Lecture 8
Fundamental Theorem for Line Integrals

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Outline

- Text: section 16.3
- The Fundamental Theorem of Calculus for line integrals.
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The Fundamental Theorem of Calculus for line integrals

- Recall the single variable FTC:

$$\int_a^b f'(x)dx = f(b) - f(a).$$

- The FTC can be generalized to oriented line integrals.
- **Theorem 1.** $\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(t_1)) - f(\mathbf{r}(t_0))$ where C is a curve parameterized by $\mathbf{r}(t),\ t_0 \leq t \leq t_1$.
- Proof. Write $\mathbf{r}(t) = a(t)\mathbf{i} + b(t)\mathbf{j}$ so that

$$I = \int_C \nabla f \cdot \mathbf{r} = \int_{t_0}^{t_1} f_x(a(t), b(t))a'(t) + f_y(a(t), b(t)b'(t))dt$$

By the chain rule,

$$\frac{d}{dt} f(a(t), b(t)) = f_x(a(t), b(t))a'(t) + f_y(a(t), b(t)b'(t))$$

Hence, by the single variable FTC, we have

$$I = \int_{t_0}^{t_1} \frac{d}{dt} f(a(t), b(t))dt = f(a(t_1), b(t_1)) - f(a(t_0), b(t_0)).$$

- A similar proof works for 3D line integrals. Indeed, the FTC of line integrals holds in all dimensions.

Example: gravitational potential

- Question: calculate the work done by the gravitation field

$$\mathbf{F} = -\frac{\mathbf{r}}{|\mathbf{r}|^3}$$

  in moving from position $\mathbf{r}_1 = (x_1, y_1, z_1)$ to position $\mathbf{r}_2 = (x_2, y_2, z_2)$.
- Recall that the gravitational field is conservative $\mathbf{F} = -\nabla U$ where the gravitational potential is $U = -1/|\mathbf{r}|$.
- Since $\mathbf{F}$ is conservative, the work done is the same regardless of the path that connects the two given points.
- Thus, by the FTC, the work done is

$$-\Delta U = -U(x_2, y_2, z_2) + U(x_1, y_1, z_1) = mMG\left(\frac{1}{|\mathbf{r}_2|} - \frac{1}{|\mathbf{r}_1|}\right).$$

- The reason for the negative sign in $\mathbf{F} = -\nabla U$ is that, in physics, positive work done by a conservative force corresponds to a loss of potential energy, and negative work to a gain of potential energy.
- For example, if $|\mathbf{r}_2| > |\mathbf{r}_1|$ (moving up the gravity well), then work is negative with a corresponding gain of potential energy $\Delta U > 0$.

Path independence

- We say that $\int \mathbf{F} \cdot d\mathbf{r}$ is **path independent** if the value of $\int_C \mathbf{F} \cdot d\mathbf{r}$ depends only on the star- and end-point of the curve C.
- We say that a curve $\mathbf{r}(t),\ t_0 \leq t \leq t_1$ is **closed** if $\mathbf{r}(t_0) = \mathbf{r}(t_1)$; i.e., if the curve begins and ends at the same location.
- An oriented line integral $\oint_C \mathbf{F} \cdot d\mathbf{r}$ where C is a closed curve is called the **circulation** of $\mathbf{F}$ around C integral.
- **Theorem 2.** The circulation $\oint_C \mathbf{F} \cdot \mathbf{r} = 0$ for all closed curves C if and only if $\int \mathbf{F} \cdot \mathbf{r}$ is a path-independent integral.
- Proof. Suppose $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ for all closed C. Let C_1, C_2 be two curves that have the same start- and end-points. Let $C = C_1 - C_2$ be the closed curve formed by following C_1 and then following C_2 in reverse. By assumption $\oint_C \mathbf{F} \cdot \mathbf{r} = 0$. By construction, $$\int_C \mathbf{F} \cdot \mathbf{r} = \int_{C_1} \mathbf{F} \cdot \mathbf{r} - \int_{C_2} \mathbf{F} \cdot \mathbf{r} = 0.$$ It follows that $\int_{C_1} \mathbf{F} \cdot \mathbf{r} = \int_{C_2} \mathbf{F} \cdot \mathbf{r}$.
- Suppose $\int \mathbf{F} \cdot d\mathbf{r}$ is path-independent. Let C be a closed curve. Write $C = C_1 - C_2$ where C_1, C_2 have the same start- and end-points. By assumption, $\int_{C_1} \mathbf{F} \cdot \mathbf{r} = \int_{C_2} \mathbf{F} \cdot \mathbf{r}$. Therefore $\int_C \mathbf{F} \cdot \mathbf{r} = \int_{C_1} \mathbf{F} \cdot \mathbf{r} - \int_{C_2} \mathbf{F} \cdot \mathbf{r} = 0$

Conservative vector fields

- We say that a vector field $\mathbf{F} = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ is **conservative** if $\mathbf{F} = \nabla f$ for some $f(x, y)$.
- Equivalently, $\mathbf{F}$ is conservative if there exists an $f(x, y)$ such that

$$Pdx + Qdy = df = f_x dx + f_y dy$$

- **Theorem 3.** A vector field $\mathbf{F}$ is conservative if and only if $\int \mathbf{F} \cdot d\mathbf{r}$ is path independent.
- Proof. Suppose that $F = \nabla f$ for some $f(x, y)$. By the FTC of line integrals, if $\mathbf{F} = \nabla f$, then

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = f(\mathbf{r}(t_1)) - f(\mathbf{r}(t_0)),$$

where $\mathbf{r}(t),\ t_0 \leq t \leq t_1$ is a parameterization of C_1. If C_2 is another curve with the same endpoints, then $\int_{C_2} \mathbf{F} \cdot d\mathbf{r}$ will have the same value as above.

