[r, tail] & \mathrm{Hom}(X,B) \arrow[r] & \mathrm{Hom}(X,C) \arrow[dll] \\
\mathrm{Ext}(X,A) \arrow[r] & \mathrm{Ext}(X,B) \arrow[r] & \mathrm{Ext}(X,C) \\
\cdots & \cdots & \cdots \\
\mathrm{Ext}^n(X,A) \arrow[r] & \mathrm{Ext}^n(X,B) \arrow[r] & \mathrm{Ext}^n(X,C) \arrow[dll] \\
\mathrm{Ext}^{n+1}(X,A) \arrow[r] & \mathrm{Ext}^{n+1}(X,B) \arrow[r] & \mathrm{Ext}^{n+1}(X,C) & \cdots
\end{tikzcd}$$

Ab

From abelian cats with enough projectives ...

AIM

$$A \rightarrowtail^{k} B \xrightarrow{h} C$$

$\mathcal{A}$ $\left\{ \begin{array}{l} \text{abelian} \\ \text{w. enough} \\ \text{projectives} \end{array} \right.$

$$\rightsquigarrow$$

$$\begin{tikzcd}
\mathrm{Hom}(X,A) \arrow[r, tail] & \mathrm{Hom}(X,B) \arrow[r] & \mathrm{Hom}(X,C) \arrow[dll] \\
\mathrm{Ext}(X,A) \arrow[r] & \mathrm{Ext}(X,B) \arrow[r] & \mathrm{Ext}(X,C) \\
\cdots & \cdots & \cdots \\
\mathrm{Ext}^n(X,A) \arrow[r] & \mathrm{Ext}^n(X,B) \arrow[r] & \mathrm{Ext}^n(X,C) \arrow[dll] \\
\mathrm{Ext}^{n+1}(X,A) \arrow[r] & \mathrm{Ext}^{n+1}(X,B) \arrow[r] & \mathrm{Ext}^{n+1}(X,C) & \cdots
\end{tikzcd}$$

Ab

From abelian cats with enough projectives ...

AIM

$$A \rightarrowtail^{k} B \overset{h}{\twoheadrightarrow} C \qquad \mathrm{Ab}(\mathcal{C}) \begin{cases} \text{not exact, } \cancel{0} \\ \cancel{\text{w. enough}} \\ \cancel{\text{projectives}} \end{cases}$$

$$\rightsquigarrow$$

$$\mathrm{Ab}$$

$$\begin{array}{ccc}
\mathrm{Hom}(X,A) \rightarrowtail & \mathrm{Hom}(X,B) \longrightarrow & \mathrm{Hom}(X,C) \\
\mathrm{Ext}(X,A) \longrightarrow & \mathrm{Ext}(X,B) \longrightarrow & \mathrm{Ext}(X,C) \\
\cdots & \cdots & \cdots \\
\mathrm{Ext}^n(X,A) \longrightarrow & \mathrm{Ext}^n(X,B) \longrightarrow & \mathrm{Ext}^n(X,C) \\
\mathrm{Ext}^{n+1}(X,A) \longrightarrow & \mathrm{Ext}^{n+1}(X,B) \longrightarrow & \mathrm{Ext}^{n+1}(X,C) \quad \cdots
\end{array}$$

... to effectively regular naturally Mal'cev cats

... to effectively regular naturally Mal'cev cats

$\mathcal{A}$

projectives

abelian

... to effectively regular naturally Mal'cev cats

$\mathcal{A}$

projectives

abelian $=$ exact

$+$

additive (0)

... to effectively regular naturally Mal'cev cats

$\mathcal{A}$	$\mathcal{C}$
projectives	
abelian $=$ exact $+$ additive (0)	

... to effectively regular naturally Mal'cev cats

$\mathcal{A}$	$\mathcal{C}$
projectives	
abelian $=$ exact	$\longrightarrow$ exact ($\mathrm{Ab}(\mathcal{C})$ abelian)
$+$ additive ($\mathbf{0}$)	

... to effectively regular naturally Mal'cev cats

$\mathcal{A}$

projectives

abelian $=$ exact

$+$

additive ($\mathbf{0}$)

$\mathcal{C}$

$\longrightarrow$ exact ($\mathrm{Ab}(\mathcal{C})$ abelian)

Barr: torsors $\longrightarrow$ $\mathbf{6}$-term e.s.

... to effectively regular naturally Mal'cev cats

$\mathcal{A}$

projectives

abelian $=$ exact

$+$

additive ($\mathbf{0}$)

$\mathcal{C}$

$\longrightarrow$ exact ($\mathbf{Ab}(\mathcal{C})$ abelian)

Barr: torsors $\longrightarrow$ $\mathbf{6}$-term e.s.

Duskin Glenn : simplicial objs $\longrightarrow$ l.e.s.

Bourn: i. n-groupoids $\longrightarrow$ l.e.s.

Bourn R. : direction $\longrightarrow$ l.e.s.

... to effectively regular naturally Mal'cev cats

$\mathcal{A}$	$\mathcal{C}$
projectives	
abelian $=$ ~~exact~~ $\longrightarrow$	effectively regular
$+$ additive (0)	

... to effectively regular naturally Mal'cev cats

$\mathcal{A}$	$\mathcal{C}$
projectives	
abelian $=$ ~~exact~~ low \$ $+$ additive (0)	$\longrightarrow$ effectively regular

... to effectively regular naturally Mal'cev cats

$\mathcal{A}$ | $\mathcal{C}$

projectives

abelian $=$ ~~exact~~ $\longrightarrow$ effectively regular

low \$

$+$

additive ($\cancel{0}$) $=$ naturally Mal'cev

... to effectively regular naturally Mal'cev cats

$\mathcal{A}$ | $\mathcal{C}$

projectives

abelian $=$ ~~exact~~ $\longrightarrow$ effectively regular

low \$

$+$

additive ($\cancel{0}$) $=$ naturally Mal'cev

\$: $X \underset{}{\overset{!}{\rightleftarrows}} 1$

... to effectively regular naturally Mal'cev cats

$\mathcal{A}$	$\mathcal{C}$
projectives	
abelian $=$ ~~exact~~ $\longrightarrow$	effectively regular
low \$	
$+$	
additive ($\cancel{0}$) $=$	naturally Mal'cev
\$: $X \xrightarrow{!} 1$	

... to effectively regular naturally Mal'cev cats

$\mathcal{A}$	$\mathcal{C}$
projectives	
abelian $=$ ~~exact~~ low \$ $\longrightarrow$	effectively regular
$+$	
additive ($\cancel{0}$) \$: $X \xrightarrow{!} 1$ global support $=\!=$	naturally Mal'cev

... to effectively regular naturally Mal'cev cats

$\mathcal{A}$ | $\mathcal{C}$

projectives

abelian $=$ ~~exact~~ $\longrightarrow$ effectively regular

low \$

$+$

additive ($\cancel{0}$) $=\!=$ naturally Mal'cev

\$: $X \xrightarrow{!} \!\!\!\twoheadrightarrow 1$

global support

$\mathcal{C}_{\#}$

... to effectively regular naturally Mal'cev cats

$\mathcal{A}$ | $\mathcal{C}$

projectives

abelian $=$ ~~exact~~ $\longrightarrow$ effectively regular

low \$

$+$

additive ($\cancel{0}$) $=\!=$ naturally Mal'cev

\$: $X \xrightarrow{!} 1$ (regular epi)

global support

$\nexists$ pbs

$\mathcal{C}_{\#}$

... to effectively regular naturally Mal'cev cats

$\mathcal{A}$		$\mathcal{C}$
~~projectives~~	$\longrightarrow$	no projectives
abelian $=$ ~~exact~~ low \$	$\longrightarrow$	effectively regular
$+$ additive ($\cancel{0}$) \$: $X \xrightarrow{!} 1$ (regular epi) global support $\nexists$ pbs	$=\!=$	naturally Mal'cev $\mathcal{C}_{\#}$

... to effectively regular naturally Mal'cev cats

$\mathcal{A}$ | $\mathcal{C}$

~~projectives~~ $\longrightarrow$ no projectives

\$: cc of extensions

abelian $=$ ~~exact~~ $\longrightarrow$ effectively regular

low \$

$+$

additive ($\cancel{0}$) $=\!=$ naturally Mal'cev

\$: $X \overset{!}{\twoheadrightarrow} 1$

global support

$\nexists$ pbs

$\mathcal{C}_{\#}$

... to effectively regular naturally Mal'cev cats

$\mathcal{A}$		$\mathcal{C}$
~~projectives~~ \$: cc of extensions	$\longrightarrow$	no projectives
abelian $=$ ~~exact~~ low \$	$\longrightarrow$	effectively regular
$+$ additive ($\cancel{0}$) \$: $X \xrightarrow{!} 1$ (regular epi) global support $\sharp$ pbs	$\Longrightarrow$	naturally Mal'cev $\mathcal{C}_{\#}$
n-chain complex	$\longrightarrow$	internal n-groupoid

... to effectively regular naturally Mal'cev cats

Direction Functor

$\mathcal{C}$

no projectives

effectively regular

naturally Mal'cev

internal n-groupoid

... to effectively regular naturally Mal'cev cats

$\mathcal{C}$

GOAL

Direction Functor

no projectives

effectively regular

naturally Mal'cev

internal n-groupoid

... to effectively regular naturally Mal’cev cats

$\mathcal{C}$

groups || Lie algebras

GOAL

Direction Functor

no projectives

effectively regular

naturally Mal’cev

internal n-groupoid

... to effectively regular naturally Mal’cev cats

$\mathcal{C}$

groups || Lie algebras

GOAL

(ab.) groups

$\downarrow$

(ab.) topological groups

(ab.) Hausdorff groups

Direction Functor

no projectives

effectively regular

naturally Mal’cev

internal n-groupoid

Effectively regular naturally Mal'cev cats

$\mathcal{A}/Y$, $\mathrm{Mal}(\mathrm{Gp}/C)$, $\mathrm{Mal}(\mathrm{R}_{\mathrm{Lie}}/A)$, (topological / Hausdorff)

Effectively regular naturally Mal'cev cats

$\mathcal{A}/Y$, $\mathrm{Mal}(\mathrm{Gp}/C)$, $\mathrm{Mal}(\mathrm{R_{Lie}}/A)$, (topological / Hausdorff)

