Lecture 1
Triple Integrals

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Outline

- Text: section 15.6
- Approximating sums
- Box domains
- Example 1
- General 3D domains
- Example 2
- Example 3

Triple integrals over box domains

- Recall that double integrals are approximated by dividing the domain of integration into squares and by selecting a sample value within each square. Similarly a triple integral is approximated by dividing the domain of integration into cubes and sampling. Smaller cubes give better approximations and in the limit converge to the true value of the integral.
- Consider a 3D box domain:

$$\begin{aligned} B &= [a, b] \times [c, d] \times [r, s] \\ &= \{(x, y, z)\colon a \le x \le b,\ c \le y \le d,\ r \le z \le s\} \end{aligned}$$

- Subdivide the sides of B into n pieces each with

$$\Delta x = \frac{b-a}{n}, \quad \Delta y = \frac{d-c}{n}, \quad \Delta z = \frac{s-r}{n}.$$

- The resulting cells are cubes

$$B_{ijk} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{k-1}, z_k], \quad \text{where}$$

  $i, j, k = 1, \ldots, n$ are index variables and where

$$x_i = a + i\Delta x, \quad y_j = c + j\Delta y, \quad z_k = r + \Delta z.$$

Approximating 3D integrals

- The volume of each cell is

$$\Delta V = \Delta x \times \Delta y \times \Delta z = \frac{(b-a)(d-c)(f-e)}{n^3}.$$

- For each triple (i, j, k) where $i, j, k = 1, \ldots, n$ choose a sample point $(x_{ijk}, y_{ijk}, z_{ijk}) \in B_{ijk}$ and form the approximation

$$\iiint_B f(x, y, z) dV \approx \sum_{i,j,k=1}^{n} f(x_{ijk}, y_{ijk}, z_{ijk}) \Delta V.$$

In the limit, as $n \to \infty$ the right side will tend to the value of the triple integral.

Iterated triple integrals

- Just like double integrals, the value of a triple integrals over a box domain can be calculated by using a triply iterated integral.
- For domain $B = [a, b]x[c, d]x[r, s]$ and integrand $f(x, y, z)$ we have

$$I = \iiint_B f(x, y, z)dV = \int_{z=r}^{z=s} \left(\int_{y=c}^{y=d} \left(\int_{x=a}^{x=b} f(x, y, z)dx \right) dy \right) dz$$

- The value of I can, in principle, be calculated by any of the other 6 integrals corresponding to the 6 possible permutations of x, y, z. For example, using the order z, x, y we have

$$I = \iiint_B f(x, y, z)dV = \int_{y=c}^{y=d} \left(\int_{x=a}^{x=b} \left(\int_{z=r}^{z=s} f(x, y, z)dz \right) dx \right) dy$$

Example 1.

- Calculate the value of $I = \iiint_B xyz^2 dV$ where $B = [0,1]x[-1,2]x[0,3]$.
- Using the order x, y, z, we have

$$I = \int_{z=0}^{z=3} \int_{y=-1}^{y=2} \int_{x=0}^{x=1} xyz^2 dxdydz$$
$$= \int_0^3 \left(\int_{-1}^2 \frac{1}{2} yz^2 dy \right) dz$$
$$= \int_0^3 \frac{3}{4} z^2 dz = \frac{1}{4} z^3 \Big|_0^3 = \frac{27}{4}$$

- Using the order z, x, y, we have

$$I = \int_{y=-1}^{y=2} \int_{x=0}^{x=1} \int_{z=0}^{z=3} xyz^2 dz\, dx\, dy$$
$$= \int_{-1}^2 \int_0^1 \left(\frac{1}{3} xyz^3 \Big|_{z=0}^{z=3} \right) dxdy = \int_{-1}^2 \int_0^1 9xy\, dx\, dy$$
$$= \int_{-1}^2 \frac{9}{2} ydy = \frac{9}{4} y^2 \Big|_{-1}^2 = 9 - \frac{9}{4} = \frac{27}{4}$$

The product formula

- The product formula for double integral generalizes readily to triple integrals. If $B = [a, b] \times [c, d] \times [r, s]$ is a box domain, then

$$\iiint_B f(x)g(y)h(z)dV = \int_a^b f(t)dt \int_c^d g(t)dt \int_r^s h(t)dt$$

- Consider again the integral from Example 1, $I = \iiint_B xyz^2 dV$ where $B = [0, 1]x[-1, 2]x[0, 3]$. Applying the product formula:

$$I = \int_0^1 xdx \int_{-1}^2 ydy \int_0^3 z^2 dz = \frac{1}{2} \times \frac{3}{2} \times 9 = \frac{27}{4}.$$

Triple integrals over general domains.

