Lecture 4
Spherical Coordinates

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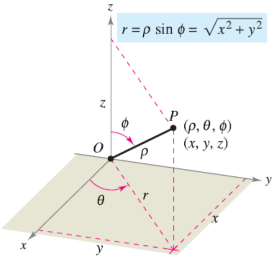
Outline

- Text: section 15.8
- Longitude and lattitude
- Spherical scale factor
- Examples

Spherical coordinates

- Lattitude $0 \leq \phi \leq \pi$ and longitude $0 \leq \theta \leq 2\pi$ specify the position of a point on the surface of the earth. In mathematics, it is customary to measure lattitude, not from the equator, but from the north pole.
- We add another variable $\rho \geq 0$ that measures the distance from the origin. Thus, $\rho = k$ is the equation of the sphere $x^2 + y^2 + z^2 = k^2$



- The relation between spherical and cartesian coordinates is given by

$$x = \rho \sin\phi \cos\theta, \quad y = \rho \sin\phi \sin\theta, \quad z = \rho \cos\phi.$$

- To derive the above relations we observe that $z = \rho \cos\phi$, $r = \rho \sin\phi$, where $r^2 = x^2 + y^2$ is the distance from the z-axis. Polar and Cartesian coordinates are related by

$$x = r \cos\theta, \quad y = r \sin\theta$$

from which the spherical coordinate formulas follow.

Cartesian to Spherical

- The relation between spherical and cartesian coordinates is given by

$$x = \rho \sin\phi \cos\theta, \quad y = \rho \sin\phi \sin\theta, \quad z = \rho \cos\phi.$$

- To obtain the inverse formulas that convert Cartesian to spherical coordinates we begin by calculating r, θ in the usual fashion:

$$r = \sqrt{x^2 + y^2}, \quad \theta = \arctan(y/x), \quad x > 0.$$

  By definition, we have

$$\rho = \sqrt{x^2 + y^2 + z^2}.$$

  Inverting the equation $z = \rho \cos\phi$ gives

$$\phi = \arccos(z/\rho).$$

Example 1

- Calculate the spherical coordinates of the point with Cartesian cordinates $(0, 2\sqrt{3}, -2)$
- Answer. In the x, y plane, the point $(0, 2\sqrt{3})$ lies on the border between the 1st and 2nd quadrant, and therefore $\theta = \pi/2$. Note: the formula $\theta = \arctan(y/x)$ does not apply because $x = 0$. We then have

$$\rho = \sqrt{0^2 + (2\sqrt{3})^2 + (-2)^2} = \sqrt{12 + 4} = 4$$

Finally,

$$\phi = \arccos(z/\rho) = \arccos(-1/2) = 2\pi/3.$$

Example 2

- Question: what is the distance between Halifax (44.65 deg N, 63.57 deg W) to Beijing (39.92 deg N, 116.38 deg E)?
- The spherical coordinates of Halifax and Beijing (in radians) are

$$\theta_1 = \frac{\pi}{180}(360 - 63.57) = 5.174, \quad \phi_1 = \frac{\pi}{180}(90 - 44.65) = 0.7915$$
$$\theta_2 = \frac{\pi}{180}116.38 = 2.031, \quad \phi_2 = \frac{\pi}{180}(90 - 39.92) = 0.8741$$

- Let's assume that earth is perfect sphere and use $\rho = 6370$ km as the radius of the earth. Using the formulas above, the Cartesian coordinates of the two cities are

$$\mathbf{u_1} = \rho \langle \cos\theta_1 \sin\phi_1, \sin\theta_1 \sin\phi_1, \cos\phi_1 \rangle = \rho \langle 0.32 - 0.64, 0.70 \rangle$$
$$\mathbf{u_2} = \rho \langle \cos\theta_2 \sin\phi_2, \sin\theta_2 \sin\phi_2, \cos\phi_2 \rangle = \rho \langle -0.34, 0.69, 0.64 \rangle$$

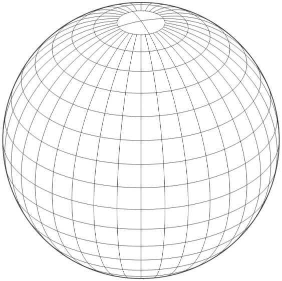
- The angle α between two vectors is given by the cosine law

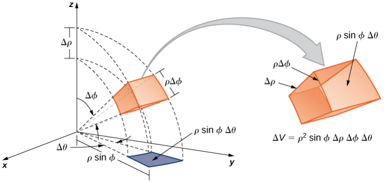
$$\cos\alpha = \frac{\mathbf{u_1} \cdot \mathbf{u_2}}{|\mathbf{u_1}||\mathbf{u_2}|}$$

  For the present case, we have $\cos\alpha = -0.10$. Taking the arc-cosine gives $\alpha = 1.67$ radians.
- The distance between the two cities is therefore $\rho\alpha \approx 10600$ km.

The spherical scale factor.

- Recall that integration using polar coordinates amounts to samping using the cells formed by the r, θ grid. The area of these polar cells is approximately $\Delta A = r \Delta r \Delta \theta$
- Analogously, we would like to evaluate triple integrals by sampling the 3-dimensional cells formed by the spherical grid. At small scales, these spherical cells resemble a box with dimensions $\Delta\rho \times \rho\Delta\phi \times \rho \sin\phi \Delta\theta$.
- The above formula is based on the observation that the meridians are proportional to ρ, but that the lines of longitude are proportional to $r = \rho \sin \phi$.





- As the above remarks demonstrate, integration in spherical coordinates requires an scale factor of $\rho^2 \sin\phi$ to account for the volume distortion of spherical coordinate cell. In other words,

$$dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta.$$

Example 3.

