Lecture 10
Curl

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Outline

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2D Curl

- Let $\mathbf{F} = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ be a 2D vector field. The quantity $Q_x - P_y$ is called the curl of $\mathbf{F}$.
- We may therefore restate Green's Theorem as

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D \operatorname{curl} \mathbf{F} \, dA.$$

- Physically, circulation is the amount of "pushing" force along a path. Curl is the amount of "twisting" force present at any given point, or the circulation when we shrink the path down to a single point.
- Thus, intuitively, Green's Theorem asserts that the total circulation of vector field $\mathbf{F}$ around a closed curve C is the total amount of curl (twisting) produced by $\mathbf{F}$ at the points located within C.

Irrotational 2D vector fields

- A 2D vector field $\mathbf{F} = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ has zero curl if $Q_x = P_y$. Such a vector field is called **irrotational**.
- **Theorem.** A conservative vector field is necessarily irrotational.
- Proof. A conservative v.f. has the form $\mathbf{F} = \nabla f = f_x\mathbf{i} + f_y\mathbf{j}$. Calculating the curl gives $\operatorname{curl}\mathbf{F} = f_{xy} - f_{yx} = 0$.
- **Theorem.** Let $\mathbf{F}$ be an irrotational 2D vector field. If $\mathbf{F}$ is defined on a simply connected domain, then $\mathbf{F}$ is conservative.
- Proof. Let $C \subset \text{dom}(\mathbf{F})$ be a simple closed curved, and let D be the interior of C. Since $\text{dom}(\mathbf{F})$ is simply connected, we have $D \subset \text{dom}(\mathbf{F})$. Hence, $Q_x = P_y$ for all points in D. Hence, by Green's Theorem, $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$. In the preceding lecture, we showed that if the circulation of $\mathbf{F}$ around every simple closed curve is zero, then $\mathbf{F}$ must be conservative.
- Consequently, if $\mathbf{F}$ is not irrotational, then the integral $\int \mathbf{F} \cdot d\mathbf{r}$ is **path dependent**. The converse is not true; the condition $Q_x = P_y$ **does not guarantee** path independence if $\text{dom}(\mathbf{F})$ is not simply connected.

Example 1

- Claim: the vector field $\mathbf{F} = (x - y)\mathbf{i} + (x - 2)\mathbf{j}$ is not conservative.
- Here $P = x - y,\ Q = x - 2$ and hence $P_y = -1, Q_x = 1$ are not equal. Therefore $\mathbf{F}$ is not irrotational, and therefore not conservative.
- We can also show that $\mathbf{F}$ fails to be conservative by demonstrating that $I_C = \int_C \mathbf{F} \cdot d\mathbf{r}$ is path dependent.
- Consider the curve C_1 defined by $(x, y) = (t, t),\ 0 \leq t \leq 1$.

$$I_{C_1} = \int_0^1 0dt + (t-2)dt = t^2 - 2t\Big|_0^1 = -1.$$

- Consider the curve C_2 defined by $(x, y) = (t, t^2),\ 0 \leq t \leq 1$.

$$I_{C_2} = \int_0^1 (t - t^2)dt + \int_0^1 (t-2)2tdt = \int_0^1 -2 + 2t - t^2$$
$$= \Big[-2t + t^2 - \frac{t^3}{3}\Big]_0^1 = -2 + 1 - \frac{1}{3} = -\frac{4}{3}$$

- Both curves have endpoints $(0, 0)$ and $(1, 1)$. However $I_{C_1} \neq I_{C_2}$.

Example 2

- Claim: the vector field $\mathbf{F} = (3 + 2xy)\mathbf{i} + (x^2 - 3y^2)\mathbf{j}$ is conservative.
- Here $P = 3 + 2xy,\ Q = x^2 - 3y^2$. Observe that $Q_x = 2x,\ P_y = 2x$ are equal. Hence, $\mathbf{F}$ is irrotational.
- There are no singularities, so $\text{dom}(\mathbf{F}) = \mathbb{R}^2$ is simply connected. Therefore, $\mathbf{F}$ is conservative.
- We will now determine the $f(x, y)$ such that $\mathbf{F} = \nabla f$.
- Since $P = f_x$ we begin by integrating $P(x, y)$ with respect to x. This allows us to determine $f(x, y)\pm$ a function of y.

$$f(x, y) = \int^x (3 + 2xy)dx = 3x + x^2y + g(y).$$

- It follows that $f_y = 2x^2 + g'(y)$. Hence,

$$g'(y) = -3y^2, \quad g(y) = -y^3 + C.$$

- Therefore, $f(x, y) = 3x^2 + 2x^2y - y^3 + C$, where C is a constant of integration.

