Lecture 3
Cylindrical Coordinates

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Outline

- Text: section 15.7
- Definition
- Volume scale factor
- Rotational symmetry
- Examples

Definition

- Cylindrical coordinates are an extension of polar coordinates to 3D. We use $r = \sqrt{x^2 + y^2}$ an and $\theta = \tan^{-1}(y/x)$ to represent the horizontal position, and use z to represent the altitude.
- Here are some examples of surfaces expressed as equations in parabolic coordinates.
  - The equation $r = 1$ describes an infinite right cylinder of radius 1
  - The equation $z = r$ describes a right circular (double) cone.
- Recall that in polar coordinates the 2D area element has a scale factor:

  $$dA = r\ dr\, d\theta.$$

  It stands to reason that the 3D volume element has the same scale factor:

  $$dV = r\ dz\, dr\, d\theta.$$

Example.

- Example. Consider the region E bounded by the cylinder (walls) $x^2 + y^2 = 1$, the plane (roof) $z = 4$, and the paraboloid (floor) $z = 1 - x^2 - y^2$
- In Cartesian coordinates we have

$$E = \{(x, y, z) : x^2 + y^2 \leq 1,\ 1 - x^2 - y^2 \leq z \leq 4\}.$$

In cylindrical coordinates this simplifies to

$$E = \{(r, \theta, z) : 0 \leq r \leq 1, 0 \leq \theta \leq 2\pi,\ 1 - r^2 \leq z \leq 4\}$$

- Let's calculate the mass of a solid corresponding to E with a density given by $\rho = Kr$, where K is a constant. The mass integral is

$$\begin{aligned} M &= \iiint_E \rho dV = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} \int_{z=1-r^2}^{z=4} \rho r\ dz\, dr\, d\theta \\ &= \int_0^{2\pi} \int_0^1 Kr^2(4 - 1 + r^2) dr d\theta \\ &= 2K\pi \Big[r^3 + r^5/5 \Big]_0^1 = \frac{12}{5} K\pi \end{aligned}$$

Rotational symmetry

- An integral where the domain and integrand have axial symmetry are prime candidates for integration in cylindrical coords.
- Indeed, sometimes a difficult iterated integral in Cartesian coordinates can be evaluated by rewriting it as an iterated integral in cylindrical coords.
- Example. Evaluate

$$I = \int_{x=-2}^{x=2} \int_{y=-\sqrt{4-x^2}}^{y=\sqrt{4-x^2}} \int_{z=\sqrt{x^2+y^2}}^{z=2} (x^2 + y^2)\, dz\, dy\, dx.$$

- We rewrite the above as the triple integral

$$I = \iiint_E (x^2 + y^2) dV$$

where

$$E = \{(x, y, z) : -2 \le x \le 2,\ -\sqrt{4-x^2} \le y \le \sqrt{4-x^2},\ \sqrt{x^2+y^2} \le z \le 2\}$$
$$= \{(x, y, z) : x^2 + y^2 \le 4,\ \sqrt{x^2+y^2} \le z \le 2\}$$

is the region bounded by the right circular cone lying and the plane $z = 2$.

Symmetry example cont.

- The above domain has a simplified expression in cylindrical coordinates:

$$E = \{(r, \theta, z) : 0 \leq \theta \leq 2\pi, 0 \leq r \leq 2,\ r \leq z \leq 2\}$$

- Finally, we re-express I as an iterated integral in cylindrical coordinates:

$$\begin{aligned} I &= \int_{\theta=0}^{2\pi} \int_{r=0}^{r=2} \int_{z=r}^{z=2} r^2 \times\ r\ dz\, dr\, d\theta \\ &= 2\pi \int_0^2 r^3(2-r)dr \\ &= 2\pi \Big[\frac{r^4}{2} - \frac{r^5}{5}\Big]_0^2 = \frac{16}{5}\pi \end{aligned}$$