Lecture 14
Stokes' Theorem

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Outline

- Text: section 16.8
- The formula
- Reduction to 2D
- Example
- Proof
- Circulation and curl

Statement of Theorem

- Stokes theorem is the 3D generalization of Green's Theorem that asserts an equality between a circulation integral and a flux integral.
- The precise statement is

  $$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot \mathrm{d}\mathbf{S}$$

  where S is an oriented surface and C a closed curve with matching orientation. This means that if our right hand matches the circulation along C, then the thumb will match the direction of the normal vector to S.

2D version of Stoke's Theorem

- Suppose that C is a 2D simple closed curve and that S is a parameterization of a 2D domain D corresponding to the interior of C. We orient C counter-clockwise. To obtain a matching orientation for S we take $\mathbf{N} = \mathbf{k}$; i.e. we count an upward flux as positive.
- For a 2D vector field $\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$, the left-side of Stokes' formula is the circulation integral

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_C P(x, y)dx + Q(x, y)dy.$$

- We have $\operatorname{curl} \mathbf{F} = (Q_x - P_y)\mathbf{k}$, so the right side of Stokes' Formula is $\iint_D (Q_x - P_y)dA$.
- Thus, in 2D, Stoke's Theorem reduces to Green's Theorem:

$$\oint_C Pdx + Qdy = \iint_D (Q_x - P_y)dA.$$

Example 1

- Consider the vector field $\mathbf{F} = -y^2\mathbf{i} + x\mathbf{j} + z^2\mathbf{k}$ and let S be the part of the plane $y + z = 2$ bounded by the cylinder $x^2 + y^2 = 1$.
- We may regard S as the graph of the function $z = f(x, y) = 2 - y$ restricted to the domain $D = \{(x, y) \colon x^2 + y^2 < 1\}$. This gives us

$$\mathrm{d}\mathbf{S} = -f_x\mathbf{i} - f_y\mathbf{j} + \mathbf{k} = (\mathbf{j} + \mathbf{k})\mathrm{d}A$$

- To evaluate the RHS of the Stokes' formula, we have

$$\operatorname{curl}\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ D_x & D_y & D_z \\ -y^2 & x & z^2 \end{vmatrix} = (1 + 2y)k$$

$$\operatorname{curl}\mathbf{F} \cdot \mathrm{d}\mathbf{S} = (1 + 2y)dA$$

- Using polar coordinates, we obtain

$$\iint_S \operatorname{curl}\mathbf{F} \cdot \mathrm{d}\mathbf{S} = \int_0^{2\pi}\int_0^1 (1 + 2r\sin t)r\,dr\,d\theta = \pi + 0 = \pi$$

Example 1 cont.

- Parameterizing C as

$$\mathbf{r}(t) = \cos(t)\mathbf{i} + \sin(t)\mathbf{j} + (2 - \sin(t))\mathbf{k}, \quad 0 \leq t \leq 2\pi$$

  we obtain

$$\begin{aligned}
\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} &= \int_C -y^2 dx + x dy + z^2 dz \\
&= \int_0^{2\pi} \left((-\sin^2 t)(-\sin t) + \cos^2 t + (2 - \sin t)^2 (-\cos t) \right) dt \\
&= \int_0^{2\pi} \left(\sin^3 t + \cos^2 t - 4\cos t + 4 \sin t \cos t - \sin^2 t \cos t \right) dt \\
&= 0 + \pi + 0 + 0 + 0 = \pi
\end{aligned}$$

- The calculated values of the LHS and the RHS of the Stokes' formula agree.