- Integration over general 3D domains requires that we describe them in an iterated fashion using a particular ordering of x, y, z.
- A common possibility involves a domain of the form

$$B = \{(x, y, z) : (x, y) \in D \text{ and } g_1(x, y) \leq z \leq g_2(x, y)\}$$

  where D is a 2D region in the xy-plane.
- Intuitively, such B may be visualized as a dwelling with a 2D "floor plan" D with $g_2(x, y)$ the height of the roof and $g_1(x, y)$ the depth of the floor at each point $(x, y) \in D$.
- If D is also non-rectangular, we have to represent it in an iterated fashion such as

$$D = \{(x, y) : a \leq x \leq b, \ f_1(x) \leq y \leq f_2(y)\}$$

General domains cont.

- Thus, if B can be described using an x, y, z ordering as

  $$B = \{(x, y, z) : a \leq x \leq b,\ f_1(x) \leq y \leq f_2(x),\ g_1(x, y) \leq z \leq g_2(x, y)$$

  then a triple integral

  $$I = \iiint_B f(x, y, z)dV$$

  can be evaluated as the iterated integral

  $$I = \int_{x=a}^{x=b} \left(\int_{y=f_1(x)}^{y=f_2(x)} \left(\int_{z=g_1(x,y)}^{z=g_2(x,y)} f(x, y, z)dz \right) dy \right) dx.$$

- Other orderings of the variables are possible, but the key principle is that the order of the variables in the domain description dictates the variable order in the iterated integral.

Example 2

- Determine the value of the triple integral

$$I = \iiint_E z dV$$

  where E is the tetrahedron formed by the origin and the points $(1,0,0), (0,1,0), (0,0,1)$.
- The projection to the xy plane is the triangle D with vertices $(0,0,0), (1,0,0), (0,1,0)$. The borders of this triangle are the lines $x = 0$, $y = 0$ and $x + y = 1$.
- The iterated description of the triangle D is therefore

$$D = \{(x, y, 0) : 0 \leq x \leq 1,\ 0 \leq y \leq 1 - x\}.$$

Example 2 cont.

- The "floor" of the tetrahedron is given by the plane $z = 0$, while the "ceiling" by the plane $x + y + z = 1$. Rewriting the last equation as $z = 1 - x - y$ we obtain the following iterated description

$$E = \{(x, y, z) : 0 \leq x \leq 1,\ 0 \leq y \leq 1 - x,\ 0 \leq z \leq 1 - x - y\}.$$

- The triple integral may now be converted to the iterated integral

$$\begin{aligned}
I &= \int_{x=0}^{1} \int_{y=0}^{y=1-x} \int_{z=0}^{z=1-x-y} z dz dy dx \\
&= \int_0^1 \int_0^{1-x} \frac{1}{2}(1 - x - y)^2 dy dx \\
&= -\frac{1}{6} \int_0^1 (1 - x - y)^3 \Big|_{y=0}^{y=1-x} dx \\
&= \frac{1}{6} \int_0^1 (1 - x)^3 dx = -\frac{1}{24}(1 - x)^4 \Big|_0^1 = \frac{1}{24}
\end{aligned}$$

Example 3

- Evaluate the triple integral $I = \iiint_E \sqrt{x^2 + z^2} dV$ where E is bounded by paraboloid $y = x^2 + z^2$ and the plane $y = 4$.
- Projecting onto the xz plane we obtain the disk

$$D = (x, z) : x^2 + z^2 \leq 4\}$$

- The domain of integration may now be described as

$$E = \{(x, y, z) : (x, z) \in D,\ x^2 + y^2 \leq y \leq 4\}.$$

- The integral now becomes

$$\begin{aligned} I &= \iint_D \left(\int_{y=x^2+z^2}^{y=4} \sqrt{x^2 + z^2} dy \right) dA \\ &= \iint_D (4 - x^2 - z^2) \sqrt{x^2 + z^2} dA. \end{aligned}$$

Example 3 cont.

- To finish the evalution we switch to polar coordinates $x = r\cos\theta,\ z = r\sin\theta$.
- The disk D is now described as

$$D = \{(r,\theta) : 0 \leq r \leq 2, ; 0 \leq \theta \leq 2\pi\}.$$

- Recall also that $dA = rdrd\theta$.
- The integral now reduces to

$$I = \int_{\theta=0}^{\theta=2pi} \int_{r=0}^{2} (4 - r^2) r^2 drd\theta$$
$$= 2\pi \left[\frac{4}{3}r^3 - \frac{1}{5}r^5\right]_0^2 = \frac{128\pi}{15}$$