Lecture 6
Line Integrals

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Outline

- Text: sections 10.2, 10.3, 13.2 (review) and section 16.2
- Line integrals with respect to arclength.
- Example 1.
- Mass integrals.
- Example 2.
- Oriented line integrals.
- Example 3.

Parametric curves

- A 2D parametric curve is the representation of a plane curve by means of two functions of one variable:

$$x = f(t), \quad y = g(t), \quad t_0 \leq t \leq t_1.$$

  One interprets the above equations as the trajectory of a particle with position x, y at time t. A domain $t_0 \leq t \leq t_1$ restricts the curve to the segment with endpoints:

$$(x_0, y_0) = (f(t_0), g(t_0)), \quad (x_1, y_1) = (f(t_1), g(t_1)).$$

- Example: $x = \cos(t), y = \sin(t),\ 0 \leq t \leq \pi$ is a parameterization of the upper semicircle $x^2 + y^2 = 1,\ y \geq 0$. The initial endpoint $(1, 0)$ is attained at $t = 0$. The final endpoint $(-1, 0)$ is attained at $t = \pi$.
- A change of variables such as $\tau = \cos t$ provides a different parameterization of the same curve:

$$x = \tau, y = \sqrt{1 - \tau^2},\ -1 \leq \tau \leq 1.$$

- With the reparameterization, the starting endpoint is $(-1, 0)$ and the final endpoint is $(1, 0)$. (Desmos) We therefore say the above parameterizations have opposite orientations.

Speed and arclength

- Consider a parametric curve $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$. If $\mathbf{r}(t)$ represents the position of a particle, then $\mathbf{r}'(t) = f'(t)\mathbf{i} + g'(t)\mathbf{j}$ represents the velocity vector. The magnitude of the velocity vector
$$|\mathbf{r}'(t)| = \sqrt{f'(t)^2 + g'(t)^2}$$
represents the speed of the particle.
- The arclength function represents the distance travelled by a particle
$$s(t) = \int_0^t |\mathbf{r}'(u)| du$$
- By FTC, $s'(t) = |\mathbf{r}'(t)|$. Rewriting as
$$\frac{ds}{dt} = \sqrt{\frac{dx}{dt}^2 + \frac{dy}{dt}^2},$$
gives the formula for the arclength element
$$ds = \sqrt{\frac{dx}{dt}^2 + \frac{dy}{dt}^2}\, dt$$
- The length of a parametric curve may be obtained by integrating the speed with respect to time:

Line integrals with respect to arclength

- Let C be a curve and ds the corresponding element of arclength. We consider integrals of the form

$$I = \int_C F(x, y) ds$$

- We evaluate such an integral by parameterizing the curve

$$x = f(t),\ y = g(t), \quad a \leq t \leq b$$

and using the preceding expression for the arclength. Thus,

$$I = \int_{t=a}^{t=b} F(f(t), g(t)) \sqrt{f'(t)^2 + g'(t)^2} dt$$

- As a special case, if $F(x, y) = 1$ then we obtain the integral for the length of the curve:

$$L = \int_C ds = \int_{t=a}^{t=b} \sqrt{f'(t)^2 + g'(t)^2} dt.$$

Example

- Evaluate $I = \int_C (2 + x^2 y) ds$ where C is the upper unit half-circle.
- We parameterize C as $x = \cos t,\ y = \sin t,\ 0 \le t \le \pi$.
- The arclength formula gives $ds = dt$.
- Applying the above substitutions gives

$$I = \int_0^\pi (2 + \cos^2(t)\sin(t))dt = 2\pi - \left[\frac{1}{3}\cos^3 t\right]_0^\pi = 2\pi + \frac{2}{3}.$$

- Reparameterizing a given curve C amounts to applying the substitution rule in evaluating I.
- Example: reparameterize C as $x = u,\ y = \sqrt{1 - u^2}, \quad -1 \le u \le 1$.