- Use spherical coordinates to evaluate the triple integral

$$I = \iiint_B (x^2 + y^2 + z^2)^2 dV$$

where B is the ball of radius 5 centred at the origin.
- We begin by converting the integrand into spherical coordinates:

$$(x^2 + y^2 + z^2) = \rho^4.$$

- Applying the spherical coordinate formula for dV yields

$$\begin{aligned} I &= \int_{\theta=0}^{\theta=2pi} \int_{\phi=0}^{\phi=\pi} \int_{\rho=0}^{\rho=5} \rho^4 \times \rho^2 \sin\phi \; d\rho d\phi d\theta \\ &= 2\pi \int_{\phi=0}^{\phi=\pi} \sin\phi d\phi \int_{\rho=0}^{\rho=5} \rho^6 d\rho \\ &= 2\pi \times \Big[-\cos\phi \Big]_0^\pi \times \Big[\rho^7/7 \Big]_0^5 \\ &= 312500/7\pi \end{aligned}$$

Example 4

- Use spherical coordinates to evaluate the triple integral

$$I = \iiint_E x\exp(x^2 + y^2 + z^2)dV$$

  where E is the positive portion of the unit ball $x, y, z \geq 0$.
- In spherical coords, the domain of integration is $E = \{(\rho, \theta, \phi) : 0 \leq \rho \leq 1,\ 0 \leq \phi \leq \pi/2,\ 0 \leq \theta \leq \pi/2\}$.
- The integrand is given by

$$x\exp(x^2 + y^2 + z^2) = \rho e^{\rho^2} \cos\theta \sin\phi.$$

- Taking into account the volume scale factor, we obtain

$$\begin{aligned}
I &= \int_{\theta=0}^{\theta=\pi/2} \int_{\phi=0}^{\phi=\pi/2} \int_{\rho=0}^{\rho=1} \rho^3 e^{\rho^2} \sin^2\phi \cos\theta\ d\rho d\phi d\theta \\
&= \int_{\theta=0}^{\theta=\pi/2} \cos\theta d\theta \int_{\phi=0}^{\phi=\pi/2} \sin^2\phi d\phi \int_{\rho=0}^{\rho=1} \rho^3 e^{\rho^2} d\rho \\
&= \Big[\sin\theta\Big]_0^{\pi/2} \Big[\phi/2 - \sin(2\phi)/4\Big]_0^{\pi/2} \Big[\frac{1}{2}(\rho^2 - 1)e^{\rho^2}\Big]_0^1 \\
&= 1 \times \pi/4 \times 1/2 = \pi/8
\end{aligned}$$

Example 5

- Use spherical coordinates to calculate the volume of a sphere of radius R
- We wish to evaluate the triple integral $I = \iiint_B dV$ where $B = \{(\rho, \phi, \theta) : 0 \leq \rho \leq R,\ 0 \leq \phi \leq \pi,\ 0 \leq \theta \leq 2\pi\}$.
- Taking into account the volume scale factor we have

$$\begin{aligned} I &= \int_{\theta=0}^{\theta=2\pi} \int_{\phi=0}^{\phi=\pi} \int_{\rho=0}^{\rho=R} \rho^2 \sin\phi \ d\rho d\phi d\theta \\ &= 2\pi\Big[-\cos\phi\Big]_0^\pi \Big[\frac{1}{3}\rho^3\Big]_0^R \\ &= \frac{4}{3}\pi R^3. \end{aligned}$$

Example 6

- Find the volume of the solid E bounded by the half-cone $z = \sqrt{x^2 + y^2}$ and the sphere $x^2 + y^2 + z^2 = z$.
- Rewrite the equation of the sphere as $x^2 + y^2 + (z - 1/2)^2 = 1/4$. The solid is a sphere of radius $1/2$, centered at $(0, 0, 1/2)$.
- The equation of this sphere in sperical coords is $\rho^2 = \rho \cos \phi$, which simplifies to $\rho = \cos \phi$. The equation of the cone is $\rho \cos \phi = \rho \sin \phi$, which simplifies to $\tan \phi = 1$, or $\phi = \pi/4$.
- Thus, $E = \{(\rho, \phi, \theta) : 0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \pi/4, 0 \leq \rho \leq \cos \phi\}$
- The volume integral is therefore

$$
\begin{aligned}
I &= \iiint_E dV = \int_{\theta=0}^{\theta=2\pi} \int_{\phi=0}^{\phi=\pi/4} \int_{\rho=0}^{\rho=\cos\phi} \rho^2 \sin \phi \; d\rho d\phi d\theta \\
&= 2\pi \int_0^{\pi/4} \frac{1}{3} \cos^3 \phi \sin \phi \; d\phi \\
&= -\frac{2}{3}\pi \Big[\cos \phi^4 / 4 \Big]_0^{\pi/4} \\
&= -\frac{\pi}{6} \left(\frac{1}{4} - 1 \right) = \frac{\pi}{8}
\end{aligned}
$$

Example 7

- Determine the average distance between a point inside a ball of radius R and the ball's centre.
- The average value is given by I/V where

$$I = \iiint_B \sqrt{x^2 + y^2 + z^2} dV$$

  where B is a ball of radius R centered at the origin, and where V is the volume of said ball.
- Using spherical coordinates,

$$\begin{aligned} I &= \int_{\theta=0}^{\theta=2\pi} \int_{\phi=0}^{\phi=\pi} \int_{\rho=0}^{\rho=R} \rho^3 \sin\phi \; d\phi d\theta \\ &= 4\pi \left[R^4/4 \right] = \pi R^4 \end{aligned}$$

- Since $V = 4/3\pi R^3$, the average distance is

$$\frac{\pi R^4}{V} = \frac{3}{4} R$$