- $\mathcal{C}_\#$ e. affine

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[d, shift right, two heads] & \bullet \arrow[d, shift right, two heads] \\
\bullet \arrow[r] \arrow[u, shift right, tail] & \bullet \arrow[u, shift right, tail]
\end{tikzcd}$$

Effectively regular naturally Mal'cev cats

$\mathcal{A}/Y$, $\mathrm{Mal}(\mathrm{Gp}/C)$, $\mathrm{Mal}(\mathrm{R}_{\mathrm{Lie}}/A)$, (topological / Hausdorff)

- $\mathcal{C}_{\#}$ e. affine

$$
\begin{tikzcd}
\bullet \arrow[r] \arrow[d, shift right, two heads] \arrow[dr, phantom, very near start, "\lrcorner"] & \bullet \arrow[d, shift right, two heads] \\
\bullet \arrow[r] \arrow[u, shift right, tail] & \bullet \arrow[u, shift right, tail]
\end{tikzcd}
$$

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$\mathcal{A}/Y$, $\mathrm{Mal}(\mathrm{Gp}/C)$, $\mathrm{Mal}(\mathrm{R}_{\mathrm{Lie}}/A)$, (topological / Hausdorff)

- $\mathcal{C}_{\#}$ e. affine

$$\begin{tikzcd}
\bullet \arrow[r, two heads] \arrow[d, two heads, shift right] & \bullet \arrow[d, two heads, shift right] \\
\bullet \arrow[r, two heads] \arrow[u, tail, shift right] & \bullet \arrow[u, tail, shift right]
\end{tikzcd}$$

Effectively regular naturally Mal’cev cats

$\mathcal{A}/Y$, $\mathrm{Mal}(\mathrm{Gp}/C)$, $\mathrm{Mal}(\mathrm{R_{Lie}}/A)$, (topological / Hausdorff)

- $\mathcal{C}_{\#}$ e. affine

$$\begin{tikzcd}
\bullet \arrow[r, two heads] \arrow[d, shift right, two heads] & \bullet \arrow[d, shift right, two heads] \\
\bullet \arrow[r, two heads] \arrow[u, shift right, tail] & \bullet \arrow[u, shift right, tail]
\end{tikzcd}$$

- $$\begin{array}{rcll} p : R \times_X T & \longrightarrow & X & \text{connector} \\ xRyTz & \longmapsto & p(x, y, z) & \end{array}$$

Effectively regular naturally Mal'cev cats

$\mathcal{A}/Y$, $\mathrm{Mal}(\mathrm{Gp}/C)$, $\mathrm{Mal}(\mathrm{R_{Lie}}/A)$, (topological / Hausdorff)

- $\mathcal{C}_\#$ e. affine

$$\begin{tikzcd} \bullet \arrow[r, two heads] & \bullet \\ \bullet \arrow[r, two heads] \arrow[u, shift right, tail] & \bullet \arrow[u, shift right, tail] \end{tikzcd}$$

- $$\begin{array}{rcl} p : R \times_X T & \longrightarrow & X \quad \text{connector} \\ xRyTz & \mapsto & p(x, y, z) \end{array}$$

- $$\begin{tikzcd} R \times_X T \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & T \arrow[d, shift left] \arrow[d, shift right] \\ R \arrow[r, shift left] \arrow[r, shift right] & X \end{tikzcd}$$ centralizing double equiv. relation

Effectively regular naturally Mal’cev cats

$\mathcal{A}/Y$, $\mathrm{Mal}(\mathrm{Gp}/\mathcal{C})$, $\mathrm{Mal}(\mathrm{R_{Lie}}/A)$, (topological / Hausdorff)

- $\mathcal{C}_{\#}$ e. affine

$$\begin{tikzcd} \bullet \arrow[r, two heads] \arrow[d, shift right] & \bullet \arrow[d, shift right] \\ \bullet \arrow[r, two heads] \arrow[u, shift right] & \bullet \arrow[u, shift right] \end{tikzcd}$$

- $p : R \times_X T \longrightarrow X$ connector
  $xRyTz \mapsto p(x, y, z)$

- $$\begin{tikzcd} R \times_X T \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift right] \arrow[d, shift left] & T \arrow[d, shift right] \arrow[d, shift left] \\ R \arrow[r, shift left] \arrow[r, shift right] & X \end{tikzcd}$$
  centralizing double equiv. relation

- R effective $\Rightarrow R \times_X T$ effective

Effectively regular naturally Mal'cev cats

$\mathcal{A}/Y$, $\mathrm{Mal}(\mathrm{Gp}/C)$, $\mathrm{Mal}(\mathrm{R}_{\mathrm{Lie}}/A)$, (topological / Hausdorff)

- $\mathcal{C}_\#$ e. affine

$$\begin{tikzcd} \bullet \arrow[r, two heads] \arrow[d, shift right, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & \bullet \arrow[d, shift right, two heads] \\ \bullet \arrow[u, shift right, tail] \arrow[r, two heads] & \bullet \arrow[u, shift right, tail] \end{tikzcd}$$

- $$\begin{aligned} p : R \times_X T &\longrightarrow X \quad \text{connector} \\ xRyTz &\longmapsto p(x,y,z) \end{aligned}$$

- $$\begin{tikzcd} R \times_X T \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift right] \arrow[d, shift left] \arrow[dr, phantom, "\lrcorner", very near start] & T \arrow[d, shift right] \arrow[d, shift left] \\ X \times X \arrow[r, shift left, "p_1"] \arrow[r, shift right] & X \end{tikzcd}$$

- R effective $\Rightarrow R \times_X T$ effective

Effectively regular naturally Mal'cev cats

$\mathcal{A}/Y$, $\mathrm{Mal}(\mathrm{Gp}/C)$, $\mathrm{Mal}(\mathrm{R_{Lie}}/A)$, (topological / Hausdorff)

- $\mathcal{C}_\#$ e. affine
$$\begin{tikzcd} \bullet \arrow[r] & \bullet \\ \bullet \arrow[r] \arrow[u, shift right] & \bullet \arrow[u, shift right] \arrow[from=1-1, to=2-1, shift right] \arrow[from=1-2, to=2-2, shift right] \end{tikzcd}$$

- $$\begin{array}{rcl} p : R \times_X T & \longrightarrow & X \quad \text{connector} \\ xRyTz & \longmapsto & p(x, y, z) \end{array}$$

- $$\begin{tikzcd} R \times_X T \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift right] \arrow[d, shift left] & X \times X \arrow[d, shift right, "p_0"'] \arrow[d, shift left] \\ X \times X \arrow[r, shift left, "p_1"] \arrow[r, shift right] & X \end{tikzcd}$$

- R effective $\Rightarrow R \times_X T$ effective

Effectively regular naturally Mal'cev cats

$\mathcal{A}/Y$, $\mathrm{Mal}(\mathrm{Gp}/C)$, $\mathrm{Mal}(\mathrm{R}_{\mathrm{Lie}}/A)$, (topological / Hausdorff)

- $\mathcal{C}_{\#}$ e. affine

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, shift right] & \bullet \arrow[d, shift right] \\ \bullet \arrow[r] \arrow[u, shift right] & \bullet \arrow[u, shift right] \end{tikzcd}$$

- $$\begin{array}{rcl} p : R \times_X T & \longrightarrow & X \quad \text{connector} \\ xRyTz & \longmapsto & p(x, y, z) \end{array}$$

- $$\begin{tikzcd} X{\times}X{\times}X \arrow[r, shift left, "p_2"] \arrow[r, shift right, "(p_0{,}p)"'] \arrow[d, shift right] \arrow[d, shift left] & X{\times}X \arrow[d, shift right, "p_0"'] \arrow[d, shift left] \\ X{\times}X \arrow[r, shift left, "p_1"] \arrow[r, shift right] & X \end{tikzcd}$$

- R effective $\Rightarrow$ $R \times_X T$ effective

Effectively regular naturally Mal'cev cats

$\mathcal{A}/Y$, $\mathrm{Mal}(\mathrm{Gp}/C)$, $\mathrm{Mal}(\mathrm{R}_{\mathrm{Lie}}/A)$, (topological / Hausdorff)

- $\mathcal{C}_{\#}$ e. affine

$$\begin{tikzcd} \bullet \arrow[r, two heads] \arrow[d, shift right] \arrow[dr, phantom, "\lrcorner", very near start] & \bullet \arrow[d, shift right] \\ \bullet \arrow[r, two heads] \arrow[u, tail, shift right] & \bullet \arrow[u, tail, shift right] \end{tikzcd}$$

- $p : R \times_X T \longrightarrow X$ connector

$$xRyTz \longmapsto p(x, y, z)$$

- $$\begin{tikzcd} X{\times}X{\times}X \arrow[r, shift left, "p_2"] \arrow[r, shift right, "(p_0{,}p)"'] \arrow[d, shift right] \arrow[d, shift left] \arrow[dr, phantom, "\lrcorner", very near start] & X{\times}X \arrow[r, two heads] \arrow[d, shift right, "p_0"'] \arrow[d, shift left] & \text{coequalizer} \\ X{\times}X \arrow[r, shift left, "p_1"] \arrow[r, shift right] & X & \end{tikzcd}$$

- R effective $\Rightarrow R \times_X T$ effective

The direction functor

$$\begin{tikzcd}
X \times X \times X \arrow[r, shift left, "p_2"] \arrow[r, shift right, "(p_0, p)"'] \arrow[d, shift right, two heads] & X \times X \arrow[r, two heads, "\nu_X"] \arrow[d, shift right, two heads, "p_0"'] & d(X) \\
X \times X \arrow[u, shift right, tail] \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_0"'] & X \arrow[u, shift right, tail] &
\end{tikzcd}$$