$$\begin{aligned} ds &= \sqrt{1 + \frac{u^2}{1 - u^2}}\, du = \frac{du}{\sqrt{1 - u^2}} \\ I &= \int_{u=-1}^{u=1} \left(2 + u^2\sqrt{1 - u^2}\right) \frac{du}{\sqrt{1 - u^2}} \\ &= 2\left[\arcsin(u) + \frac{u^3}{3}\right]_{-1}^{1} = 2\pi + \frac{2}{3} \end{aligned}$$

- The above evaluation corresponds to the trig. substitution $u = \cos t$

Mass integrals

- By regarding the integrand as a density function $\rho(x, y)$, we interpret a line integral as the mass of a thin wire. Note that if density is constant, then mass equals length times density.
- Example. Consider a wire C in the shape of a parabolic segment $y = x^2,\ 0 \leq x \leq 1$ with a density function $\rho(x, y) = 2x$.
- Parameterize C as $x = t, y = t^2, 0 \leq t \leq 1, \quad ds = \sqrt{1 + 4t^2}dt$
- The corresponding mass integral is

$$M = \int_C 2xds = \int_{t=0}^{t=1} t\sqrt{1 + 4t^2}dt$$
$$= \left[(1 + 4t^2)^{3/2} \times \frac{1}{2} \times \frac{1}{3}\right]_0^1 = \frac{1}{6}\left(5\sqrt{5} - 1\right).$$

- A different parameterization, say $x = \sqrt{u},\ y = u,\ 0 \leq u \leq 1$ gives

$$ds = \sqrt{1 + \frac{1}{4u}}du = \frac{1}{2}\frac{\sqrt{4u + 1}}{\sqrt{u}}du$$
$$M = \int_{u=0}^{u=1} \sqrt{1 + 4u}du = \frac{1}{4} \times \frac{2}{3} \times (1 + 4u)^{3/2}\Big|_0^1 = \frac{1}{6}\left(5\sqrt{5} - 1\right).$$

- The reparameterization is equivalent to a change of variable $u = t^2$.

Oriented line integrals

- Let C be a curve. An integral of the form

$$I = \int_C P(x,y)dx + Q(x,y)dy$$

  is called an oriented line integral.
- To evaluate I one needs to choose a parameterization $x = f(t),\ y = g(t), a \leq t \leq b$ and to apply the substitutions $dx = f'(t)dt$ and $dy = g'(t)dt$.
- The value of the integral is then

$$I = \int_{t=a}^{t=b} \left(P(f(t), g(t))f'(t) + Q(f(t), g(t))g'(t)\right) dt$$

- A reparameterization of C with the same orientation does not change the value of I. A reparameterization that reverses the orientation changes the sign of I.

Example 3.

- Let C be the parabolic segment $y = x^2,\ 0 \leq x \leq 2$. Evaluate
$$I = \int_C ydx + xdy.$$
- Our first choice of parameterization is $x = t,\ y = t^2,\ 0 \leq t \leq 2$.
$$dx = dt,\ dy = 2tdt \qquad I = \int_{t=0}^{t=2} (t^2 + 2t^2)dt = t^3\Big|_0^2 = 8$$
- The reparameterization $x = \sqrt{u},\ y = u,\ 0 \leq u \leq 4$ corresponds to the change of variables $u = t^2$. The start and endpoints are the same for both parameterizations. The value of the integral is unchanged:
$$dx = \frac{1}{2\sqrt{u}}du,\ dy = du, \qquad I = \int_{u=0}^{u=4} \frac{3}{2}\sqrt{u}du = u^{3/2}\Big|_{u=0}^{u=4} = 8$$
- Reparameterize C as $x = 2 - v, y = (2-v)^2,\ 0 \leq v \leq 2$.
$$dx = -dv,\ dy = -2(2-v)dv, \quad I = -3\int_0^2 (2-v)^2 dv = (2-v)^3\Big|_0^2 = 0-8$$
- The corresponding change of variables $t = 2 - v$ reverses the orientation. Now $(2,4)$ is the start and $(0,0)$ is the endpoint.