Example 2 cont.

- Evaluate $I_C = \int_C (3 + 2xy)dx + (x^2 - 3y^2)dy$ where C is the curve given by $\mathbf{r}(t) = e^t \sin(t)\,\mathbf{i} + e^t \cos(t)\,\mathbf{j},\ 0 \leq t \leq \pi$
- In the preceding slide we showed the corresponding vector field $\mathbf{F} = (3 + 2xy)\mathbf{i} + (x^2 - 3y^2)\mathbf{j}$ is conservative with $\mathbf{F} = \nabla f$ where $f(x, y) = 3x^2 + 2x^2y - y^3 + C$.
- Observe that $\mathbf{r}(0) = (0, 1),\ \mathbf{r}(\pi) = (0, -e^\pi)$.
- Thus, by the fundamental theorem of line integrals,

$$I_C = f(0, -e^\pi) - f(0, 1) = e^{3\pi} + C - (-1 + C) = 1 + e^{3\pi}.$$

Example 3.

- Consider $I = \int \frac{-ydx + xdy}{x^2 + y^2}$
- Claim: the corresponding vector field is irrotational.

$$P_y = \frac{-(x^2+y^2) + 2y^2}{(x^2+y^2)^2} = \frac{y^2 - x^2}{(x^2+y^2)^2}$$

$$Q_x = \frac{(x^2+y^2) - 2x^2}{(x^2+y^2)^2} = P_y$$

- Claim: the above integral is path dependent.
- Let C be the unit circle: $x = \cos t,\ y = \sin t,\ 0 \leq t \leq 2\pi$.
- Thus, $dx = -\sin t\, dt,\ dy = \cos t\, dt$.

$$\frac{-ydx + xdy}{x^2 + y^2} = \frac{\sin^2 tdt + \cos^2 tdt}{\sin^2 t + \cos^2 t} = dt, \quad I_C = \int_0^{2\pi} dt = 2\pi \neq 0.$$

Example 3 cont.

- On the previous slide we showed that the vector field

$$\mathbf{F} = \frac{-y\mathbf{i}}{x^2+y^2} + \frac{x\mathbf{j}}{x^2+y^2}$$

  is path dependent, despite being irrotational.
- Let $f(x, y) = \tan^{-1}(y/x),\ x > 0$ and observe that

$$f_x = \frac{1}{1+y^2/x^2}\left(-\frac{y}{x^2}\right) = \frac{-y}{x^2+y^2}$$
$$f_y = \frac{1}{1+y^2/x^2}\left(\frac{1}{x}\right) = \frac{x}{x^2+y^2}$$

  Therefore, $\mathbf{F} = \nabla f$.
- We just showed that $\int \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ is path dependent, and therefore not conservative. However, there is no contradiction, because $f(x, y)$ cannot be continuously defined on the same domain as $\mathbf{F}(x, y)$.
- Here dom($\mathbf{F}$) is not simply connected, because it does not include $(0, 0)$. This example illustrates that an if an irrotational vector field has singularities, then dom($\mathbf{F}$) is not simply connected. Therefore, such an $\mathbf{F}$ may fail to be conservative.

3D Curl

- Let $\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$ be a 3D vector field. We define

$$\operatorname{curl} \mathbf{F} = (R_y - Q_z)\mathbf{i} + (P_z - R_x)\mathbf{j} + (Q_x - P_y)\mathbf{k}$$

- Recall the definition of the cross product. For 3D vectors $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ we have

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

- Its convenient to express the gradient operator as $\nabla = D_x\mathbf{i} + D_y\mathbf{j} + D_z\mathbf{k}$, so that $\nabla f = f_x\mathbf{i} + f_y\mathbf{j} + f_z\mathbf{k}$.
- We can now express the curl operator as

$$\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ D_x & D_y & D_z \\ P & Q & R \end{vmatrix},$$

which agrees with our original definition.