The direction functor

$$
\begin{tikzcd}
X \times X \times X \arrow[r, shift left, "p_2"] \arrow[r, shift right, "(p_0, p)"'] \arrow[d, shift right] \arrow[dr, phantom, "\lrcorner", very near start] & X \times X \arrow[r, two heads, "\nu_X"] \arrow[d, shift right, "p_0"'] & {\color{red} d(X)} \\
X \times X \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_0"'] \arrow[u, shift right] & X \arrow[u, shift right] &
\end{tikzcd}
$$

direction
of X

The direction functor

$$
\begin{tikzcd}
X \times X \times X \arrow[r, shift left, "p_2"] \arrow[r, shift right, "(p_0{,}p)"'] \arrow[d, shift right, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & X \times X \arrow[r, two heads, "\nu_X"] \arrow[d, shift right, two heads, "p_0"'] \arrow[dr, phantom, "1"] & d(X) \arrow[d, two heads] \\
X \times X \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_0"'] \arrow[u, shift right, tail] & X \arrow[r] \arrow[u, shift right, tail] & 1
\end{tikzcd}
$$

direction of X

The direction functor

$$
\begin{tikzcd}
X \times X \times X \arrow[r, shift left, "p_2"] \arrow[r, shift right, "(p_0{,}p)"'] \arrow[d, shift right, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & X \times X \arrow[r, two heads, "\nu_X"] \arrow[d, shift right, two heads, "p_0"'] \arrow[dr, phantom, "1"] & \color{red}{d(X)} \arrow[d, two heads] \\
X \times X \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_0"'] \arrow[u, shift right, tail] & X \arrow[r] \arrow[u, shift right, tail] & 1
\end{tikzcd}
$$

direction of X

- Barr-Kock Thm $\Rightarrow$ $\begin{tikzcd}[sep=small] {} \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "1"] & {} \arrow[d, two heads] \\ {} \arrow[r] & {} \end{tikzcd}$ pb

The direction functor

$$\begin{tikzcd}
X \times X \times X \arrow[r, shift left, "p_2"] \arrow[r, shift right, "(p_0, p)"'] \arrow[d, shift right] \arrow[dr, phantom, "\lrcorner", very near start] & X \times X \arrow[r, two heads, "\nu_X"] \arrow[d, shift right, "p_0"'] \arrow[dr, phantom, "\lrcorner", very near start] \arrow[dr, phantom, "1"] & d(X) \arrow[d, shift right] \\
X \times X \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_0"'] \arrow[u, shift right] & X \arrow[r, two heads] \arrow[u, shift right] & 1 \arrow[u, shift right]
\end{tikzcd}$$

direction of X

- Barr-Kock Thm $\Rightarrow$ $\begin{tikzcd} \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "1"] & \cdot \arrow[d] \\ \cdot \arrow[r] & \cdot \end{tikzcd}$ pb
- $X \in \mathcal{C}_\# \Rightarrow d(X) \in \mathrm{Ab}(\mathcal{C})$

The direction functor

$$\begin{tikzcd}
X \times X \times X \arrow[r, shift left, "p_2"] \arrow[r, shift right, "{(p_0,p)}"'] \arrow[d, shift right] & X \times X \arrow[r, two heads, "\nu_X"] \arrow[d, shift right, "p_0"'] \arrow[dr, phantom, "1"] & d(X) \arrow[d, shift right] \\
X \times X \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_0"'] \arrow[u, shift right] & X \arrow[r, two heads] \arrow[u, shift right] & 1 \arrow[u, shift right]
\end{tikzcd}$$

direction of X

- Barr-Kock Thm $\Rightarrow$ $\begin{tikzcd}[sep=small] \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "1"] & \cdot \arrow[d] \\ \cdot \arrow[r] & \cdot \end{tikzcd}$ pb
- $X \in \mathcal{C}_{\#} \Rightarrow d(X) \in \mathrm{Ab}(\mathcal{C})$
- $\begin{tikzcd}[sep=small] \cdot \arrow[r] \arrow[dr, phantom, "1"] & \cdot \\ \cdot \arrow[r] \arrow[u] & \cdot \arrow[u] \end{tikzcd}$ po

The direction functor

$$\begin{tikzcd}
X \times X \times X \arrow[r, shift left, "p_2"] \arrow[r, shift right, "(p_0,p)"'] \arrow[d, shift right] \arrow[dr, phantom, "\lrcorner", very near start] & X \times X \arrow[r, two heads, "\nu_X"] \arrow[d, shift right, "p_0"'] \arrow[dr, phantom, "1"] & d(X) \arrow[d, shift right, two heads] \\
X \times X \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_0"'] \arrow[u, shift right] & X \arrow[r, two heads] \arrow[u, shift right] & 1 \arrow[u, shift right]
\end{tikzcd}$$

direction of X

- Barr-Kock Thm $\Rightarrow$ $\begin{array}{ccc} & \rightarrow & \\ \downarrow & 1 & \downarrow \\ & \rightarrow & \end{array}$ pb
- $X \in \mathcal{C}_{\#} \Rightarrow d(X) \in \mathrm{Ab}(\mathcal{C})$
- $\begin{array}{ccc} & \rightarrow & \\ \uparrow & 1 & \uparrow \\ & \rightarrow & \end{array}$ po
- $X \in \mathrm{Ab}(\mathcal{C}) \Rightarrow d(X) \cong X$

The direction functor

$$
\begin{tikzcd}
X \times X \times X \arrow[r, shift left, "p_2"] \arrow[r, shift right, "(p_0{,}p)"'] \arrow[d, shift right] \arrow[dr, phantom, "\lrcorner", very near start] & X \times X \arrow[r, two heads, "\nu_X"] \arrow[d, shift right, "p_0"'] \arrow[dr, phantom, "1"] & d(X) \arrow[d, shift right] \\
X \times X \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_0"'] \arrow[u, shift right] & X \arrow[r, two heads] \arrow[u, shift right] & 1 \arrow[u, shift right]
\end{tikzcd}
$$

direction of X

direction functor $\quad d : \mathcal{C}_{\#} \longrightarrow \mathrm{Ab}(\mathcal{C})$

The direction functor

$$
\begin{tikzcd}
X \times X \times X \arrow[r, shift left, "p_2"] \arrow[r, shift right, "(p_0, p)"'] \arrow[d, shift right, two heads] & X \times X \arrow[r, two heads, "\nu_X"] \arrow[d, shift right, two heads, "p_0"'] \arrow[dr, phantom, "1"] & d(X) \arrow[d, shift right, two heads] \\
X \times X \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_0"'] \arrow[u, shift right, tail] & X \arrow[r, two heads] \arrow[u, shift right, tail] & 1 \arrow[u, shift right, tail]
\end{tikzcd}
$$

direction of X

Exs: direction functor $\quad d : \mathcal{C}_{\#} \longrightarrow \mathrm{Ab}(\mathcal{C})$

The direction functor

$$\begin{tikzcd}
X \times X \times X \arrow[r, shift left, "p_2"] \arrow[r, shift right, "(p_0,p)"'] \arrow[d, shift right] \arrow[dr, phantom, "\lrcorner", very near start] & X \times X \arrow[r, two heads, "\nu_X"] \arrow[d, shift right, "p_0"'] \arrow[dr, phantom, "\lrcorner", very near start] \arrow[dr, phantom, "1"] & d(X) \arrow[d, shift right] \\
X \times X \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_0"'] \arrow[u, shift right] & X \arrow[r, two heads] \arrow[u, shift right] & 1 \arrow[u, shift right]
\end{tikzcd}$$

direction of X

Exs: direction functor $\quad d : \mathcal{C}_{\#} \longrightarrow \mathrm{Ab}(\mathcal{C})$

- $d = 1_{\mathcal{A}} : \mathcal{A} \longrightarrow \mathcal{A}$

The direction functor

$$\begin{tikzcd}
X\times X\times X \arrow[r, shift left, "p_2"] \arrow[r, shift right, "(p_0,p)"'] \arrow[d, shift right] & X\times X \arrow[r, two heads, "\nu_X"] \arrow[d, shift right, "p_0"'] \arrow[dr, phantom, "1"] & d(X) \arrow[d, shift right] \\
X\times X \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_0"'] \arrow[u, shift right] & X \arrow[r, two heads] \arrow[u, shift right] & 1 \arrow[u, shift right]
\end{tikzcd}$$

direction of X

Exs: direction functor $\quad d:\mathcal{C}_{\#}\longrightarrow \mathrm{Ab}(\mathcal{C})$

- $d = 1_{\mathcal{A}} : \mathcal{A}\longrightarrow\mathcal{A}$
- $d = K : (\mathcal{A}/Y)_{\#}\longrightarrow\mathcal{A}$

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- $d : \mathrm{Mal}(\mathrm{Gp}/C)_\# \;\longrightarrow\; \mathrm{Mod}_C \sim \mathrm{Ab}(\mathrm{Gp}/C)$

$$A \rightarrowtail G \overset{g}{\twoheadrightarrow} C \;\longmapsto\; A_{\phi_g} \qquad \sim \quad C \ltimes A \twoheadrightarrow C$$

The direction functor

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  $A \rightarrowtail G \xrightarrow{g} C \longmapsto A_{\phi_g} \sim C \ltimes A \twoheadrightarrow C$ $\quad \vdots$

Properties of d

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preserves $\times$, $\exists$ pbs, regular epis (regular)

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- $\ldots$
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$$\Pi_0(d^{-1}(A)) \in \mathbf{Ab}$$

Properties of d

d — preserves $\times$, $\exists$ pbs, regular epis (regular)

d — reflects isos ($d^{-1}(A)$ groupoid)

d — $\ldots$

d — cofibration $\Rightarrow$ $\boldsymbol{(d^{-1}(A), \otimes)}$ sym. & closed

$$\mathrm{H}^1_c(A) = \boldsymbol{\Pi_0(d^{-1}(A)) \in \mathbf{Ab}}$$

1st cohomology group

First cohomology group

Exs: $\mathrm{H}^1_c(A)$

First cohomology group

Exs: $\mathrm{H}^1_c(A)$

- $$d = K : (\mathcal{A}/Y)_{\#} \longrightarrow \mathcal{A}$$

First cohomology group

Exs: $\mathrm{H}^1_{\mathcal{C}}(A)$

- $$d = K : (\mathcal{A}/Y)_{\#} \longrightarrow \mathcal{A}$$

$\times$ $\quad A \times A \rightarrowtail G \times_X H \twoheadrightarrow Y \qquad$ $\begin{array}{c} A \times A \\ \downarrow {\scriptstyle +} \\ A \end{array}$

First cohomology group

Exs: $\mathrm{H}^1_{\mathcal{C}}(A)$

- $$d = K : (\mathcal{A}/Y)_\# \longrightarrow \mathcal{A}$$

$\times$

$$\begin{tikzcd}
A \times A \arrow[r, tail] \arrow[d, two heads, "+"'] & G \times_X H \arrow[r, two heads] \arrow[d, "u"] & Y \arrow[d, equal] \\
A \arrow[r, tail] & B \arrow[r, two heads, "b"'] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
A \times A \arrow[d, "+"] \\
A
\end{tikzcd}$$