Example 2

- Consider the vector field $\mathbf{F} = xz\mathbf{i} + yz\mathbf{j} + xy\mathbf{k}$. and let S be the spherical cap formed by $x^2 + y^2 + z^2 = 4$ inside the cylinder $x^2 + y^2 = 1$.
- We regard S as the graph of a function

$$f(x, y) = \sqrt{4 - x^2 - y^2}$$

restricted to the domain $D = \{(x, y) : x^2 + y^2 \leq 1\}$. This gives us

$$\mathrm{d}\mathbf{S} = \frac{x\mathbf{i}}{\sqrt{4 - x^2 - y^2}} + \frac{y\mathbf{j}}{\sqrt{4 - x^2 - y^2}} + \mathbf{k}.$$

- To evaluate the RHS of the Stokes' formula, we have

$$\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ D_x & D_y & D_z \\ xz & yz & xy \end{vmatrix} = (x - y)\mathbf{i} + (x - y)\mathbf{j}$$

Example 2 cont

- Switching to polar coordinates, we obtain

$$\iint_S \operatorname{curl} \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \frac{(x-y)(x+y)}{\sqrt{4-x^2-y^2}} dA$$
$$= \int_0^{2\pi} \int_0^1 \frac{r^3}{\sqrt{4-r^2}} \cos(2t) dr dt = 0$$

- We parameterize the curve C as

$$x = \cos t, \quad y = \sin t, \quad z = \sqrt{3}, \quad 0 \leq t \leq 2\pi.$$

- The LHS of the Stokes formula is then

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_C xz\mathrm{d}x + yz\mathrm{d}y + xy\mathrm{d}z$$
$$= \int_0^{2\pi} \sqrt{3} \cos t(-\sin t) + \sqrt{3} \sin t \cos t = 0$$

- Again, we have illustrated the agreement between the LHS and RHS of the Stokes' formula .

Proof of Stokes' Theorem

- Here we give the proof of Stokes' Theorem for the special case where the surface S is the graph of a function $z = f(x, y)$ restricted to a 2-dimensional domain $(x, y) \in D$. This allows us to reduce the proof of ST to that of Green's Theorem.
- For such a surface, we have a parameterization ready at hand:

  $$\mathbf{r}(x, y) = x\mathbf{i} + y\mathbf{j} + f(x, y)\mathbf{k}, \quad (x, y) \in D.$$

  We can now make use of a formula from the lecture on parameterized surfaces:

  $$\mathrm{d}\mathbf{S} = (-f_x\,\mathbf{i} + f_y\,\mathbf{j} + \mathbf{k})dA$$

- Writing $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$, we have

  $$\operatorname{curl}\mathbf{F} = (Ry - Qz)i + (Pz - Rx)j + (Qx - Py)k$$
  $$\iint_S \operatorname{curl}\mathbf{F} \cdot \mathrm{d}\mathbf{S} = \iint_D \left(f_x(Q_z - R_y) + f_y(R_x - P_z) + Q_x - P_y\right) dA$$

Proof of ST cont.

- Let C_1 be the 2D curve that serves as the boundary of D. For the LHS of the Stokes' formula, we begin by parameterizing C_1 as $x = g(t),\ y = h(t),\ a \leq t \leq b$, and then lift that to a parameterization of C by setting

$$\mathbf{r}(t) = g(t)\mathbf{i} + h(t)\mathbf{j} + f(g(t), h(t))\mathbf{k}, \quad a \leq t \leq b.$$

  Let us suppose that S has an upward orientation. In order to obtain a matching orientation of C, we must orient C_1 counter-clockwise.
- Evaluating the LHS of the Stokes' formula, we obtain

$$\begin{aligned}\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = &\int_a^b P(g(t), h(t), f(g(t), h(t)) g'(t) dt + \\ &+ \int_a^b Q(g(t), h(t), f(g(t), h(t)) h'(t) dt + \\ &+ \int_a^b R(g(t), h(t), f(g(t), h(t)) \frac{d}{dt} f(g(t), h(t)) dt\end{aligned}$$

Proof part 3.

