Lecture 7
Vector Fields and Work Integrals

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Outline

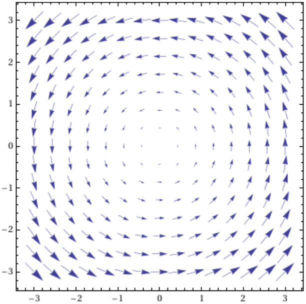
- Text: section 16.1,16.2
- Vector fields
- Conservative vector fields
- Work Integrals
- Examples

Vector fields

- A 2D vector field assigns a vector to every point in the plane. Mathematically, a plane vector field is represented by two functions of two variables: $\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$.
- A 3D vector field requies 3 functions of 3 variables:

$$\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}.$$

- From the point of view of physics, a vector field may be regarded as a chart wind velocities, or a chart of force vectors. Wind velocities at ground level correspond to a 2D vector field. If altitude is taken into account, then wind velocities correspond to a 3D vector field.
- We can visualize 2D vector fields by plotting vectors on a discrete grid. A plot of the vector field $\mathbf{F} = -y\mathbf{i} + x\mathbf{j}$ is shown on the right. See figures 6-8 in Section 16.1 of the text for additional examples.



Example: Newtons force of gravity

- Two objects of masses m and M experience a mutual gravitation attraction of magnitude mMG/r^2, where r is the distance between the two objects, and where G is the universal gravitational constant.
- If we situate object M at the origin, then direction of the force experienced by object m is given by the unit vector $-\mathbf{r}/|\mathbf{r}|$ where $\mathbf{r}$ is the position of the object m.
- If we regard M as a fixed massive object, then the gravitational force field exercised upon smaller objects in its vicinity is represented by the vector field

$$\mathbf{F}(x, y, z) = -mMG\frac{\mathbf{r}}{|\mathbf{r}|^3}, \quad \mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

Conservative vector fields

- Recall the definition of the gradient operator ∇f. If $f(x,y)$ is a function of 2 variables, then $\nabla f = f_x \mathbf{i} + f_y \mathbf{j}$.
  If $f(x,y,z)$ is a function of 3 variables, then

$$\nabla f = f_x \mathbf{i} + f_y \mathbf{j} + f_z \mathbf{k}.$$

- We say that a vector field $\mathbf{F}$ is conservative if $F = \nabla f$ for some function f. Note: in physics one typically expresses a conservative force as $\mathbf{F} = -\nabla U$, where U is called the potential function.
- Example. The gravitational vector field

$$\mathbf{F} = -\frac{\mathbf{r}}{|\mathbf{r}|^3}, \quad \mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

  is conservative. The corresponding gravitational potential is

$$U(x,y,z) = -\frac{1}{\sqrt{x^2+y^2+z^2}} = -\frac{1}{|\mathbf{r}|}.$$

- A direct calculation shows that $\mathbf{F} = -\nabla U$, that is

$$U_x = \frac{x}{|\mathbf{r}|^3}, \quad U_y = \frac{y}{|\mathbf{r}|^3}, \quad U_z = \frac{z}{|\mathbf{r}|^3}.$$

Work integrals

- Let $F(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ be a vector field that represents a field of force vectors. Let C be an oriented curve. The integral

$$W = \int_C \mathbf{F} \cdot \mathbf{T} ds$$

  where $\mathbf{T}$ is the unit tangent vector of C represents the work done by an object moving along trajectory C.
- The value of $\mathbf{T}$ and hence of W will change sign if we reverse the orientation of C. Therefore, a work integral is an instance of an oriented line integral.
- The work integral generalizes the physics principle $W = FD$; work is force times distance. The basic principle applies when force is applied uniformly. Integration is required when that isn't the case.

Work integrals cont.

- In order to calculate a work integral $W = \int_C \mathbf{F} \cdot \mathbf{T} ds$ we must parameterize C as $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j},\ a \leq t \leq b$
- Observe that

$$\mathbf{T} = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}, \qquad ds = |\mathbf{r}'(t)|dt$$

  It follows that $\mathbf{T}ds = dr$, and hence $W = \int_C \mathbf{F} \cdot d\mathbf{r}$.
- Writing $\mathbf{F} = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$ we obtain

$$W = \int_a^b P(f(t), g(t))f'(t) + Q(f(t), g(t))g'(t)dt$$

  This means that the work integral could also be written as

$$W = \int_C Pdx + Qdy.$$

- Consequently, we will call an oriented line integral of the form

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C Pdx + Qdy$$

  a line integral with respect to the vector field $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$.

Example

- Find the work done the by force $\mathbf{F} = x^2\mathbf{i} - xy\mathbf{j}$ on a praticle that moves along the quater-circle $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j},\ 0 \leq t \leq \pi/2$.
- We have $d\mathbf{r} = (-\sin t\,\mathbf{i} + \cos t\,\mathbf{j})dt$. Hence,

$$
\begin{aligned}
I = \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{\pi/2} (\cos^2 t\ \mathbf{i} - \cos t \sin t\ \mathbf{j}) \cdot (-\sin t\ \mathbf{i} + \cos t\ \mathbf{j}) dt \\
&= \int_0^{\pi/2} -2\cos^2 t \sin t\, dt \\
&= \frac{2}{3} \cos^3 t \Big|_0^{\pi/2} = -\frac{2}{3}
\end{aligned}
$$

- An equivalent formulation is $I = \int_C (x^2 dx - xy dy)$
  Using this notation, we would write

$$
\begin{aligned}
x = \cos t,\ y &= \sin t,\ 0 \leq t \leq \pi/2 \\
dx = -\sin t dt,\ dy &= \cos t dt \\
I &= \int_0^{\pi/2} (-\cos^2 t \sin t - \cos t \sin t \cos t) dt \\
&= -2 \int_0^{\pi/2} \cos^2 t \sin t\, dt
\end{aligned}
$$

3D line integrals

- 3D work and line integrals are a straight-forward generalization of the 2D concept.
- Example. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F} = xy\mathbf{i} + yz\mathbf{j} + zx\mathbf{k}$ and C is the segment of the twisted cubic $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k},\ 0 \leq t \leq 1$
- We have $d\mathbf{r} = (\mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k})dt$. Hence,

$$
\begin{aligned}
I &= \int_0^1 (t^3\mathbf{i} + t^5\mathbf{j} + t^4\mathbf{k}) \cdot (\mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k})dt \\
&= \int_0^1 (t^3 + 2t^6 + 3t^6)dt = \left[\frac{1}{4}t^4 + \frac{5}{7}t^7\right]_0^1 = \frac{27}{28}
\end{aligned}
$$

- An equivalent formulation is $I = \int_C xydx + yzdy + zxdz$. Using this notation we would write

$$
\begin{gathered}
x = t,\ y = t^2,\ z = t^3,\ 0 \leq t \leq 1 \\
dx = dt,\ dy = 2tdt,\ dz = 3t^2dt \\
xydx = t^3dt,\ yzdy = 2t^6dt,\ zxdt = 3t^6dt \\
I = \int_0^1 (t^3 + 2t^6 + 3t^6)dt
\end{gathered}
$$