$\otimes$

First cohomology group

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$\otimes =$ Baer sum

First cohomology group

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$$\begin{tikzcd} A \times A \arrow[r, tail] \arrow[d, "+"'] \arrow[dr, phantom, "\lrcorner"] & G \times_X H \arrow[r, two heads] \arrow[d, "u"] & Y \arrow[d, equal] \\ A \arrow[r, tail] \arrow[ur, phantom, "\ulcorner"] & B \arrow[r, two heads, "b"'] & Y \end{tikzcd} \qquad \begin{tikzcd} A \times A \arrow[d, "+"] \\ A \end{tikzcd}$$

$$\mathrm{H}^1_{\mathcal{A}/Y}(A) = \mathrm{Ext}(Y, A)$$

(AbTop, AbHaus)

First cohomology group

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A \arrow[r, tail] & B \arrow[r, two heads, "b"'] \arrow[ul, phantom, "\ulcorner", very near start] & Y
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- $$\mathrm{H}^1_{\mathrm{Mal}(\mathrm{Gp}/C)}(A_\phi) = \mathrm{Opext}(C, A, \phi)$$

$\mathrm{R}_{\mathrm{Lie}}$

- $$\mathrm{H}^1_{\mathrm{Mal}(\mathrm{GpTop}/C)}(A_\phi) = \mathrm{TOpext}(C, A, \phi)$$

First cohomology group

Exs: $\mathrm{H}^1_{\mathcal{C}}(A)$

- $\mathcal{E}$ exact (e.r.), $A \in \mathrm{Ab}(\mathcal{E})$

$$\mathrm{H}^1_{\mathcal{E}}(A) = \Pi_0(\underline{\mathrm{PLO}}(A)), \quad A\text{-torsors}$$

First cohomology group

Exs: $\mathrm{H}^1_{\mathcal{C}}(A)$

- $\mathcal{E}$ exact (e.r.), $A \in \mathbf{Ab}(\mathcal{E})$

$$\mathrm{H}^1_{\mathcal{E}}(A) = \Pi_0(\underline{\mathbf{PLO}}(A)), \quad A\text{-torsors}$$

$$\updownarrow$$

aut. Mal'cev ops w/ direction A

First cohomology group

Exs: $\mathrm{H}^1_{\mathcal{C}}(A)$

- $\mathcal{E}$ exact (e.r.), $A \in \mathbf{Ab}(\mathcal{E})$

$$\begin{array}{ccccc} \mathbf{H}^1_{\mathcal{E}}(\boldsymbol{A}) & = & \boldsymbol{\Pi}_0(\underline{\mathbf{PLO}}(\boldsymbol{A})), & & \boldsymbol{A}\text{-torsors} \\ \| & & \| & & \updownarrow \\ \mathbf{H}^1_{\mathbf{AutM}(\mathcal{E})}(\boldsymbol{A}) & = & \boldsymbol{\Pi}_0(\boldsymbol{d}^{-1}(\boldsymbol{A})), & & \text{aut. Mal'cev ops w/ direction } \boldsymbol{A} \end{array}$$

First cohomology group

Exs: $\mathrm{H}^1_{\mathcal{C}}(A)$

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$$
\begin{array}{ccccc}
\mathrm{H}^1_{\mathcal{E}}(A) & = & \Pi_0(\underline{\mathbf{PLO}}(A)), & & A\text{-torsors} \\
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\end{array}
$$

$\rightsquigarrow$ alternative description of H^1

First cohomology group

Exs: $\mathrm{H}^1_{\mathcal{C}}(A)$

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$\rightsquigarrow$ alternative description of H^1

$\rightsquigarrow$ same 6-term e.s.

Level 2

- 6-term e.s. $\longrightarrow$ longer e.s.

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- 6-term e.s. $\xrightarrow{\text{i. groupoids}}$ longer e.s.

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- $$\mathrm{Grd}(\mathcal{C})$$
  $$\underline{X}_1 : X_1 \rightrightarrows X_0$$

Level 2

- 6-term e.s. $\xrightarrow{\text{i. groupoids}}$ longer e.s.

- Lawvere condition: reflexive graphs = i. groupoids

- $$\begin{array}{rcl} ()_0 : \mathrm{Grd}(\mathcal{C}) & \longrightarrow & \mathcal{C} \\ \underline{X}_1 : X_1 \rightrightarrows X_0 & & X_0 \end{array}$$

Level 2

- $\mathrm{Grd}(\mathcal{C}_{\#}) \xrightarrow{d_1} \mathrm{Grd}(\mathrm{Ab}(\mathcal{C}))$

$$X_1 \rightleftarrows X_0 \qquad d(X_1) \rightleftarrows d(X_0)$$

Level 2

- $$\begin{tikzcd} \mathrm{Grd}(\mathcal{C}_{\#}) \arrow[r, "d_1"] & \mathrm{Grd}(\mathrm{Ab}(\mathcal{C})) \end{tikzcd}$$

$$\begin{tikzcd} X_1 \arrow[r, shift left=2] \arrow[r, shift right=2] & X_0 \arrow[l] \end{tikzcd} \qquad \begin{tikzcd} d(X_1) \arrow[r, shift left=2] \arrow[r, shift right=2] & d(X_0) \arrow[l] \end{tikzcd}$$

Level 2

- $\mathrm{Grd}(\mathcal{C}_{\#}) \xrightarrow{\underline{d_1}} \mathrm{Grd}(\mathrm{Ab}(\mathcal{C}))$

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Level 2

- $$\begin{array}{ccc} \mathrm{Grd}(\mathcal{C}_{\#}) & \xrightarrow{d_1} & \mathrm{Grd}(\mathrm{Ab}(\mathcal{C})) \\ X_1 \rightleftarrows X_0 & & d(X_1) \rightleftarrows d(X_0) \end{array}$$

$\mathrm{Grd}(\mathcal{C})$ e.r.n.M

Level 2

- $$\mathbf{Grd}(\mathcal{C}_{\#}) \xrightarrow{d_{\mathbf{Grd}}} \mathbf{Grd}(\mathbf{Ab}(\mathcal{C}))$$ $\mathbf{Grd}(\mathcal{C})$ e.r.n.M

  $$\boldsymbol{X_1} \rightrightarrows \boldsymbol{X_0} \qquad \boldsymbol{d(X_1)} \rightrightarrows \boldsymbol{d(X_0)}$$

Level 2

- $\mathrm{Grd}(\mathcal{C}_{\#}) \xrightarrow{d_{\mathrm{Grd}}} \mathrm{Grd}(\mathrm{Ab}(\mathcal{C})) \xrightarrow{\text{kernel}} \mathrm{Ab}(\mathcal{C})$

$$X_1 \rightleftarrows\!\!\!\rightarrow X_0 \qquad d(X_1) \rightleftarrows\!\!\!\rightarrow d(X_0)$$

$$\begin{tikzcd} A \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & d(X_1) \arrow[d] \\ 1 \arrow[r] & d(X_0)\times d(X_0) \end{tikzcd}$$

Level 2

- $\mathbf{Asp}(\mathcal{C}_{\#}) \xrightarrow{d_{\mathbf{Grd}}} \mathbf{Asp}(\mathrm{Ab}(\mathcal{C})) \xrightarrow{\mathbf{kernel}} \mathrm{Ab}(\mathcal{C})$

  $X_1 \rightrightarrows X_0 \qquad d(X_1) \rightrightarrows d(X_0)$

$$\begin{tikzcd} A \arrow[r, tail] \arrow[d] & d(X_1) \arrow[d] \\ 1 \arrow[r] & d(X_0) \times d(X_0) \end{tikzcd}$$

aspherical

Level 2

d_1 1-dimensional direction

- $$\mathrm{Asp}(\mathcal{C}_{\#}) \xrightarrow{d_{\mathrm{Grd}}} \mathrm{Asp}(\mathrm{Ab}(\mathcal{C})) \xrightarrow{\text{kernel}} \mathrm{Ab}(\mathcal{C})$$

$$X_1 \rightrightarrows X_0 \qquad d(X_1) \rightrightarrows d(X_0)$$

$$\begin{tikzcd} A \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & d(X_1) \arrow[d] \\ 1 \arrow[r] & d(X_0)\times d(X_0) \end{tikzcd}$$

aspherical

Properties of d_1

$$
\begin{tikzcd}
 & \mathbf{Asp}(\mathcal{C}) \arrow[dl, "()_0"'] \arrow[dr, "d_1"] & \\
\mathcal{C}_{\#} & & \mathbf{Ab}(\mathcal{C})
\end{tikzcd}
$$

Properties of d_1

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- $()_0$, d_1 preserve $\times$, $\exists$ pbs

Properties of d_1

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$$

- $()_0$, d_1 preserve $\times$, $\exists$ pbs
- $()_0$ fibration, d_1 cofibration

Properties of d_1

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- $()_0$ fibration, d_1 cofibration
- $()_0$-cartesian iff d_1-invertible

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 & \mathrm{Asp}(\mathcal{C}) \arrow[dl, "()_0"'] \arrow[dr, "d_1"] & \\
\mathcal{C}_\# & & \mathrm{Ab}(\mathcal{C})
\end{tikzcd}
$$

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- $()_0$-cartesian iff d_1-invertible
- d_1-cocartesian iff $()_0$-invertible $\cdots$

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$$\Pi_0(d_1^{-1}(A)) \in \mathbf{Ab}$$

Properties of d_1

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- $()_0$-cartesian iff d_1-invertible
- d_1-cocartesian iff $()_0$-invertible $\cdots$

$$H^2_{\mathcal{C}}(A) = \Pi_0(d_1^{-1}(A)) \in \mathrm{Ab}$$

2nd cohomology group

Second cohomology group

Exs: $\mathrm{H}^1_{\mathcal{C}}(A)$

- $\mathrm{H}^1_{\mathcal{A}/Y}(A) = \mathbf{Ext}(Y, A)$ (**AbTop**, **AbHaus**)
- $\mathrm{H}^1_{\mathrm{Mal(Gp}/C)}(A_\phi) = \mathbf{Opext}\ (C, A, \phi)$
- $\mathrm{H}^1_{\mathrm{Mal(GpTop}/C)}(A_\phi) = \mathbf{TOpext}\ (C, A, \phi)$

