Lecture 13
Surface and Flux Integrals

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Math 2002, Winter 2020

Outline

- Text: section 16.7
- Unoriented surface integrals
- Example
- Graph parameterization
- The normal vector and orientation
- Flux integrals
- Examples
- Flux integrals using graph parameterization

Unoriented surface integrals

- A surface S is a 2-dimensional subset of 3-dimensional space, that can be parameterized using 3 functions of 2 variables:

$$\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}, \quad (u,v) \in D.$$

- An unoriented surface integral

$$I = \iint_S f(x,y,z)dS$$

is a weighted generalization of the surface area integral $\iint_S dS$.

- Indeed, if we intepret $f(x,y,z)$ as a density function, then I represents the mass of the 2-dimensional object in question.
- In the previous lecture, we demonstrated that

$$dS = |\mathbf{r}_u \times \mathbf{r}_v| \mathrm{d}A(u,v).$$

Therefore, $I = \displaystyle\iint_D f(x(u,v), y(u,v), z(u,v)) |\mathbf{r}_u \times \mathbf{r}_v| \mathrm{d}A(u,v).$

Example 1.

- Evaluate $I = \iint_S x^2 dS$ where S is the unit sphere.
- We will first evaluate I using spherical coordinates:

$$\mathbf{r}(\phi, \theta) = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi), \quad 0 \le \phi \le \pi,\ 0 \le \theta \le 2\pi$$

- In the last lecture, we showed that $dS = \sin\phi\,\mathrm{d}A(\phi, \theta)$.
- Since $x^2 = \sin^2\phi\cos^2\theta$, we have

$$I = \int_0^{\theta=2\pi}\int_0^{\phi=\pi} \sin^3\phi\cos^2\theta d\phi d\theta = \int_0^{\pi}\sin^3\phi d\phi \int_0^{2\pi}\cos^2\theta d\theta$$

$$\int_0^{\pi}\sin\phi(1-\cos^2\phi)d\phi = \Big[-\cos\phi + \frac{1}{3}\cos^3\phi\Big]_0^{\pi} = \frac{4}{3}$$

$$\int_0^{2\pi}\cos^2\theta d\theta = \Big[\frac{1}{4}\cos 2t + \frac{t}{2}\Big]_0^{2\pi} = \pi$$

$$I = \frac{4}{3}\pi.$$

Graph parameterization

- Consider a surface S given as the graph of $z = F(x, y),\ (x, y) \in D$.
- We introduce the graph parameterization

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + f(x, y)\mathbf{k}$$

- Recall that the surface element is given by $dS = |\mathbf{r}_x \times \mathbf{r}_v| \mathrm{d}A$.
- Calculating further,

$$\begin{aligned}
\mathbf{r}_x &= \mathbf{i} + F_x\mathbf{k} \\
\mathbf{r}_y &= \mathbf{j} + F_y\mathbf{k} \\
\mathbf{r}_x \times \mathbf{r}_y &= -F_x\mathbf{i} - F_y\mathbf{j} + \mathbf{k} \\
dS &= |\mathbf{r}_x \times \mathbf{r}_y| \mathrm{d}A = \sqrt{1 + F_x^2 + F_y^2}\mathrm{d}A
\end{aligned}$$

$$I = \iint_S f(x, y, z)dS = \iint_D f(x, y, F(x, y))\sqrt{1 + F_x^2 + F_y^2}\,\mathrm{d}A.$$

Example 1 cont.

