Lecture 9
Green's Theorem

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Green's Theorem

- A simple closed curve is a closed curve without self-intersections.
- A region $D \subset \mathbb{R}^2$ that is bounded by a simple closed curve is called simply connected. A region that **isn't** simply connected has a complicated boundary consisting of multiple closed curves, interior arcs, and point-punctures.
- **Theorem.** Suppose that $P(x, y)$ and $Q(x, y)$ and their partial derivatives are non-singular on a region $D \subset \mathbb{R}^2$. Then,

  $$\oint_C Pdx + Qdy = \iint_D (Q_x - P_y)dA,$$

  where C is the boundary of D oriented in such a way that D lies to the left, as C is traversed.
- Note, if D is a simply connected region, then the "left-side" rule means that it's boundary C should be given a counter-clockwise orientation.

Example 1.

- Consider the circulation integral $I = \oint_C x\,dy$ where C is the circle of radius R centered at the origin with a counterclockwise orientation.
- Taking $x = R\cos t,\ y = r\sin t,\ 0 \le t \le 2\pi$ as the parameterization of C we calculate

$$dx = -R\sin t, \quad dy = R\cos t dt,$$
$$I = R^2\int_0^{2\pi}\cos^2 t dt = \pi R^2$$

- In the above integral, $P = 0, Q = x$. Thus, by Green's theorem

$$I = \iint_D (Q_x - P_y) dA = \iint_D dA = \pi R^2$$

because the interior of the circle C is the disk of radius R.

Proof of GT

- We will begin by proving a special case of Green's Theorem where C bounds a type I region $D = \{(x, y) : a \leq x \leq b, f(x) \leq y \leq g(x)\}$ and where the integrand has the form Pdx.
- Our goal is to prove that $\oint_C Pdx = -\iint_D P_y dA$ where the border of D is the compound curve $C = C_1 + C_2 - C_3 - C_4$ where

$$C_1\colon x = t,\ y = f(t),\ a \leq t \leq b$$
$$C_2\colon x = b,\ y = t,\ f(b) \leq t \leq g(b)$$
$$C_3\colon x = t,\ y = g(t),\ a \leq t \leq b$$
$$C_4\colon x = a,\ y = t,\ f(a) \leq t \leq g(a)$$

- The corresponding integrals are

$$I_1 = \int_{C_1} Pdx = \int_a^b P(t, f(t))dt$$
$$I_2 = \int_{C_2} Pdx = 0 \quad \text{because } dx = 0$$
$$I_3 = \int_{C_3} Pdx = \int_a^b P(t, g(t))dt$$
$$I_4 = \int_{C_4} Pdx = 0$$

Proof cont.

- On the preceding slide we showed that the border of a type I region is a compound curve $C = C_1 + C_2 - C_3 - C_4$ and calculated the value of the corresponding circulation integral:

$$I = \oint_C P dx = \int_a^b (P(t, f(t)) - P(t, g(t)))dt$$

- Turning to the double integral, we have

$$\iint_D P_y \, dA = \int_{x=a}^{x=b} \int_{y=f(x)}^{y=g(x)} P_y dy dx$$
$$= \int_a^b (P(x, g(x)) - P(x, f(x)))dx$$

  where in the last step we used the fundamental theorem of calculus.
- Comparing the two calculations we see that

$$\oint_C P dx = -\iint_D P_y \, dA.$$

- If D is a type II region, then a similar proof shows that

$$\int_C Q dy = \iint_D Q_x dA$$

Example 2.

- Use Green's Theorem to evaluate $I = \oint_C x^4 dx + xydy$ where C is the triangle with vertices $(0,0),(1,0),(0,1)$ oriented counter-clockwise.
- The interior of C is the domain D, bounded by the x and y-axes and by the line $y = 1 - x$.
- This domain is a type I region:

$$D = (x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1 - x$$

- In this example $P = x^4$ and $Q = xy$. Therefore the curl is

$$Q_x - P_y = y.$$

- Applying Green's Theorem gives

$$\begin{aligned} I = \iint_D ydA &= \int_{x=0}^{x=1} \int_{y=0}^{y=1-x} ydydx \\ &= \int_0^1 \left[\frac{y^2}{2}\right]_0^{1-x} \\ &= \frac{1}{2}\int_0^1 (1-x)^2 dx = -\frac{1}{6}(1-x)^3\Big|_0^1 = \frac{1}{6} \end{aligned}$$

Example 3

- Evaluate $I = \oint_C \left(3y - e^{\sin x}\right) dx + \left(7x + \sqrt{1 + y^4}\right) dy$ where C is the circle $x^2 + y^2 = 9$ oriented counter-clockwise.
- Note: the line integral in Example 1 could be evaluated with some effort. However, the line integral is next to impossible to evaluate directly.
- Instead, we have $Q_x - P_y = 7 - 3 = 4$.
- Hence, by Green's theorem:

$$I = \iint_D 4dA$$

  where D is the disk $x^2 + y^2 \leq 9$.
- Therefore, by the $A = \pi r^2$ formula for the area of a disk, we get

$$I = 4\pi \times 3^2 = 36\pi.$$

Proof of Green's Theorem for general domains

- The proof that
$$\int_C P dx = -\iint_D P_y dA$$
where D is not a type I domain requies that we draw a finite number of auxilliary vertical segments that subdivide D into type I subregions. This is accomplished by drawing one or more vertical lines at the points of C where the slope is vertical or undefined.
- We then subdivide the boundary as $C = C_1 + C_2 + \cdots + C_n$ where each C_i is a closed curve that encloses a type I region D_i with $D = D_1 \cup \cdots \cup D_n$. The integrals along the supporting verticals are repeated twice with opposite orientations and so cancel.
- Likewise, the general proof that
$$\int_C Q dx = \iint_D Q_x dA$$
requires the division of D into type II regions by drawing auxilliary horizontal segments.

Example 4

- Use Green's Theorem to evaluate $I = \oint_C y dx$ where C is the left-side boundary of the annulus bounded by $x^2 + y^2 = 4$ and $x^2 + y^2 = 1$.
- Here $P = y, Q = 0$; so $Q_x - P_y = -1$. Hence,

$$I = -\iint_D dA = -\mathrm{Area}(D).$$

- The area of the annulus is the area of the large disk minus the area of the smaller disk. Therefore, $I = -4\pi + \pi = -3\pi$.
- Let us evaluate I directly. Here $C = C_2 - C_1$ where C_2 is the outer circle and C_1 is the inner circle, both oriented counter-clockwise.
- We parameterize C_2 as $(x, y) = (2\cos t, 2\sin t)$, $0 \leq t \leq 2\pi$. Hence

$$dx = -2\sin t dt, \; y dx = -4\sin^2 t dt = -2(1 - \cos(2t))dt$$

$$\int_{C_2} y dx = \int_0^{2\pi} -2(1 - \cos(2t))dt = -4\pi$$

- We parameter C_1 as $(x, y) = (\cos t, \sin t)$, $0 \leq t \leq 2\pi$. Hence

$$\int_{C_2} y dx = \int_0^{2\pi} -\frac{1}{2}(1 - \cos(2t))dt = -\pi$$

- Putting it all together, $\displaystyle\int_C y dx = \int_{C_1} y dx - \int_{C_2} y dx = -3\pi$.