Second cohomology group

Exs: $\mathrm{H}^2_{\mathcal{C}}(A)$

- $\mathrm{H}^2_{\mathcal{A}/Y}(A) = \mathrm{Ext}^2(Y, A)$ (AbTop, AbHaus)
- $\mathrm{H}^2_{\mathrm{Mal}(\mathrm{Gp}/C)}(A_\phi) = \mathrm{Opext}^2(C, A, \phi)$
- $\mathrm{H}^2_{\mathrm{Mal}(\mathrm{GpTop}/C)}(A_\phi) = \mathrm{TOpext}^2(C, A, \phi)$

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- $\mathrm{H}^{2}_{\mathrm{Mal}(\mathrm{Gp}/C)}(A_{\phi}) = \mathrm{Opext}^{2}(C, A, \phi)$
- Gp $\mathrm{Grd}(\mathrm{Gp}) \sim \mathrm{XMod}$

Second cohomology group

Exs: $\mathrm{H}^2_{\mathcal{C}}(A)$

- $\mathrm{H}^2_{\mathcal{A}/Y}(A) = \mathrm{Ext}^2(Y, A)$ (AbTop, AbHaus)
- $\mathrm{H}^2_{\mathrm{Mal}(\mathrm{Gp}/C)}(A_\phi) = \mathrm{Opext}^2(C, A, \phi)$
- Gp/C $\mathrm{Grd}(\mathrm{Gp}) \sim \mathrm{XMod}$ $\mathrm{Asp}(\mathrm{Gp}/C) \sim 2\text{-}\mathrm{XMod}_C$

$$A \rightarrowtail C \xrightarrow[\mathrm{XMod}]{} G \twoheadrightarrow C$$

Second cohomology group

Exs: $\mathrm{H}^2_{\mathcal{C}}(A)$

- $\mathrm{H}^2_{\mathcal{A}/Y}(A) = \mathrm{Ext}^2(Y, A)$ (AbTop, AbHaus)

- $\mathrm{H}^2_{\mathrm{Mal}(\mathrm{Gp}/C)}(A_\phi) = \mathrm{Opext}^2(C, A, \phi)$

- Gp/C $\quad \mathrm{Grd}(\mathrm{Gp}) \sim \mathrm{XMod} \qquad \mathrm{Asp}(\mathrm{Gp}/C) \sim \text{2-}\mathrm{XMod}_C$

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$$A \rightarrowtail C \xrightarrow[\mathrm{XMod}]{} G \twoheadrightarrow C$$

$\downarrow$

$\mathrm{Mal}(\mathrm{Gp}/C)$ e.r.n.M.

Second cohomology group

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- $\mathrm{H}^2_{\mathcal{A}/Y}(A) = \mathrm{Ext}^2(Y, A)$ (AbTop, AbHaus)

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- Gp/C $\quad \cancel{\mathrm{Grd}(\mathrm{Gp}) \sim \mathrm{XMod} \quad \mathrm{Asp}(\mathrm{Gp}/C) \sim 2\text{-}\mathrm{XMod}_C}$

$$A \rightarrowtail C \xrightarrow[\mathrm{XMod}]{} G \twoheadrightarrow C$$

$$\downarrow$$

$\mathrm{Mal}(\mathrm{Gp}/C)$ e.r.n.M.

$\mathrm{Asp}(\mathrm{Mal}(\mathrm{Gp}/C)) \quad \sim$

$$\begin{tikzcd} \underline{X}_1 : X_1 \arrow[r, shift left=2] \arrow[r, shift right=2] \arrow[dr] & X_0 \arrow[l] \arrow[d, "\delta"] \\ & C \end{tikzcd}$$

Second cohomology group

Exs: $\mathrm{H}^2_{\mathcal{C}}(A)$

- $\mathrm{H}^2_{\mathcal{A}/Y}(A) = \mathrm{Ext}^2(Y, A)$ (AbTop, AbHaus)
- $\mathrm{H}^2_{\mathrm{Mal}(\mathrm{Gp}/C)}(A_\phi) = \mathrm{Opext}^2(C, A, \phi)$
- Gp/C $\quad \mathrm{Grd}(\mathrm{Gp}) \sim \mathrm{XMod} \qquad \mathrm{Asp}(\mathrm{Gp}/C) \sim 2\text{-}\mathrm{XMod}_C$

$$A \rightarrowtail C \underset{\mathrm{XMod}}{\longrightarrow} G \twoheadrightarrow C$$

$$\mathrm{Gp}/C \longrightarrow \mathrm{Mal}(\mathrm{Gp}/C)$$

$\mathrm{Mal}(\mathrm{Gp}/C)$ e.r.n.M.

$\mathrm{Asp}(\mathrm{Mal}(\mathrm{Gp}/C)) \quad \sim \quad$ exact sequences

$$\begin{tikzcd}
\underline{X}_1 : X_1 \arrow[r, shift left=2] \arrow[r, shift right=2] \arrow[dr] & X_0 \arrow[l] \arrow[d, "\delta"] \\
& C
\end{tikzcd}$$

$$\begin{tikzcd}
A \arrow[r, tail] & B_1 \arrow[rr] \arrow[dr, "\mathrm{Mod}_C"'] & & X_0 \arrow[r, two heads, "\delta"] & C \\
& & B \arrow[ur, tail] & &
\end{tikzcd}$$

Second cohomology group

Exs: $\mathrm{H}^2_{\mathcal{C}}(A)$

- $\mathrm{H}^2_{\mathcal{A}/Y}(A) = \mathrm{Ext}^2(Y, A)$ (AbTop, AbHaus)

- $\mathrm{H}^2_{\mathrm{Mal}(\mathrm{Gp}/C)}(A_\phi) = \mathrm{Opext}^2(C, A, \phi)$

- Gp/C $\quad \cancel{\mathrm{Grd}(\mathrm{Gp}) \sim \mathrm{XMod} \qquad \mathrm{Asp}(\mathrm{Gp}/C) \sim 2\text{-}\mathrm{XMod}_C}$

$$A \rightarrowtail C \xrightarrow[\mathrm{XMod}]{} G \twoheadrightarrow C$$

$$\begin{array}{c} \mathrm{Gp}/C \\ \downarrow \\ \mathrm{Mal}(\mathrm{Gp}/C) \end{array}$$

$\mathrm{Mal}(\mathrm{Gp}/C)$ e.r.n.M.

$$\mathrm{Asp}(\mathrm{Mal}(\mathrm{Gp}/C)) \quad \sim \quad \text{exact sequences}$$

$$\begin{tikzcd} \underline{X}_1 \arrow[d, "d_1"'] & : X_1 \arrow[rr, shift left] \arrow[rr, shift right] \arrow[dr] & & X_0 \arrow[ll] \arrow[dl, "\delta"] \\ A_\phi & & C & \end{tikzcd}$$

$$\begin{tikzcd} A \arrow[r, tail] & B_1 \arrow[rr] \arrow[dr, two heads, "\mathrm{Mod}_C"'] & & X_0 \arrow[r, two heads, "\delta"] & C \\ & & B \arrow[ur, tail] & & \end{tikzcd}$$

Second cohomology group

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- Gp/C $\quad \mathrm{Grd}(\mathrm{Gp}) \sim \mathrm{XMod} \qquad \mathrm{Asp}(\mathrm{Gp}/C) \sim 2\text{-}\mathrm{XMod}_C$

$$\begin{tikzcd} A \arrow[r, tail] & C \arrow[r, "\mathrm{XMod}"'] & G \arrow[r, two heads] & C \end{tikzcd}$$

$$\mathrm{Gp}/C \longrightarrow \mathrm{Mal}(\mathrm{Gp}/C)$$

$\mathrm{Mal}(\mathrm{Gp}/C)$ e.r.n.M.

$\mathrm{Asp}(\mathrm{Mal}(\mathrm{Gp}/C)) \quad \sim \quad$ exact sequences

$$\begin{tikzcd} \underline{X}_1 \arrow[d, "d_1"'] & : & X_1 \arrow[rr, shift left=2] \arrow[rr, shift right=2] \arrow[dr] & & X_0 \arrow[ll] \arrow[dl, "\delta"] \\ A_\phi & & & C & \end{tikzcd}$$

$$\begin{tikzcd} A_\phi \arrow[r, tail] & B_1 \arrow[rr] \arrow[dr] & & X_0 \arrow[r, two heads, "\delta"] & C \\ & \mathrm{Mod}_C & B \arrow[ur, tail] & & \end{tikzcd}$$

Second cohomology group

Exs: $\mathrm{H}^2_{\mathcal{C}}(A)$

- $\mathrm{H}^2_{\mathcal{A}/Y}(A) = \mathrm{Ext}^2(Y, A)$ (AbTop, AbHaus)
- $\mathrm{H}^2_{\mathrm{Mal}(\mathrm{Gp}/C)}(A_\phi) = \mathrm{Opext}^2(C, A, \phi)$
- Gp/C $\quad \cancel{\mathrm{Grd}(\mathrm{Gp}) \sim \mathrm{XMod}} \qquad \cancel{\mathrm{Asp}(\mathrm{Gp}/C) \sim 2\text{-}\mathrm{XMod}_C}$

$$A \rightarrowtail C \xrightarrow[\mathrm{XMod}]{} G \twoheadrightarrow C$$

$$\begin{tikzcd} \mathrm{Gp}/C \arrow[d] \\ \mathrm{Mal}(\mathrm{Gp}/C) \end{tikzcd}$$

$\mathrm{Mal}(\mathrm{Gp}/C)$, $\mathrm{R}_{\mathrm{Lie}}, \cdots$

$\mathrm{Asp}(\mathrm{Mal}(\mathrm{Gp}/C)) \quad \sim \quad$ exact sequences

$$\begin{tikzcd} \underline{X}_1 \arrow[d, "d_1"'] & : & X_1 \arrow[rr, shift left=2] \arrow[rr, shift right=2] \arrow[dr] & & X_0 \arrow[ll] \arrow[dl, "\delta"] \\ A_\phi & & & C & \end{tikzcd}$$

$$\begin{tikzcd} A_\phi \arrow[r, tail] & B_1 \arrow[rr] \arrow[dr] & & X_0 \arrow[r, two heads, "\delta"] & C \\ & \mathrm{Mod}_C & B \arrow[ur, tail] & & \end{tikzcd}$$

Level 1 vs level 2

- d reflects isos $\Rightarrow d^{-1}(A)$ groupoid

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- d_1-invertible $\Leftrightarrow ()_0$-cartesian

  $\Rightarrow$ connected $\neq$ isomorphic

- X connected to Y:

$$\begin{tikzcd}
 & {} \arrow[dl] \arrow[dr] & \cdots & {} \arrow[dl] \arrow[dr] & \\
X & {} & \cdots & {} & Y
\end{tikzcd}$$