Proof of the converse.

- Suppose that $\int_C Pdx + Qdy$ is path independent.
- Fix a point (x_0, y_0) and define the function

$$f(x,y) = \int_{(x_0,y_0)}^{(x,y)} Pdx + Qdy = \int_C Pdx + Qdy$$

  Here C is any curve that connects (x_0, y_0) to (x, y). The value of the integral is the same for all C by the path-independence assumption.
- Fix an arbitrary point (x_1, y_1). We claim that $P(x_1, y_1) = f_x(x_1, y_1)$.
- Observe that $f_x(x_1, y_1) = \lim_{h\to 0} \dfrac{f(x_1, y_1) - f(x_1 - h, y_1)}{h}$
- Observe that $f(x_1, y_1) - f(x_1 - h, y_1) = \displaystyle\int_{(x_1-h,y_1)}^{(x_1,y_1)} Pdx + Qdy$
- Let C be the curve $x = x_1 + t, y = y_1, \ -h \leq t \leq 0$.
- We have $dx = dt, dy = 0$. Hence, $\displaystyle\int_{(x_1-h,y_1)}^{(x_1,y_1} Pdx + Qdy = \int_{-h}^{0} P(x_1 + t, y_1)dt$
- Hence, $\dfrac{f(x_1, y_1) - f(x_1 - h, y_1)}{h}$ is the average value of $P(x, y)$ along C. As $h \to 0$, this average converges to $P(x_1, y_1)$.
- This proves that $P(x_1, y_1) = f_x(x_1, y_1)$.
- The proof that $Q = f_y$ is similar.

Example 2.

- Consider the line integral $\int (x-y)dx + (x-2)dy = \int \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F} = (x-y)\mathbf{i} + (x-2)\mathbf{j}$
- Join the points $(0,0)$ and $(1,1)$ by the curve C_1, given by $x = t,\ y = t,\ 0 \leq t \leq 1$, and by C_2, given by $x = t,\ y = t^2,\ 0 \leq t \leq 1$.
- Evaluating the two integrals gives

$$\int_{C_1} (x-y)dx + (x-2)dy = \int_0^1 (t-t)dt + \int_0^1 (t-2)dt$$

$$= t^2 - 2t\Big|_0^1 = -1$$

$$\int_{C_2} (x-y)dx + (x-2)dy = \int_0^1 (t-t^2)dt + \int_0^1 (t-2)2tdt$$

$$= \int_0^1 (-2 + 2t - t^2)dt = \left[-2t + t^2 - \frac{t^3}{3}\right]_0^1$$

$$= -2 + 1 - \frac{1}{3} = -\frac{4}{3}$$

- The value of the two integrals is different, and therefore the given $\mathbf{F}$ is not conservative.

Example 3.

- We will show that the vector field $\mathbf{F} = (3 + 2xy)\mathbf{i} + (x^2 - 3y^2)\mathbf{j}$ is conservative.
- We are searching for a function $f(x, y)$ such that

$$df = f_x dx + f_y dy = (3 + 2xy)dx + (x^2 - 3y^2)dy.$$

- Integrating the first component yields

$$f(x, y) = \int (3 + 2xy)dx = 3x + x^2 y + g(y),$$

  where $g(y)$ is an unknown function of y.
- Observe that $f_y = x^2 + g'(y)$. This implies that

$$g'(y) = -3y^2, \qquad g(y) = -y^3 + C.$$

- Therefore $\mathbf{F} = \nabla f$ is conservative with

$$f(x, y) = 3x^2 + 2x^2 y - y^3 + C.$$

- The constant of integration doesn't matter when we evaluate integral because $dC = 0$, just like the constant of integration in the single variable anti-derivative.

Example 4.

- Evaluate $I = \int_C (3+2xy)dx + (x^2 - 3y^2)dy$ where C is given by $\mathbf{r}(t) = e^t \sin t\,\mathbf{i} + e^t \cos t\,\mathbf{j},\ 0 \leq t \leq \pi$.
- Now that we know that the above integral is path-independent, we don't actuall have to do any integration. By the FTC we have

$$\mathbf{r}(0) = (0,1) \quad \mathbf{r}(\pi) = (0, e^\pi),$$
$$I = f(0, e^\pi) - f(0,1),$$

  where $f(x,y) = 3x^2 + 2x^2y - y^3 + C$.
- Performing the necessary substitutions gives

$$f(0,1) = -1 + C,$$
$$f(0, -e^\pi) = e^{3\pi} + C,$$
$$I = e^{3\pi} + 1$$