Lecture 12
Parametric Surfaces

R. Milson
Math 2002, Winter 2020

Outline

- Text: section 16.6
- Review: parameterized curves
- Definition
- Example
- Normal vector and tangent plane
- Surface area
- Example

Parameterization

- A curve embedded in the 2D Euclidean plane is described by an equation in 2 variables $F(x, y) = 0$.
- Similarly, a surface embedded in 3D Euclidean space is described by an equation in 3 variables: $F(x, y, z) = 0$
- A parameterization of a curve is a 2-vector valued function of one variable
$$(x, y) = \mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$$
such that $F(f(t), g(t)) = 0$
- Analogously, a parameterization of a surface is a 3-vector valued function of **two** variables
$$(x, y, z) = \mathbf{r}(u, v) = f(u, v)\mathbf{i} + g(u, v)\mathbf{j} + h(u, v)\mathbf{k}$$
such that $F(f(u, v), g(u, v), h(u, v)) = 0$.

Examples

- A parametrized curve is commonly interpreted as a particle moving along the curve.
- By contrast, a parameterized surface may be regarded as a grid imposed on the surface. The grid lines are curves obtained by setting one of the parameters u and v to a.
- The unit circle $x^2 + y^2 = 1$ can parameterized as $(x, y) = (\cos(t), \sin(t))$. This represents a particle moving counter-clockwise around the circle with unit speed.
- The 2-sphere $x^2 + y^2 + z^2 = 1$ may be parameterized using spherical coordinates:

$$(x, y, z) = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$$

- The lines of lattitude are the curves $\phi = \phi_0$ The meridians (lines of longitude) are the curves $\theta = \theta_0$. 3D demonstration

Example 2

- Identify the parameterized surface

$$\mathbf{r}(u, v) = 2\cos u\mathbf{i} + v\mathbf{j} + 2\sin u\mathbf{k}.$$

- Rewrite the parameterization as

$$x = 2\cos u, \quad y = v, \quad z = 2\sin u,$$

  and eliminate u and v to obtain an equation in x, y, z.
- Since $y = v$, we can eliminate v making the substitution $v \to y$.
- To eliminate u, we square and add the 1st and 3rd equations. The result is $x^2 + z^2 = 4$. There is no v in this equation, so we are done.
- The equation $x^2 + z^2 = 4$ represents a cylinder of radius 2 in the direction of the y-axis. (Demonstration)

Example 3

- Use polar coordinates to parameterize the half-cone surface $z = 2\sqrt{x^2 + y^2}$
- Making the substitutions $x = r\cos\theta, y = r\sin\theta$, and setting $r = u, \theta = v$ we obtain the parameterization

$$\mathbf{r}(u, v) = (u\cos v, u\sin v, 2u).$$

(Demonstration)

Graph parameterizations

- One way to parameterize surface with equation $F(x, y, z) = 0$ is to solve for one of the variables, say $z = f(x, y)$. The resulting graph represets a subset of the surface in question.
- Example. Consider the unit sphere $x^2 + y^2 + z^2 = 1$.
- Solving for z we obtain

$$\mathbf{r}(u, v) = (u, v, \sqrt{1 - u^2 - v^2}).$$

- That the above parameterization gives only the upper hemisphere. If we wanted to represent the lower hemisphere, we would us $z = -\sqrt{1 - x^2 - y^2}$.
- Note that the grid corresponding to this parameterization is very different from the spherical coordinate grid. (Demonstration)

The tangent plane

- Recall that the tangent plane is the best approximating plane to a surface at a fixed point of that surface. A surface at high magnification resembles its tangent plane.
- To calculate the tangent plane to a parametric surface $(x, y, z) = \mathbf{r}(u, v)$, we use the fact that the vectors

$$\mathbf{r}_u = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} + \frac{\partial z}{\partial u}\mathbf{k}, \quad \mathbf{r}_v = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j} + \frac{\partial z}{\partial v}\mathbf{k}$$

- This is true because the gridline curve $U(t) = \mathbf{r}(t, v_0)$ is trapped inside the surface. Hence, the velocity vector $U'(t_0) = \mathbf{r}_u(t_0, v_0)$ is tangent to the surface. Similarly, $\mathbf{r}_v(u_0, t_0)$ is a tangent vector because it is the velocity of the gridline curve $V(t) = \mathbf{r}(u_0, t)$.
- Therefore, the equation of the tangent plane at the point $(x_0, y_0, z_0) = \mathbf{r}(u_0, v_0)$ is given by

$$((X, Y, Z) - \mathbf{r}(u_0, v_0)) \cdot \mathbf{N}(u_0, v_0) = 0,$$

where $\mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v$ is a normal vector of the given surface.