2D curl vs 3D curl

- A 2D vector field may always be regarded as a 3D vector field by writing $\mathbf{F}(x, y, z) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$.
- Applying the definition of curl, we have

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ D_x & D_y & D_z \\ P & Q & 0 \end{vmatrix} = (Q_x - P_y)\mathbf{k}$$

- The name curl comes from the physics of fluid flow. The flow of a liquid in a shallow bed is goverened by a 2D vector field $P\mathbf{i} + Q\mathbf{j}$ which represents the velocity of the fluid.
- Imagine putting a small vertically oriented turbine at each point of the flow. A small difference in the flow at different points of the turbine will cause it to rotate with angular velocity $Q_x - P_y$ and with an axis of rotation parallel to $\mathbf{k}$.
- Thus 2D curl $Q_x - P_y$ is merely the magnitude of the 3D curl $(Q_x - P_y)\mathbf{k}$; the two concepts are largely equivalent.

Conservative implies irrotational

- **Theorem 1.** Let $f = f(x, y, z)$ be a function of 3 variables. Then, $\text{curl}\,\nabla f = 0$. In other words, a conservative v.f. is irrotational.
- It is tempting to rewrite the above identity ans $\nabla \times \nabla f$ and argue that $\nabla \times \nabla = 0$. While, this is a useful mnemonique, it is not a valid proof.
- Proof. We apply the definition of ∇f and of curl to obtain

$$\begin{aligned} \text{curl}\,\nabla f &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ D_x & D_y & D_z \\ f_x & f_y & f_z \end{vmatrix} \\ &= (f_{zy} - f_{yz})\mathbf{i} + (f_{xz} - f_{zx})\mathbf{j} + (f_{yx} - f_{xy})\mathbf{k} = 0 \end{aligned}$$

- Theorem 1 provides test for conservative vector fields, in the sense that if $\text{curl}\,\mathbf{F} \neq 0$, then $\mathbf{F}$ isnt conservative. Under certain conditions, it is possible to strengthen the test for a conservative vector field.
- **Theorem 2.** Suppose that $\mathbf{F}$ is irrotational; i.e., $\text{curl}\,\mathbf{F} = 0$. Also suppose that $\mathbf{F}$ and the partial derivatives are defined on all of $\mathbb{R}^3$, without singularities. Then, $\mathbf{F}$ is conservative; i.e. $\mathbf{F} = \nabla f$ for some $f(x, y, z)$.

Additional remarks

- In 3D space, the interpretation of curl is similar to the 2D interpretation. The difference is that instead of a vertically oriented turbine we imagine a small sphere that is free to rotate about a fixed point.
- A non-uniform flow will cause the sphere to spin with the direction of curl F giving the axis of rotation and with the magnitude of curl F giving angular velocity. Follow this curl idea link for additional explanations and visualizations.
- To put it another way, curl is the circulation per unit area, circulation density, or rate of rotation — the amount of twisting at a single point. In 3D space that circulation depends on orientation of the plane. and that's why 3D curl is a vector quantity. The following article elaborates on the physical intuition behind 2D and 3D curl.

Example 4

- Calculate the curl of $\mathbf{F} = xz\mathbf{i} + xyz\mathbf{j} - y^2\mathbf{k}$.
- We have

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ D_x & D_y & D_z \\ xz & xyz & -y^2 \end{vmatrix} = (-xy - 2y)\mathbf{i} + x\mathbf{j} + yz\mathbf{k}.$$

- Since the given vector field has non-vanishing curl, it is not conservative.

Example 5

- Show that $\mathbf{F} = y^2z^3\mathbf{i} + 2xyz^3\mathbf{j} + 3xy^2z^2\mathbf{k}$ is a conservative v.f.
- There are no singularities. Enough to establish that $\mathbf{F}$ is irrotational.

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ D_x & D_y & D_z \\ y^2z^3 & 2xyz^3 & 3xy^2z^2 \end{vmatrix}$$
$$= (6xyz^2 - 6xyz^2)\mathbf{i} + (3y^2z^2 - 3y^2z^2)\mathbf{j} + (2yz^3 - 2yz^3)\mathbf{k} = 0$$

- Find the potential function $f(x, y, z)$ such that $\mathbf{F} = \nabla f$.
- Since $f_x = P, f_y = Q, f_z = R$, we have

$$f = \int y^2z^3dx = xy^2z^3 + g(y, z)$$

- We now apply the condition $f_y = Q$ to obtain

$$f_y = 2xyz^3 + g_y = 2xyz^3.$$

  Hence, $g_y = 0$ and $g(y, z) = h(z)$ is a function of z only.
- Finally, we apply the condition $f_z = R$ to obtain

$$f_z = 3xy^2z^2 + h'(z) = 3xy^2z^2.$$

  Hence, $h'(z) = 0$ and therefore $f = xy^2z^3 + C$, where C is a constant of integration.