Lecture 5
Transformations and Changes of Variables

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Outline

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The one-to-one property

- We say that a function $y = f(x)$ is **one-to-one** if distinct values of x are mapped to distinct values of y.
- The one-to-one property means that we can solve for x in terms of y. The formula that expresses x as a function of y is called the **inverse transformation**. We will denote it as $x = f^{-1}(y)$.
- Example. The function $f(x) = x^2$ is not one-to-one; e.g., $f(-2) = f(2) = 4$. In other words, there is no way to uniquely solve the equation $y = x^2$.
- However, if we restrict the domain to $x > 0$, then we obtain a one-to-one function whose inverse is $x = f^{-1}(y) = \sqrt{y}$.

2D Transformations

- A 2D transformation

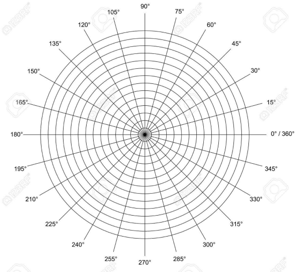
$$(x, y) = T(u, v) = (f(u, v), g(u, v))$$

  is a rule for mapping a 2D point to another 2D point. We also say that (x, y) is the image of (u, v) under the transforamtion.
- Just as above, we say that T is one-to-one if distinct points produce distinct images.
- If T is a one-to-one transformation then we can speak of an inverse transformation $(u, v) = T^{-1}(x, y)$ that gives (u, v) as an image of (x, y).
- Even though, conceptually 2D transformations have the same definition as 1D transformations, the algebraic manipulations tend to be much more involved.
- We can also regard a 2D transformation as the relation between two sets of coordinate systems.
- The curves $u =$ const. and $v =$ const. generate a grid in the x, y plane that allows us to visualize the transformation and the corresponding coordinates.

Example: polar coordinates

- The transformation from polar to Cartesian coorinates is the function

$$(x, y) = T(r, \theta) = (r \cos \theta, r \sin \theta).$$



Polar Grid In Degrees With Radius 15°

- As such, the above transformation is not one-to-one because

$$T(r, \theta) = T(r, \theta + 2\pi).$$

- However, if we restrict the domain to $r > 0$ and $-\pi/2 < \theta < \pi/2$ we obtain a one-to-one transformation with inverse

$$(r, \theta) = T^{-1}(x, y) = \left(\sqrt{x^2 + y^2}, \tan^{-1}\left(\frac{y}{x}\right) \right).$$

- The image of the strip $U = \{(r, \theta) : r > 0, -\pi/2 < \theta < \pi/2\}$ is the right half-plane $T(U) = \{(x, y) : x > 0\}$.
- Example. The inverse image of the $\frac{1}{4}$-disk $D = \{(x, y) : x^2 + y^2 < 1, x > 0, y > 0\}$ is the rectangle

$$T^{-1}(D) = \{(r, \theta) : 0 < r < 1,\ 0 < \theta < \pi/2\}.$$

Linear transformations

- A transformation of the form

$$(x, y) = L(u, v) = (au + bv, cu + dv), \quad ad - bc \neq 0$$

  is called a linear transformation.
- The inverse transformation is

$$(u, v) = L^{-1}(x, y) = \left(\frac{dx - by}{ad - bc}, \frac{ay - cx}{ad - bc} \right)$$

  The denominator $ad - bc$ in the above is the reason why we include the condition $ad - bc \neq 0$.
- The calculations below suffice to verify the inverse formula:

$$a(dx - by) + b(ay - cx) = (ad - bc)x$$
$$c(dx - by) + d(ay - cx) = (ad - bc)y$$

- The image of a rectangle under a linear transformations is a parallelogram. In particular the image $L(S)$ of the unit square $S = [0, 1] \times [0, 1]$ in the u, v plane is the parallelogram formed by the vectors $\langle a, c \rangle$ and $\langle b, d \rangle$.

Change of variables

- We study transformations and coordinates for the same reason we study the substitution rule (change of variables) in 1D integrals.
- Recall that
$$\int_a^b f(x)dx = \int_c^d f(g(u))g'(u)du$$
where $x = g(u)$ is a change of variables, where $u = g^{-1}(x)$ is the inverse function, and where $c = g^{-1}(a), d = g^{-1}(b)$.
- In order to derive an analogous rule for double integrals we must observe that a change of variables introduces a distortion of area.
- We define the **Jacobian** of a transformation $(x, y) = T(u, v)$ to be the 2×2 determinant
$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$
- The cells of a finely meshed u, v grid are approximated by a parallelogram whose area is scaled by the above Jacobian.