Example 4

- Find the equation of the tangent planes to the unit sphere $x^2 + y^2 + z^2 = 1$ using spherical coordinates.
- We are parameterizing the sphere S as

$$\mathbf{r}(\phi, \theta) = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi), \quad 0 \le \phi \le \pi,\ 0 \le \theta \le 2\pi,$$
$$\mathbf{r}_\phi = \cos\phi\cos\theta\,\mathbf{i} + \cos\phi\sin\theta\,\mathbf{j} - \sin\phi\,\mathbf{k}$$
$$\mathbf{r}_\theta = -\sin\phi\sin\theta\,\mathbf{i} + \sin\phi\cos\theta\,\mathbf{j}$$

- Taking the cross product gives

$$\mathbf{N} = \mathbf{r}_\phi \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\phi\cos\theta & \cos\phi\sin\theta & -\sin\phi \\ -\sin\phi\sin\theta & \sin\phi\cos\theta & 0 \end{vmatrix}$$
$$= \sin^2\phi\cos\theta\,\mathbf{i} + \sin^2\phi\sin\theta\,\mathbf{j} + \cos\phi\sin\phi\,\mathbf{k}$$
$$= \sin\phi\,\mathbf{r}(\phi, \theta)$$

- This makes sense geometrically, because on a sphere the normal at x, y, z points in the same direction as (x, y, z)
- Therefore, the tangent plane at $(x_0, y_0, z_0) \in S$ has the equation

$$x_0(X - x_0) + y_0(Y - y_0) + z_0(Z - z_0) = 0.$$

Surface area of a graph

- Recall the formula for surface area of a graph $z = f(x, y)$ raised above a region D in the xy plane:

$$A = \iint_D \sqrt{1 + f_x^2 + f_y^2}\, dA.$$

- The above integrand represents the area of a small parallelogram at position $(x, y, f(x, y))$ lying above a small rectangle in the xy plane.
- The unit vector $\mathbf{i}$ lifts to the vector $\mathbf{i} + f_x\mathbf{k}$, while the unit vector $\mathbf{j}$ lifts to the vector $\mathbf{j} + f_y\mathbf{k}$.
- Therefore, the rectangle spanned by $\Delta x\mathbf{i}$ and $\Delta y\mathbf{j}$ lifts to the parallelogram spanned by $\Delta x(\mathbf{i} + f_x\mathbf{k})$ and $\Delta y(\mathbf{j} + f_y\mathbf{k})$.
- The area of a parallelogram is equal to the magnitude of the cross product of the generating vectors. For the parallelogram in question,

$$(\mathbf{i} + f_x\mathbf{k}) \times (\mathbf{j} + f_y\mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix} = (-f_x\mathbf{i} - f_y\mathbf{j} + \mathbf{k})$$

$$|(\mathbf{i} + f_x\mathbf{k}) \times (\mathbf{j} + f_y\mathbf{k})| = \sqrt{1 + f_x^2 + f_y^2}.$$

- Therefore, the area of each small parallelogram is $\sqrt{1 + f_x^2 + f_y^2}\Delta A$ where $\Delta A = \Delta x \Delta y$. In the limit, as $\Delta x, \Delta y \to 0$ we obtain the above integral.

Surface area of a parameterized surface

- We generalize the preceding formula to an arbitrary parameterized surface $(x, y, z) = \mathbf{r}(u, v),\ (u, v) \in D$.
- The parameterization maps a small square with sides $\Delta u\mathbf{i}$ and $\Delta v\mathbf{j}$ to a parallelogram generated by the vectors $\Delta u\mathbf{r}_u$ and $\Delta v\mathbf{r}_v$. This parallelogram has area $|\mathbf{r}_u \times \mathbf{r}_v|\Delta A$ where $\Delta A = \Delta u\Delta v$.
- Taking the limit as $\Delta u, \Delta v \to 0$ we obtain the following formula for the area of the parameterized surface:

$$A = \iint_D |\mathbf{r}_u \times \mathbf{r}_v| dA(u, v).$$

Example 4 cont.

- Let's apply the formula $A = \iint_D |\mathbf{r}_u \times \mathbf{r}_v| dA(u, v)$ to calculate the area of a sphere of radius R by using the spherical coordinate parameterization.

$$\mathbf{r}(\phi, \theta) = R(\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi), \quad 0 \le \phi \le \pi,\ 0 \le \theta \le 2\pi,$$

- In a preceding slide, we calculated

$$\mathbf{N} = \mathbf{r}_\phi \times \mathbf{r}_\theta = R^2 \sin\phi\,(\sin\phi\cos\theta\,\mathbf{i} + \sin\phi\sin\theta\,\mathbf{j} + \cos\phi\,\mathbf{k}).$$

- It follows that

$$\begin{aligned} |\mathbf{N}| &= R^2 \sin\phi \\ A &= R^2 \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi} \sin\phi\, d\phi\, d\theta \\ &= 2\pi R^2 \times \Big[-\cos\phi \Big]_0^{\pi} = 4\pi R^2 \end{aligned}$$