Level 1 vs level 2

- d reflects isos $\Rightarrow d^{-1}(A)$ groupoid

  $\Rightarrow$ connected = isomorphic

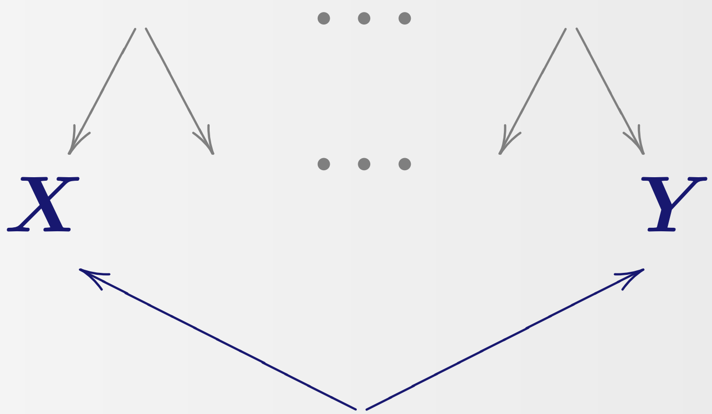
- d_1-invertible $\Leftrightarrow ()_0$-cartesian

  $\Rightarrow$ connected $\neq$ isomorphic

- X connected to Y:

  $\nexists$ pbs

  $>$ levels



Level $n+1$

- 6-term e.s. $\xrightarrow{\text{i.g.}}$ 9-term e.s.

Level $n + 1$

- 6-term e.s. $\xrightarrow{\text{i. g.}}$ 9-term e.s. $\xrightarrow{\text{i. } n\text{−groupoids}}$ l.e.s

Level $n+1$

- $\mathbf{6}$-term e.s. $\xrightarrow{\textbf{i. g.}}$ $\mathbf{9}$-term e.s. $\xrightarrow{\textbf{i. } n\textbf{–groupoids}}$ l.e.s

- Lawvere condition: reflexive graphs = i. groupoids

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- Lawvere condition: reflexive graphs = i. groupoids

- $$n - \mathrm{Grd}(\mathcal{C})$$

$$\underline{X}_n : X_n \rightrightarrows X_{n-1} \cdots X_1 \rightrightarrows X_0$$

Level $n+1$

- 6-term e.s. $\xrightarrow{\text{i. g.}}$ 9-term e.s. $\xrightarrow{\text{i. } n\text{−groupoids}}$ l.e.s
- Lawvere condition: reflexive graphs = i. groupoids
- $$()_{n-1} : n - \mathrm{Grd}(\mathcal{C}) \longrightarrow (n-1) - \mathrm{Grd}(\mathcal{C})$$

$$\underline{X}_n : \underbrace{X_n \rightrightarrows X_{n-1} \cdots X_1 \rightrightarrows X_0}_{} \longmapsto \underline{X}_{n-1}$$

Level $n+1$

$$
\begin{array}{ccc}
n\text{-}\mathrm{Grd}(\mathcal{C}_{\#}) & \overset{\underline{d}_n}{\longrightarrow} & n\text{-}\mathrm{Grd}(\mathrm{Ab}(\mathcal{C})) \\
\underline{X}_n & & d(\underline{X}_n)
\end{array}
$$

Level $n+1$

$$\begin{array}{ccc} n\text{-}\mathrm{Grd}(\mathcal{C}_{\#}) & \xrightarrow{\underline{d}_n} & n\text{-}\mathrm{Grd}(\mathrm{Ab}(\mathcal{C})) \\ \underline{X}_n & & d(\underline{X}_n) \end{array}$$

Level $n+1$

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n-$\mathrm{Grd}(\mathcal{C})$ e.r.n.M

Level $n+1$

$$\boldsymbol{n}\text{-}\mathrm{Grd}(\mathcal{C}_{\#}) \xrightarrow{d_{n\text{-}\mathrm{Grd}}} \boldsymbol{n}\text{-}\mathrm{Grd}(\mathrm{Ab}(\mathcal{C}))$$

$$\underline{\boldsymbol{X}}_n \qquad\qquad d(\underline{\boldsymbol{X}}_n)$$

n-$\mathrm{Grd}(\mathcal{C})$ e.r.n.M

Level $n+1$

$$n\text{-}\mathbf{Grd}(\mathcal{C}_{\#}) \xrightarrow{d_{n\text{-}\mathbf{Grd}}} n\text{-}\mathbf{Grd}(\mathbf{Ab}(\mathcal{C})) \xrightarrow{\text{kernel}} \mathbf{Ab}(\mathcal{C})$$

$$\underline{X}_n \qquad\qquad d(\underline{X}_n)$$

$$\begin{tikzcd} A \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & d(X_n) \arrow[d] \\ 1 \arrow[r] & d(X_{n-1}^{\hat{=}}) \end{tikzcd}$$

Level $n+1$

$$n\text{-}\mathrm{Asp}(\mathcal{C}_{\#}) \xrightarrow{d_{n\text{-}\mathrm{Grd}}} n\text{-}\mathrm{Asp}(\mathrm{Ab}(\mathcal{C})) \xrightarrow{\text{kernel}} \mathrm{Ab}(\mathcal{C})$$

$$\underline{X}_n \qquad d(\underline{X}_n)$$

$$\begin{array}{ccc} A & \rightarrowtail & d(X_n) \\ \downarrow & & \downarrow \\ 1 & \longrightarrow & d(X_{n-1}^{\hat{=}}) \end{array}$$

aspherical

Level $n + 1$

$$\begin{tikzcd}[column sep=large] n\text{-}\mathrm{Asp}(\mathcal{C}_{\#}) \arrow[r, "d_{n\text{-Grd}}"] \arrow[rr, bend left, "d_n"] & n\text{-}\mathrm{Asp}(\mathrm{Ab}(\mathcal{C})) \arrow[r, "\text{kernel}"] & \mathrm{Ab}(\mathcal{C}) \end{tikzcd}$$

n-dimensional direction

$\underline{X}_n \qquad d(\underline{X}_n)$

$$\begin{tikzcd} A \arrow[r, tail] \arrow[d] & d(X_n) \arrow[d, two heads] \\ 1 \arrow[r] & d(X^{\hat{=}}_{n-1}) \end{tikzcd}$$

aspherical

Properties of d_n

$$\begin{tikzcd}[column sep=small] & \mathbf{Asp}(\mathcal{C}) \arrow[dl, "()_0"'] \arrow[dr, "d_1"] & \\ \mathcal{C}_{\#} & & \mathbf{Ab}(\mathcal{C}) \end{tikzcd}$$

Properties of d_n

$$
\begin{tikzcd}
 & n\text{-}\mathrm{Asp}(\mathcal{C}) \arrow[dl, "()_{n-1}"'] \arrow[dr, "d_n"] & \\
(n-1)\text{-}\mathrm{Asp}(\mathcal{C}) & & \mathrm{Ab}(\mathcal{C})
\end{tikzcd}
$$

Properties of d_n

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- $()_{n-1}$, d_n preserve $\times$, $\exists$ pbs

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- $()_{n-1}$ fibration, d_n cofibration

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- $()_{n-1}$-cartesian iff d_n-invertible

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$$\Pi_0(d_n^{-1}(A)) \in \mathbf{Ab}$$

Properties of d_n

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$$H^{n+1}_{\mathcal{C}}(A) = \Pi_0(d_n^{-1}(A)) \in \mathbf{Ab}$$

$(n+1)$-th cohomology group

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Exs: $\mathrm{H}^2_{\mathcal{C}}(A)$

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- $\mathrm{H}^2_{\mathrm{Mal}(\mathrm{GpTop}/C)}(A_\phi) = \mathrm{TOpext}^2(C, A, \phi)$

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$\mathrm{Asp}(\mathrm{Mal}(\mathrm{Gp}/C)) \quad \sim \quad$ exact sequences

$$\begin{tikzcd} X_1 \arrow[rr, shift left=2] \arrow[rr, shift right=2] \arrow[dr] & & X_0 \arrow[ll] \arrow[dl, "\delta"] \\ & C & \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r] & B_1 \arrow[rr] \arrow[dr, two heads, "\mathrm{Mod}_C"'] & & X_0 \arrow[r, two heads, "\delta"] & C \\ & & B \arrow[ur, tail] & & \end{tikzcd}$$

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n- $\mathrm{Asp}(\mathrm{Mal}(\mathrm{Gp}/C))$ $\sim$ exact sequences

$$\begin{tikzcd}
X_n \arrow[r, shift left=2] \arrow[r, leftarrow] \arrow[r, shift right=2] \arrow[drr] & \cdots \arrow[r, shift left=2] \arrow[r, leftarrow] \arrow[r, shift right=2] & X_0 \arrow[dl, "\delta"] \\
& C &
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, tail] & B_{n-1} \arrow[r] & \cdots B_1 \arrow[rr] \arrow[dr, two heads, "\mathrm{Mod}_C"'] & & X_0 \arrow[r, two heads, "\delta"] & C \\
& & & B \arrow[ur, tail] & &
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A_\phi & C &
\end{tikzcd}
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$$\begin{tikzcd} A_\phi \arrow[r, tail] & B_{n-1} \arrow[r] & \cdots B_1 \arrow[rr] \arrow[dr, two heads, "\mathrm{Mod}_C"'] & & X_0 \arrow[r, two heads, "\delta"] & C \\ & & & B \arrow[ur, tail] & & \end{tikzcd}$$

$\mathrm{R}_{\mathrm{Lie}}, \cdots$

The long exact sequence

$\mathcal{C}$ e.r.n.M.