Derivation of the Jacobian

- The cells of a finely meshed u, v grid are approximated by a parallelogram whose area is scaled by the above Jacobian.
- Express $(x, y) = T(u, v)$ as $x = f(u, v),\ y = g(u, v)$.
- Linear approximation then gives us

$$\Delta x \approx \frac{\partial x}{\partial u}\Delta u + \frac{\partial x}{\partial v}\Delta v$$
$$\Delta y \approx \frac{\partial y}{\partial u}\Delta u + \frac{\partial y}{\partial v}\Delta v$$

  The above approximate improves as $\Delta u, \Delta v$ shrink to zero.
- Using the above, we approximate the corners of a u, v cell as follows:

$$T(u + \Delta u, v) \approx T(u, v) + T_u\,\Delta u, \quad T_u = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u} \right\rangle$$
$$T(u, v + \Delta v) \approx T(u, v) + T_v\,\Delta v \quad T_v = \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v} \right\rangle$$

- For small $\Delta u, \Delta v$ the cell resembles a pallelogram generated by the vectors $T_u\Delta u$, $T_v\Delta v$ having area

$$|T_u \times T_v|\Delta u \Delta v = \left|\frac{\partial(x, y)}{\partial(u, v)}\right| \Delta u \Delta v.$$

Change of variables in double integrals

- Let $(x, y) = T(u, v)$ be a 2D transformation and R a domain in the (x, y) plane. Then,

$$\iint_R F(x,y)dxdy = \iint_S F(T(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv$$

where $S = T^{-1}(R)$ is the inverse image of R in the (u, v) plane.

- The Jacobian is present as a scale-factor in the above formula to account for the distortion introduced by switching to the (u, v) grid.
- Example. Consider the Jacobian of the polar coordinate transformation $x = r\cos\theta,\ y = r\sin\theta$.

$$\begin{aligned}\frac{\partial(x,y)}{\partial(r,\theta)} &= \frac{\partial x}{\partial r}\frac{\partial y}{\partial \theta} - \frac{\partial x}{\partial \theta}\frac{\partial y}{\partial r} \\ &= r\cos\theta \times \cos\theta - (-r\sin\theta) \times \sin\theta = r\end{aligned}$$

- The above value for the Jacobian is reflected in the change of variables formula for polar coordinates:

$$\iint_R f(x,y)dxdy = \iint_S f(r\cos\theta, r\sin\theta)rdrd\theta$$

where S is the pre-image of the domain R in the r, θ plane.

Example

- Evaluate the double integral $I = \iint_R \exp((x+y)/(x-y))dA$ where R is the trapezoid with vertices $(1,0),(2,0),(0,-2),(0,-1)$
- We employ the change of variables $u = x+y, v = x-y$.
- The inverse transformation is $x = 1/2(u+v),\ y = 1/2(u-v)$.
- The integrand is now $\exp((x+y)/(x-y)) = e^{u/v}$.
- The vertices of R, when expressed using (u,v) coordinates correspond to $(1,1),(2,2),(-2,2),(-1,1)$. Since the transformation is linear, the image of R in the (u,v) plane is the trapezoid

$$S = \{(u,v) : 1 \leq v \leq 2, -v \leq u \leq v\}.$$

- The Jacobian is $\dfrac{\partial(x,y)}{\partial(u,v)} = |-1/4 - 1/4| = 1/2$.
- Finally, applying the change of variables formula gives

$$I = \frac{1}{2}\int_{v=1}^{v=2}\int_{u=-v}^{u=v} e^{u/v}dudv = \frac{1}{2}\int_1^2 \left(\left[ve^{u/v}\right]_{u=-v}^{u=v}\right)dv$$
$$= \frac{1}{2}\int_1^2 v(e-e^{-1})dv = (e-e^{-1})\left[v^2/4\right]_{v=1}^{v=2} = \frac{3}{4}(e-e^{-1})$$

Change of variables in triple integrals

- A change of variables in a triple integral follows the same logic as for double integrals.
- A 3D transformation $(x, y, z) = T(u, v, w)$ distorts the volume of the reference cells by a scale factor corresponding to the 3×3 Jacobian

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

- Accordingly, the change of variables for triple integrals takes the form

$$\iint_R F(x,y,z) dxdydz = \iiint_S F(T(u,v,w)) \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| dudvdw$$

where $S = T^{-1}(R)$ is the inverse image of R in (u, v, w) space.

Application: spherical coordinate volume element

- We re-derive the spherical coord. scale-factor as a 3×3 Jacobian.
- The spherical coordinate transformation is

$$x = \rho \cos\theta \sin\phi, \quad y = \rho \sin\theta \sin\phi, \quad z = \rho \cos\phi$$

- To simplify things we set $\mathbf{r} = \rho\cos\theta\sin\phi\,\mathbf{i} + \rho\sin\theta\sin\phi\,\mathbf{j} + \rho\cos\phi\,\mathbf{k}$ and express the Jacobian as a triple product

$$\begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \theta} \end{vmatrix} = \mathbf{r}_\rho \cdot (\mathbf{r}_\phi \times \mathbf{r}_\theta)$$

$$\mathbf{r}_\rho = \cos\theta\sin\phi\mathbf{i} + \sin\theta\sin\phi\mathbf{j} + \cos\phi\mathbf{k}$$

$$\mathbf{r}_\phi \times \mathbf{r}_\theta = \begin{vmatrix} \rho\cos\theta\cos\phi & \rho\sin\theta\cos\phi & -\rho\sin\phi \\ -\rho\sin\theta\sin\phi & \rho\cos\theta\sin\phi & 0 \\ \mathbf{i} & \mathbf{j} & \mathbf{k} \end{vmatrix}$$

$$= \rho^2\sin\phi(\cos\theta\sin\phi\mathbf{i} + \sin\theta\sin\phi\mathbf{j} + \cos\phi\mathbf{k}) = \rho^2\sin\phi\mathbf{r}_\rho$$

- Once we observe that $\mathbf{r}_\rho$ is a unit vector, we conclude that

$$\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} = \rho^2\sin\phi$$

in agreement with the formula for triple integrals in spherical coordinates.