- Evaluate $I = \iint_S x^2 dS$ where S is the unit sphere using the graph parameterization $z = \pm\sqrt{1 - x^2 - y^2}$ where $x^2 + y^2 \le 1$.
- Observe that $F_x = \frac{\partial z}{\partial x} = -\frac{x}{z}, \quad F_y = \frac{\partial z}{\partial x} = -\frac{y}{z}$.
- Thus, for the upper hemi-sphere S_+, we have

$$I_+ = \iint_{S_+} = \iint_D x^2 \sqrt{1 + \frac{x^2}{z^2} + \frac{y^2}{z^2}} \mathrm{d}A = \iint_D \frac{x^2}{z} \mathrm{d}A$$

where D is the unit disk $x^2 + y^2 \le 1$ and where $z = \sqrt{1 - x^2 - y^2}$.
- Switching to polar coordinates, $I_+ = \int_0^{2\pi} \cos^2\theta d\theta \int_{r=0}^{r=1} \frac{r^3}{\sqrt{1-r^2}} dr$
- Making the substitition, $s = \sqrt{1 - r^2}$, $ds = -\frac{r}{\sqrt{1-r^2}} dr$, we have

$$\int_{r=0}^{r=1} \frac{r^3}{\sqrt{1-r^2}} dr = -\int_{s=1}^{s=0} (1 - s^2) ds = \frac{2}{3}$$

- Therefore, $I_+ = \frac{2}{3}\pi$. By symmetry, $I = 2I_+ = \frac{4}{3}\pi$, in agreement with the preceding calculation.

Example 2

- Evaluate $I = \iint_S ydS$ where S is the graph of $z = x + y^2,\ 0 \leq x \leq 1,\ 0 \leq y \leq 2.$
- Letting $f(x, y) = x + y^2$, we have

$$f_x = 1 \quad f_y = 2y$$

$$\mathrm{d}A = \sqrt{1 + f_x^2 + f_y^2} = \sqrt{2 + 4y^2} = \sqrt{2}\sqrt{1 + 2y^2}$$

- Therefore,

$$\begin{aligned} I &= \sqrt{2} \int_{x=0}^{x=1} \int_{y=0}^{y=2} y\sqrt{1 + 2y^2} dydx \\ &= \sqrt{2} \times \frac{1}{6} (1 + 2y^2)^{3/2} \Big|_0^2 \\ &= \sqrt{2} \times \frac{27 - 1}{6} = \sqrt{2} \times \frac{13}{3} \end{aligned}$$

Orientation

- Let $\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}$ be a parameterized surface. Recall that $\mathbf{N}(u,v) = \mathbf{r}_u \times \mathbf{r}_v$ is a normal vector to the surface at $\mathbf{r}(u,v)$.
- The unit normal vector at $\mathbf{r}(u,v)$ is

$$\mathbf{n}(u,v) = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|},$$

  which has the same direction as $\mathbf{N}$ but is normalized so that $|\mathbf{n}| = 1$.
- By exchanging the order of u and v we obtain the vector

$$\frac{\mathbf{r}_v \times \mathbf{r}_u}{|\mathbf{r}_u \times \mathbf{r}_v|} = -\mathbf{n}(u,v),$$

  which is also a unit normal.
- Indeed every point on the surface has two unit normals, for the reason that a surface has 2 sides. A choice of unit normal amounts to a choice of orientation of the surface.
- Reversing the order of u and v reverses the sign of the unit normal, and thereby reverses the orientation.

Flux integrals

- Let $\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$ be a vector field on a surface S. The integral $\iint_S \mathbf{F} \cdot \mathbf{n}\, dS$ is called a flux integral, because it measure the flow (flux) of $\mathbf{F}$ across S.
- Whether the flow is accounted as positive or negative depends on the choice of orientation of S.
- Let $\mathbf{r}(u, v),\ (u, v) \in D$ be a parameterization of S. Recall that

$$dS = |\mathbf{r}_u \times \mathbf{r}_v| \mathrm{d}A \qquad \mathbf{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}$$

  It follows that

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, \mathrm{d}S = \iint_D \mathbf{F}(r(u, v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, \mathrm{d}A.$$

- In working with flux integrals, it is convenient to define $\mathrm{d}\mathbf{S} = (\mathbf{r}_u \times \mathbf{r}_v)\mathrm{d}A$, which allows us to express a flux integral as

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, \mathrm{d}S = \iint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S}.$$