$\mathbf{Ab}(\mathcal{C})$

$$\boldsymbol{A} \overset{k}{\rightarrowtail} \boldsymbol{B} \overset{h}{\twoheadrightarrow} \boldsymbol{C}$$

$$\rightsquigarrow$$

$\mathbf{Ab}$

$$\begin{tikzcd}
\mathrm{H}^0_{\mathcal{C}}(A) \arrow[r, tail] & \mathrm{H}^0_{\mathcal{C}}(B) \arrow[r] & \mathrm{H}^0_{\mathcal{C}}(C) \arrow[dll] \\
\mathrm{H}^1_{\mathcal{C}}(A) \arrow[r] & \mathrm{H}^1_{\mathcal{C}}(B) \arrow[r] & \mathrm{H}^1_{\mathcal{C}}(C) \\
\cdots & \cdots & \cdots \\
\mathrm{H}^n_{\mathcal{C}}(A) \arrow[r, tail] & \mathrm{H}^n_{\mathcal{C}}(B) \arrow[r] & \mathrm{H}^n_{\mathcal{C}}(C) \arrow[dll, "\delta_n"'] \\
\mathrm{H}^{n+1}_{\mathcal{C}}(A) \arrow[r] & \mathrm{H}^{n+1}_{\mathcal{C}}(B) \arrow[r] & \mathrm{H}^{n+1}_{\mathcal{C}}(C)
\end{tikzcd}$$

END

Motivation: Affine Geometry

Motivation: Affine Geometry

Back

- A-torsor

Motivation: Affine Geometry

Back

- $\boldsymbol{A}$-torsor $= (\ \boldsymbol{A}\ ,\ \mathbf{0}\ )$

Motivation: Affine Geometry

Back

- $\boldsymbol{A}$-torsor $= (\boldsymbol{A} , \cancel{0})$

Motivation: Affine Geometry

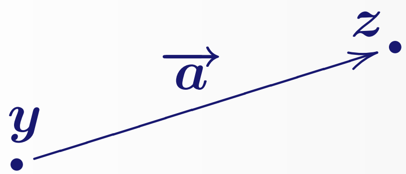
Back

- $\boldsymbol{A}$-torsor $= (\; \boldsymbol{A} \, , \; \cancel{0} \;)$
- Ex: Affine space $=$ (vector space , $\cancel{0}$)

Motivation: Affine Geometry

Back

- A-torsor $= (\ A\ ,\ \cancel{0}\ )$
- X Affine space



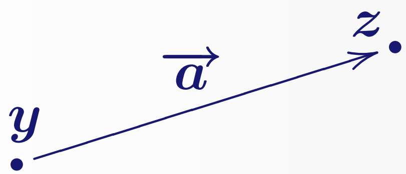
Motivation: Affine Geometry

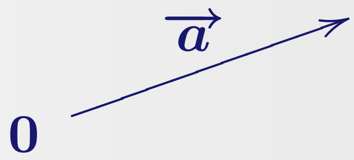
Back

- A-torsor $= (A , \cancel{0})$

direction of X

- X Affine space — A vector space





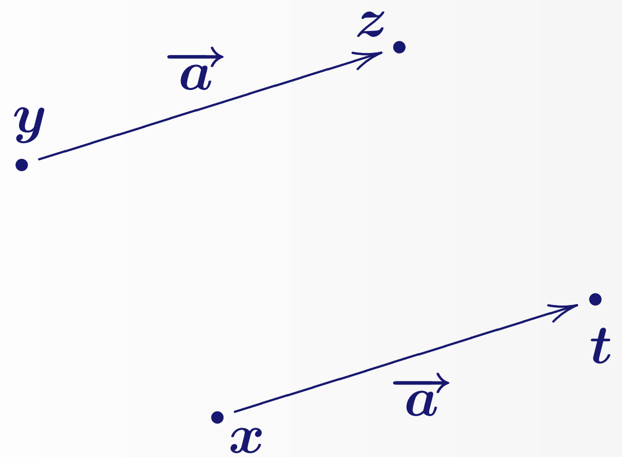
Motivation: Affine Geometry

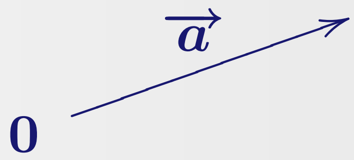
Back

- A-torsor $= (A, \cancel{0})$

direction of X

- X Affine space $\qquad$ A vector space





Motivation: Affine Geometry

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direction of X

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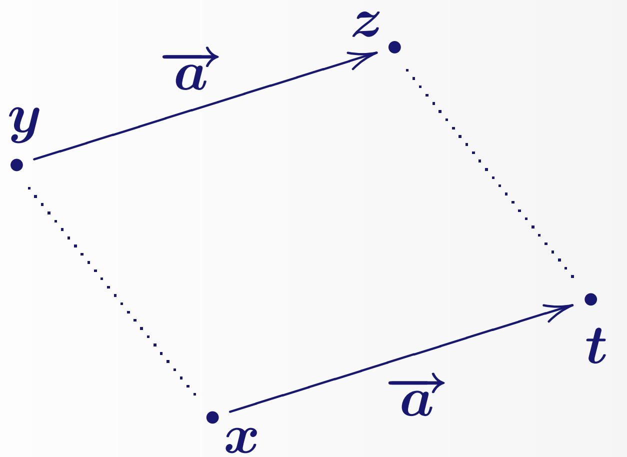

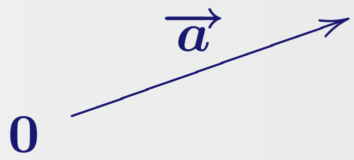

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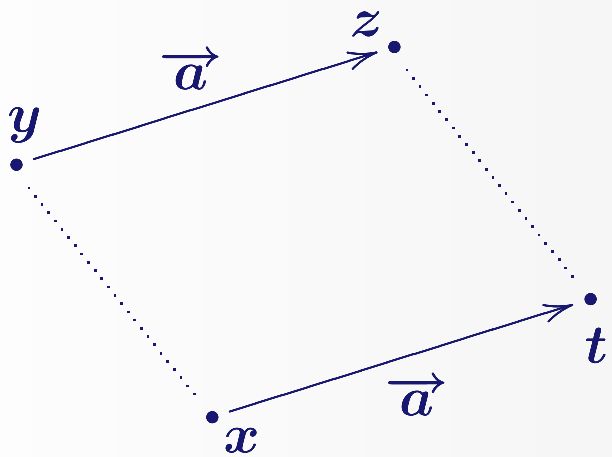
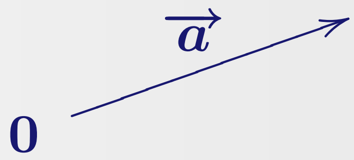




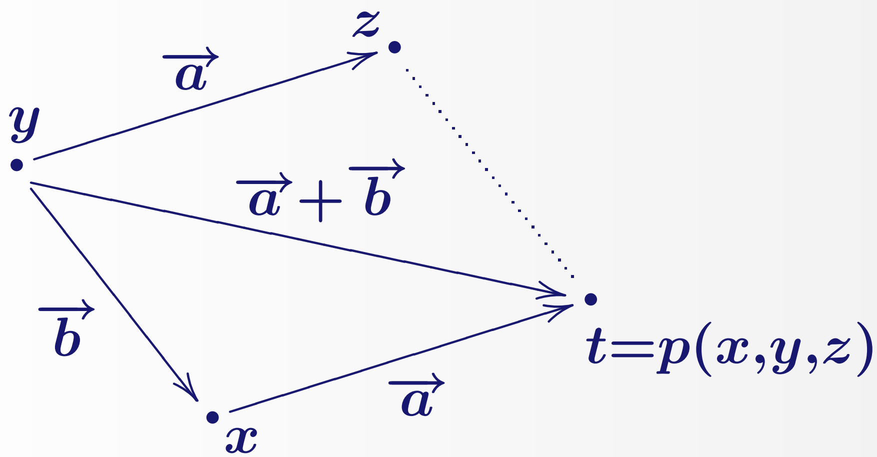
$$\overrightarrow{a} = [(y,z)]_\sim = [(x,t)]_\sim$$

Motivation: Affine Geometry

Back

- A-torsor $= \ (\ A \, , \ \cancel{0} \ )$

- X Affine space



Motivation: Affine Geometry

Back

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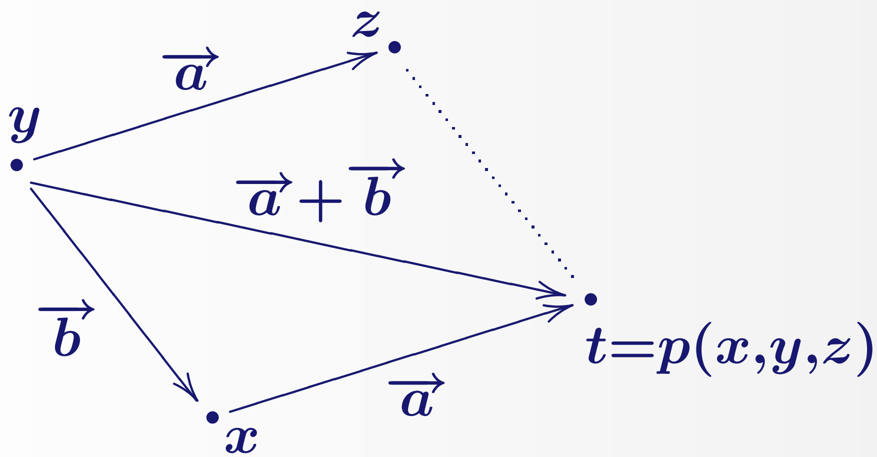
- X Affine space



Maltsev operation

$$p(x,x,y) = y$$
$$p(x,y,y) = x$$

Motivation: Affine Geometry

Back

- A-torsor $=$ $(\; A \; , \; \cancel{0} \;)$

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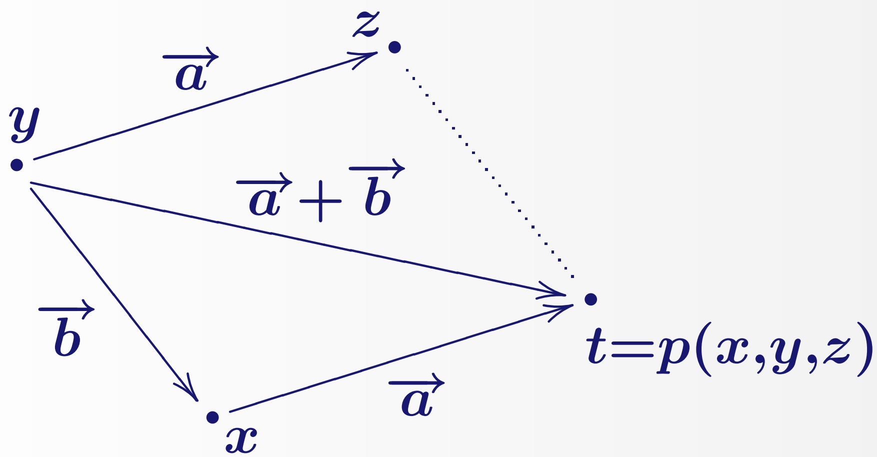


