Lecture 11
Divergence

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Outline

- Text: section 16.5
- Definition
- Example
- Incompressible flow
- Laplacian
- The product rule

Definition of divergence

- Divergence transforms a vector field into a function.
- The divergence of a 3D vector field

$$\mathbf{F} = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$$

  is defined as

$$\operatorname{div} \mathbf{F} = P_x + Q_y + R_z.$$

- Using the notation $\nabla = D_x\mathbf{i} + D_y\mathbf{j} + D_z\mathbf{k}$, the divergence operat may be given symbolically as

$$\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F}$$

- Example. If $\mathbf{F} = xz\mathbf{i} + xyz\mathbf{j} - y^2\mathbf{k}$, then

$$\operatorname{div} F = D_x(xz) + D_y(xyz) - D_z(y^2) = z + xz$$

Incompressible vector fields

- If **F** represents the velocity of a fluid flow, then div $F(x, y, z)$ measures the tendency of the fluid to compress (positive divergence) or rareify (negative divergence) at the point (x, y, z).
- For this reason, a vector feild such that $\operatorname{div} \mathbf{F} = 0$ is called **incompressible**.
- If P, Q, R are constant, then **F** is incompressible. However, the class of incompressible vector fields is much more general.
- **Theorem.** The curl of a given vector field is incompressible. Formally, $\operatorname{div} \operatorname{curl} \mathbf{F} = 0$.
- Proof. Let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$. Recall that

  $$\operatorname{curl} \mathbf{F} = (R_y - Q_z)\mathbf{i} + (P_z - Q_x)\mathbf{j} + (Q_x - P_y)\mathbf{k}$$

  It follows that

  $$\operatorname{div} \operatorname{curl} \mathbf{F} = D_x(R_y - Q_z) + D_y(P_z - Q_x) + D_z(Q_x - P_y) = 0$$

  by Clairault's Theorem.

Test for curl

- **Theorem.** The curl of a given vector field is incompressible. Formally, $\operatorname{div} \operatorname{curl} \mathbf{F} = 0$.
- The above theorem furnishes a useful test: if $\mathbf{F}$ is not incompressible, then it cannot be the curl of another vector field.
- Example. Let $\mathbf{F} = xz\mathbf{i} + xyz\mathbf{j} - y^2\mathbf{k}$
- Observe that $\operatorname{div} \mathbf{F} = z + xz \neq 0$.
- Since $\mathbf{F}$ is not incompressible, there does not exists a vector field $\mathbf{G}$ such that $\mathbf{F} = \operatorname{curl} \mathbf{G}$.

The Laplacian

- The gradient, the curl, and divergence are all important first order operators.
- The gradient transforms a function f into a vector field ∇f.
- The curl transforms a vector field $\mathbf{F}$ into another vector field $\nabla \times \mathbf{F}$.
- The divergence transforms a vector field $\mathbf{F}$ into a function $\nabla \cdot \mathbf{F}$.
- We now introduce the Laplacian, a second order operator that transforms a function f into another function $\nabla^2 f$.
- Here ∇^2 means $\nabla \cdot \nabla$. Thus,

$$\nabla^2 f = \nabla \cdot (f_x \mathbf{i} + f_y \mathbf{j} + f_z \mathbf{k}) = f_{xx} + f_{yy} + f_{zz}.$$

The product rule

- The divergence and curl operators obey a number of identities. Here we describe and prove the product rule for these operators.
- Identity 1. Let f be a function and $\mathbf{F}$ a vector field. Then,

$$\nabla \cdot (f\mathbf{F}) = f\nabla \cdot \mathbf{F} + \mathbf{F} \cdot \nabla f.$$

- Proof. Write $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$. Then, $f\mathbf{F} = (fP)\mathbf{i} + (fQ)\mathbf{j} + (fR)\mathbf{k}$. It follows that

$$\begin{aligned}\nabla \cdot (f\mathbf{F}) &= D_x(fP) + D_y(fQ) + D_z(fR)\\ &= (f_x P + f_y Q + f_z R) + f(P_x + Q_y + R_z)\\ &= \nabla f \cdot \mathbf{F} + f\nabla \cdot \mathbf{F}\end{aligned}$$

- Identity 2. We have $\nabla \times (f\mathbf{F}) = f\nabla \times \mathbf{F} + \nabla f \times \mathbf{F}$.
- Proof. By the definition of curl,

$$\nabla \times (fF) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ D_x & D_y & D_z \\ fP & fQ & fR \end{vmatrix} = f\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ D_x & D_y & D_z \\ P & Q & R \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ f_x & f_y & f_z \\ P & Q & R \end{vmatrix}$$