Example

- Calculate the flux of $\mathbf{F}(x, y, z) = z\mathbf{i} + y\mathbf{j} + x\mathbf{k}$ across the unit sphere $x^2 + y^2 + z^2 = 1$.
- As before, we use the spherical coordinate parameterization

$$\mathbf{r}(\phi, \theta) = \sin\phi\cos\theta\mathbf{i} + \sin\phi\sin\theta\mathbf{j} + \cos\phi\mathbf{k}, \quad 0 \leq \phi \leq \pi,\ 0 \leq \theta \leq 2\pi$$

- In Lecture 12, we showed $\mathbf{n} = \sin\phi\,\mathbf{r}(\phi, \theta)$, and $\mathrm{d}S = \sin\phi dA$. Note that, $\mathbf{n}(\pi/2, 0) = \mathbf{i}$, which is an outward pointing normal. This means that we are counting outward flux as positive. If we re-ordered the parameters as θ, ϕ, then the unit normal would point inside the sphere, and the outward flux would count as negative.
- Putting it all together,

$$\mathbf{F} \cdot \mathbf{r} = (z\mathbf{i} + y\mathbf{j} + x\mathbf{k}) \cdot (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = 2xz + y^2$$

$$\iint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \iint_S \sin\phi(2\sin\phi\cos\phi\cos\theta + \sin^2\phi\sin^2\theta)dA$$

$$= 2\int_{\theta=0}^{2\pi} \cos\theta \mathrm{d}\theta \int_{\phi=0}^{\pi} \sin^2\phi\cos\phi \mathrm{d}\phi + \int_0^{2\pi} \sin^2\theta \mathrm{d}\theta \int_0^{\pi} \sin^3\phi \mathrm{d}\phi$$

$$= 0 + \frac{4}{3}\pi = \frac{4}{3}\pi \text{ (a net outward flow).}$$

Graph parameterizations.

- Consider a flux integral over a graph surface $z = f(x, y), (x, y) \in D$.
- As before, we parameterize the graph as

$$\begin{aligned} \mathbf{r}(x, y) &= x\mathbf{i} + y\mathbf{j} + f(x, y)\mathbf{k}, \\ \mathrm{d}\mathbf{S} &= (\mathbf{r}_x \times \mathbf{r}_y)\mathrm{d}A = (-f_x\mathbf{i} - f_y\mathbf{j} + \mathbf{k})\mathrm{d}A. \end{aligned}$$

- Example 5. Consider $\mathbf{F} = y\mathbf{i} + x\mathbf{j} + z\mathbf{k}$ and S the union of $z = 1 - x^2 - y^2$ and the plane $z = 0$ restricted to $x^2 + y^2 \leq 1$.
- We will orient S so that outward flux is positive. Since $\mathbf{r}_x \times \mathbf{r}_y = 2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k}$ points upward, the outward orientation is compatible with the order x, y.
- Let S_1 be the top portion of S and S_2 the bottom portion. We have

$$I_1 = \iint_{S_1} \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \iint_D (4xy + (1 - x^2 - y^2))dA(x, y).$$

Example 5 cont.

- Switching to polar coords,

$$\begin{aligned} I_1 &= \int_0^{2\pi}\int_0^1 (4r^2\cos\theta\sin\theta + 1 - r^2) r dr d\theta \\ &= \int_0^1 4r^3 dr \int_0^{2\pi} \cos\theta\sin\theta d\theta + 2\pi\int_0^1 (r - r^3) dr \\ &= 0 + 2\pi\left(\frac{1}{2} - \frac{1}{4}\right) = \frac{\pi}{2} \end{aligned}$$

- The bottom portition S_2 is just D, the unit disk in the x, y plane. Thus, the parameterization is $\mathbf{r}_2(x, y) = x\mathbf{i} + y\mathbf{j} + 0\mathbf{k}$ so that $\mathbf{N} = \mathbf{k}$ points in the wrong direction. We need to switch the order of the x,y variables so that $\mathbf{N} = \mathbf{r}_y \times \mathbf{r}_x = -\mathbf{k}$.
- Thus,

$$I_2 = \iint_{S_2} \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \iint_D 0 dA = 0$$

- The total flux is therefore $I_1 + I_2 = \pi/2$.