Lecture 2
Applications of triple integrals

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Outline

- Text: section 15.6
- Volume integrals
- Mass integrals
- Average value and Centre of mass

Volumes as triple integrals

- A triple integral $\iiint_B dV$ represents the volume of the 3D region B.
- Example: calculate the volume of the tetrahedron B bounded by the planes $x + 2y + z = 2, x = 2y, x = 0, z = 0$.
- The projection of the above tetrahedron onto the xy plane gives the triangle T bounded by the lines $x = 0$, $x = 2y$, and $x + 2y = 2$
- Solving for y gives $y = \frac{x}{2}$ and $y = -\frac{x}{2} + 1$. An iterative description of this triangle is therefore

$$T = \left\{(x, y) : 0 \leq x \leq 1,\ \frac{x}{2} \leq y \leq 1 - \frac{x}{2}\right\}.$$

- Solving for z in the first of the above plane equations gives $z = 2 - x - 2y$. The iterative description of the tetrahedron is therefore

$$B = \left\{(x, y, z) : 0 \leq x \leq 1,\ \frac{x}{2} \leq y \leq 1 - \frac{x}{2},\ 0 \leq z \leq 2 - x - 2y\right\}.$$

Volume example continued.

- The iterative description of the tetrahedron is

$$B = \left\{ (x, y, z) : 0 \le x \le 1, \ \frac{x}{2} \le y \le 1 - \frac{x}{2}, \ 0 \le z \le 2 - x - 2y \right\}.$$

- The volume integral may now be given as

$$\begin{aligned} V &= \int_{x=0}^{x=1} \int_{y=x/2}^{y=1-x/2} \int_{z=0}^{z=2-x-2y} dzdydx \\ &= \int_0^1 \int_{x/2}^{1-x/2} (2 - x - 2y) dydx \\ &= \int_0^1 \left[(2-x)y - y^2 \right]_{y=x/2}^{y=1-x/2} dx \\ &= \int_0^1 (2-x)(1-x) - \left(1 - \frac{x}{2}\right)^2 + \frac{x^2}{4} dx \\ &= \int_0^1 x^2 - 2x + 1 = \frac{1}{3} \end{aligned}$$

Mass

- The mass of a 3D solid is given by the triple integral of a given density function over the corresponding domain.
- Example. Find the mass of a solid enclosed by the cylinder $y = x^2$ and the planes $z = 0$ and $y + z = 1$ and density $\rho = y^2$.
- The domain in question is

$$E = \{(x, y, z) : -1 \leq x \leq 1,\ x^2 \leq y \leq 1,\ 0 \leq z \leq 1 - y\}.$$

- The mass integral is therefore

$$\begin{aligned} M &= \int_{x=-1}^{x=1} \int_{y=x^2}^{y=1} \int_{z=0}^{z=1-y} y^2 dzdydx \\ &= \int_{-1}^{1} \int_{x^2}^{1} (y^2 - y^3) dydx \\ &= \int_{-1}^{1} \left(\frac{1}{12} - \frac{x^6}{3} + \frac{x^8}{4} \right) dx = \frac{8}{63} \end{aligned}$$

Average Value and Centre of Mass

- If we divide the value of a triple integral by the volume of the region of integration, we obtain the average value of the integrand function in that region.
- The centre of mass of a solid may be determined by calculating the average values of the x, y, z coordinates over the domain in question.
- Example. Let E be the solid bounded by the parabolic cylinder $x = y^2$ and the planes $x = z, z = 0, x = 1$ having unit density. Calculate the centre of mass of this solid.
- The above domain can be described as

$$E = \{(x, y, z) : -1 \leq y \leq 1 : y^2 \leq x \leq 1,\ 0 \leq z \leq x\}$$

- The volume is given by

$$\begin{aligned} V &= \int_{-1}^{1} \int_{y^2}^{1} \int_{0}^{x} dzdxdy = \int_{-1}^{1} \int_{y^2}^{1} x\, dxdy \\ &= \int_{-1}^{1} \left[x^2/2\right]_{y^2}^{1} dy = \int_{-1}^{1} (1/2 - y^4/2) dy \\ &= \left[y/2 - y^5/10\right]_{-1}^{1} = 1 - 1/5 = 4/5 \end{aligned}$$

Centre of Mass example cont.

- The average values of the x, y, z coordinates are given by

$$\bar{x} = \frac{5}{4}\int_{-1}^{1}\int_{y^2}^{1}\int_{0}^{x} x\, dzdxdy$$
$$\bar{y} = \frac{5}{4}\int_{-1}^{1}\int_{y^2}^{1}\int_{0}^{x} y\, dzdxdy$$
$$\bar{z} = \frac{5}{4}\int_{-1}^{1}\int_{y^2}^{1}\int_{0}^{x} z\, dzdxdy$$

- We have

$$\bar{x} = \frac{5}{4}\int_{-1}^{1}\int_{y^2}^{1} x^2\, dxdy = \frac{5}{4}\int_{-1}^{1}\left[x^3/3\right]_{y^2}^{1} dy$$
$$= \frac{5}{4}\int_{-1}^{1}(1/3 - y^6/3)dy = \frac{5}{4}\times\frac{1}{3}\times\left(2-\frac{2}{7}\right) = \frac{5}{7}$$
$$\bar{z} = \frac{5}{4}\int_{-1}^{1}\int_{y^2}^{1}\frac{x^2}{2}\, dxdy = \frac{5}{14}$$

- By an explicit calculation, or by using the symmetry of the domain, we have $\bar{y} = 0$.