Maltsev operation

$$p(x,x,y) = y$$
$$p(x,y,y) = x$$

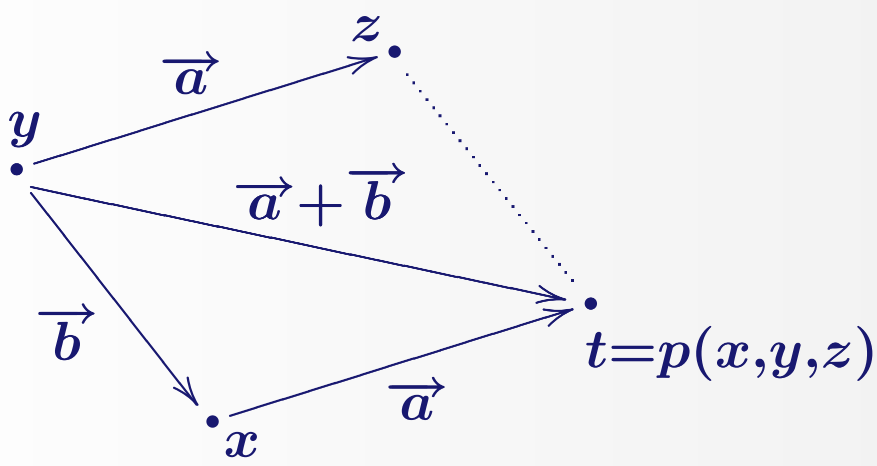
- $(\; x \; , \; p(x,y,z) \;) \sim (y,z)$

Chasles relation

Motivation: Affine Geometry

Back

- A-torsor $= \ (\ A \ , \ \cancel{0} \ )$
- X Affine space



Maltsev operation

$$p(x,x,y) = y$$
$$p(x,y,y) = x$$

- $(\ x\ ,\ p(x,y,z)\ )\ \sim\ (y,z)$

Chasles relation

- $d(X) = (X \times X)\ /\ \sim$

Naturally Mal’cev cats

Naturally Mal'cev cats

$$X \times X \times X \xrightarrow{p_X} X$$

internal

Mal'cev op

Back

Naturally Mal'cev cats

$$X \times X \times X \xrightarrow{p_X} X$$

autonomous

Mal'cev op

Back

Naturally Mal'cev cats

Back

$$X \times X \times X \xrightarrow{p_X} X$$

autonomous

Mal'cev op

$$\|$$

$$(X, \cancel{0}) \in \mathrm{Ab}(\mathcal{C})$$

abelian object

Naturally Mal'cev cats

$$X \times X \times X \xrightarrow{p_X} X$$

autonomous

Mal'cev op

Back

$$\|$$

$$(X, \cancel{0}) \in \mathrm{Ab}(\mathcal{C})$$

abelian object

Exs

- additive

Naturally Mal'cev cats

$$X \times X \times X \xrightarrow{p_X} X$$

autonomous

Mal'cev op

$$\|$$

$$(X, \cancel{0}) \in \mathrm{Ab}(\mathcal{C})$$

abelian object

Back

Exs

- additive $=$ n. Mal'cev $+\ 0$

Naturally Mal'cev cats

$$X \times X \times X \xrightarrow{p_X} X$$

autonomous

Mal'cev op

$$\|$$

$$(\,X, \cancel{0}\,) \in \mathrm{Ab}(\mathcal{C})$$

abelian object

Back

Exs

- additive $=$ n. Mal'cev $+\ 0$
- $\mathrm{Mal}(\mathcal{E})$, $\mathcal{E}$ proto

Naturally Mal'cev cats

$$X \times X \times X \xrightarrow{p_X} X$$

autonomous

Mal'cev op

Back

$$\|$$

$$(X, \cancel{0}) \in \mathrm{Ab}(\mathcal{C})$$

abelian object

Exs

- additive $=$ n. Mal'cev $+\ \mathbf{0}$
- $\mathrm{Mal}(\mathcal{E})$, $\mathcal{E}$ proto

  $\mathrm{Mal}(\mathrm{Gp}/C)$

Naturally Mal'cev cats

Back

$$X \times X \times X \xrightarrow{p_X} X$$

autonomous

Mal'cev op

$$\|$$

$$(X, \cancel{0}) \in \mathrm{Ab}(\mathcal{C})$$

abelian object

Exs

- additive $=$ n. Mal'cev $+\ 0$
- $\mathrm{Mal}(\mathcal{E})$, $\mathcal{E}$ proto

$\mathrm{Mal}(\mathrm{Gp}/C)$

$\mathrm{Mal}(\mathrm{GpTop}/C)$ $\mathbf{R}_{\mathrm{Lie}}$

$\mathrm{Mal}(\mathrm{GpHaus}/C)$

Effectively regular cats

Effectively regular cats

regular

Back

Effectively regular cats

regular

Back

$$+$$

$$\boldsymbol{S} \rightarrowtail^{\text{effective}}_{\text{equalizer}} \boldsymbol{E}$$

$\boldsymbol{E}$ effective

Effectively regular cats

Back

$$\left.\begin{array}{l}\text{regular}\\ + \\ \boldsymbol{S} \underset{\text{equalizer}}{\overset{\text{effective}}{\rightarrowtail}} \boldsymbol{E} \\ \boldsymbol{E}\ \text{effective}\end{array}\right\} \Rightarrow \quad \boldsymbol{S}\ \text{effective}$$

Effectively regular cats

Back

$$\left.\begin{array}{l} \text{regular} \\ + \\ S \underset{\text{equalizer}}{\overset{\text{effective}}{\rightarrowtail}} E \\ E \text{ effective} \end{array}\right\} \Rightarrow \quad S \text{ effective}$$

Exs

- exact

Effectively regular cats

Back

regular

$+$

$$\left.\begin{array}{l} \boldsymbol{S} \underset{\text{equalizer}}{\overset{\text{effective}}{\rightarrowtail}} \boldsymbol{E} \\ \boldsymbol{E} \text{ effective} \end{array}\right\} \Rightarrow \quad \boldsymbol{S} \text{ effective}$$

Exs

- exact
- $\mathbf{GpTop}$, $\mathbf{GpHaus}$

Effectively regular cats

Back

$$\left.\begin{array}{l}\text{regular}\\ + \\ S \underset{\text{equalizer}}{\overset{\text{effective}}{\rightarrowtail}} E \\ E \text{ effective}\end{array}\right\} \Rightarrow \quad S \text{ effective}$$

Exs

- exact
- $\mathbf{GpTop}$, $\mathbf{GpHaus}$
- $\mathbf{GpTop}/C$, $\mathbf{GpHaus}/C$

Effectively regular cats

Back

regular

$+$

$$\left.\begin{array}{l} S \underset{\text{equalizer}}{\overset{\text{effective}}{\rightarrowtail}} E \\ E \text{ effective} \end{array}\right\} \Rightarrow \quad S \text{ effective}$$

Exs

- exact
- $\mathbf{GpTop}$, $\mathbf{GpHaus}$
- $\mathbf{GpTop}/C$, $\mathbf{GpHaus}/C$

$\mathbf{R}_{\mathbf{Lie}}$

Effectively regular cats

regular

Back

$+$

$$\left.\begin{array}{l} S \underset{\text{equalizer}}{\overset{\text{effective}}{\rightarrowtail}} E \\ E \text{ effective} \end{array}\right\} \Rightarrow \quad S \text{ effective}$$

Exs

- exact
- $\mathbf{GpTop}$, $\mathbf{GpHaus}$
- $\mathbf{GpTop}/C$, $\mathbf{GpHaus}/C$
- $\mathbf{Ab}(\mathcal{E})$, $\mathbf{Gp}(\mathcal{E})$, $\quad \mathcal{E}$ e. regular

$\mathbf{R}_{\mathbf{Lie}}$

Effectively regular cats

Back

regular

$+$

$$\left.\begin{array}{l} S \underset{\text{equalizer}}{\overset{\text{effective}}{\rightarrowtail}} E \\ E \text{ effective} \end{array}\right\} \Rightarrow \quad S \text{ effective}$$

Exs

- exact
- $\mathbf{GpTop}$, $\mathbf{GpHaus}$
- $\mathbf{GpTop}/C$, $\mathbf{GpHaus}/C$
- $\mathbf{Ab}(\mathcal{E})$, $\mathbf{Gp}(\mathcal{E})$, $\quad \mathcal{E}$ e. regular
- $\mathcal{A}$ regular + lex + additive

$\mathbf{R}_{\mathbf{Lie}}$

Effectively regular cats

Back

regular

$+$

$$\left.\begin{array}{l} S \underset{\text{equalizer}}{\overset{\text{effective}}{\rightarrowtail}} E \\ E \text{ effective} \end{array}\right\} \Rightarrow \quad S \text{ effective}$$

Exs

- exact
- $\mathbf{GpTop}$, $\mathbf{GpHaus}$
- $\mathbf{GpTop}/C$, $\mathbf{GpHaus}/C$
- $\mathbf{Ab}(\mathcal{E})$, $\mathbf{Gp}(\mathcal{E})$, $\mathcal{E}$ e. regular
- $\mathcal{A}$ regular + lex + additive

$\mathbf{R}_{\mathbf{Lie}}$

$\mathcal{A}$ e. regular iff
$$\begin{tikzcd} \cdot \arrow[r, tail] \arrow[rr, tail, bend right] & \cdot \arrow[r, tail] & \cdot \end{tikzcd}$$

Effectively regular cats

Back

regular

$+$

$$\left.\begin{array}{l} S \underset{\text{equalizer}}{\overset{\text{effective}}{\rightarrowtail}} E \\ E \text{ effective} \end{array}\right\} \Rightarrow \quad S \text{ effective}$$

Exs

- exact
- $\mathbf{GpTop}$, $\mathbf{GpHaus}$
- $\mathbf{GpTop}/C$, $\mathbf{GpHaus}/C$
- $\mathbf{Ab}(\mathcal{E})$, $\mathbf{Gp}(\mathcal{E})$, $\quad \mathcal{E}$ e. regular
- $\mathcal{A}$ regular + lex + additive

$\mathcal{A}$ e. regular $\quad$ iff $\quad \rightarrowtail\; \rightarrowtail$

$\mathbf{R}_{\mathbf{Lie}}$

$\mathbf{AbTop}$

$\mathbf{AbHaus}$