- Observe that

$$\frac{d}{dt} f(g(t), h(t)) = f_x(g(t), h(t)) g'(t) + f_y(g(t), h(t)) h'(t).$$

- Using the calculations on the preceding slide, we can now rewrite the LHS of the Stokes' formula as

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_{C_1} \left(P(x, y, f(x, y)) + R(x, y, f(x, y)) f_x(x, y)\right) dx + \\ + \int_{C_1} \left(Q(x, y, f(x, y)) + R(x, y, f(x, y)) f_y(x, y)\right) dy$$

- Finally, we apply Green's Theorem to conclude that

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \iint_D \frac{\partial}{\partial x} \left(Q(x, y, f(x, y)) + R(x, y, f(x, y)) f_y(x, y)\right) dA \\ - \iint_D \frac{\partial}{\partial y} \left(P(x, y, f(x, y)) + R(x, y, f(x, y)) f_x(x, y)\right) dA$$

- Expanding the above and cancelling the f_{xy} terms, we obtain the same expression we did when evaluating the RHS.

Example 1 revisited

- Let's return to example 1. This time, instead of fully evaluating the LHS let us leave it as

$$\begin{aligned}\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} &= \int_C -y^2 dx + x dy + z^2 dz \\ &= \int_{C_1} -y^2 dx + x dy + (2-y)^2(-2dy) \\ &= \int_{C_1} -y^2 dx + (x - 8 + 8y - 2y^2) dy\end{aligned}$$

  Here we are using the substitution $z = f(x, y) = 2 - y,\ dz = -2y$.
- Let us leave the RHS as

$$\iint_S \operatorname{curl} \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \iint_D (1 + 2y) dA.$$

- We re-expressed the LHS as a 2D line integral with $P = -y^2,\ Q = x - 8 + 8y - 2y^2$. This gives $Q_x - P_y = 1 + 2y$.
- Now we clearly see how LHS and RHS of Stokes' theorem agree because, in the case of a graph, the relation reduces to the 2-dimensional Green's Theorem.

Curl via Stokes' Theorem

- Stokes Theorem provides a geometric interpretation of curl.
- Let S_a be a disk of radius a around a fixed point $\mathbf{p}_0$ oriented so as to be perpendicular to a fixed unit vector $\mathbf{n}$, and C_a be the boundary of this disk. For a given 3D vector field $\mathbf{F}$, Stokes' Theorem tells us that

$$\int_{Cr} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \iint_{S_r} \operatorname{curl} \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \iint_{S_a} (\operatorname{curl} \mathbf{F} \cdot \mathbf{n})\, \mathrm{d}S$$

- If $\mathbf{p}$ is close to $\mathbf{p}_0$, then $\operatorname{curl} \mathbf{F}(\mathbf{p}) \approx \operatorname{curl} \mathbf{F}(\mathbf{p}_0)$. Hence,

$$\int_{C_r} \mathbf{F} \cdot \mathrm{d}\mathbf{r} \approx (\operatorname{curl} \mathbf{F}(\mathbf{p}_0) \cdot \mathbf{n}) \iint_{S_a} dS = (\operatorname{curl} \mathbf{F}(\mathbf{p}_0) \cdot \mathbf{n})\pi a^2$$

  where the right-side factor represents the area of S_a. In the limit,

$$\operatorname{curl} \mathbf{F}(\mathbf{p}_0) \cdot \mathbf{n} = \lim_{a \to 0} \frac{1}{\pi a^2} \int_{C_r} \mathbf{F} \cdot \mathrm{d}\mathbf{r}$$

- The above expression is maximized if $\mathbf{n}$ points in the direction of $\operatorname{curl} \mathbf{F}(\mathbf{p}_0)$. We may therefore interpret the magnitude of $\operatorname{curl} \mathbf{F}(\mathbf{p}_0)$ as the amount of circulation caused by $\mathbf{F}$ near $\mathbf{p}_0$, and interpret the direction of $\operatorname{curl} \mathbf{F}(\mathbf{p}_0)$ as the orientation of that circulation.