Lecture 15
Divergence Theorem

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Outline

- Text: section 16.9
- The formula
- Examples
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- Flux and divergence
- Reduction to 2D

The divergence theorem

- Let S be a closed surface enclosing a 3D region E oriented so that the normal vector points outward. Let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ be a 3D vector field. Then,

$$\iint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \iiint_E \operatorname{div} \mathbf{F} \, \mathrm{d}V.$$

- In physics, the above relation is known as Gauss' law. This fundamental principle asserts that *The total of the electric flux out of a closed surface is proportional to the charge enclosed.*
- Recall that $\mathbf{F}$ is called *incompressible* if $\operatorname{div} \mathbf{F} = 0$. As a particular case of the divergence theorem, we conclude that for an incomressible $\mathbf{F}$, the flux across any closed membrane is 0.
- More generally, the divergence theorem should be regarded as a conservation law for fluxes. A positive divergence $\operatorname{div} \mathbf{F}$ represents an expanding flow $\mathbf{F}$, while a negative divergence represents compression. In the first case, the net outward flux is positive; the material flows out of the membrane. A negative divergence implies that the total flux is negative and that the "stuff" represented by $\mathbf{F}$ is concentrating inside the membrane.

Example 1

- Let's verify the divergence theorem for the flux of $\mathbf{F} = z\mathbf{i} + y\mathbf{j} + x\mathbf{k}$ across the boundary of the unit ball B.
- A direct calculation shows that div $F = 0 + 1 + 0 = 1$. Hence, the RHS of the divergence law is equal to the volume of the unit sphere:

$$\iiint_B dV = 4/3\pi.$$

- To calculate the LHS, we parameterize using spherical coordinates:

$$\mathbf{r}(\phi, \theta) = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi), \quad 0 \le \phi \le \pi,\ 0 \le \theta \le 2\pi.$$

- On S, the unit sphere, $\mathbf{n} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $\mathrm{d}S = \sin(\phi) dA(\phi, \theta)$.

$$\begin{aligned}\iint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S} &= \iint_S (\mathbf{F} \cdot \mathbf{n})\mathrm{d}S = \iint_S (2zx + y^2)\mathrm{d}S \\ &= \int_0^{2\pi} \int_0^{\pi} (2\sin^2(\phi)\cos(\phi)\cos(\theta) + \sin^3(\phi)\sin^2(\theta))\mathrm{d}\phi\mathrm{d}\theta \\ &= 0 + \int_0^{2\pi} \sin^3(\phi) d\phi \int_0^{\pi} \sin^2\theta d\theta \\ &= \pi\left[\frac{1}{3}\cos^3\phi - \cos\phi\right]_0^{\pi} = \frac{4}{3}\pi.\end{aligned}$$

Proof of the Divergence Theorem

- We will prove the Divergence Theorem for the special case where E is a domain bounded by two graphs

$$E = \{(x, y, z) : (x, y) \in D,\ f(x, y) \leq z \leq g(x, y)\}, \quad D \subset \mathbb{R}^2,$$

  and the vector field has the special form $\mathbf{F}(x, y, z) = R(x, y, z)\mathbf{k}$.
- The general proof relies on a dissection argument similar to the dissection argument we employed when discussing Green's Theorem. A general region E can be subdivided into a finite union of special regions $E = E_1 \cup \cdots \cup E_n$ with fluxes between the sub-regions cancelling each other out, and only the flux across the outer boundary of E contributing to the overall total.
- The general proof for vector fields with all 3 components can then be obtained by permuting the coordinates x, y, z.

Proof part 2.

- The boundary S of a domain

$$E = \{(x, y, z) : (x, y) \in D,\ f(x, y) \le z \le g(x, y)\}$$

  naturally breaks up into 3 components:

$$\begin{aligned} S_1 &= \{(x, y, z) \colon (x, y) \in D,\ z = g(x, y)\} \\ S_2 &= \{(x, y) \in C \colon f(x, y) \le z \le g(x, y)\} \\ S_3 &= \{(x, y, z) \colon (x, y) \in D,\ z = f(x, y)\} \end{aligned}$$

  Intuitively, S_1 is the "ceiling, S_3 is the "floor", and S_2 represents the "wall" with C a curve that is the boundary of the 2D region D.
- Now the normal vector to S_2 is parallel to xy plane, and hence the flux of $\mathbf{F} = R\,\mathbf{k}$ across S_2 is zero.
- Choosing an upward orientation for both S_1 and S_3 we obtain

$$\iint_S (R\mathbf{k}) \cdot \mathrm{d}\mathbf{S} = \iint_{S_1} (R\mathbf{k}) \cdot \mathrm{d}\mathbf{S} - \iint_{S_3} (R\mathbf{k}) \cdot \mathrm{d}\mathbf{S}$$

  because we wish to count an outward flux across S as positive.

Proof part 3.

- To calculate the flux across the "ceiling" S_1 and the "floor" S_3 we introduce parameterizations

$$\mathbf{r}_1(x, y) = x\mathbf{i} + y\mathbf{j} + g(x, y)\mathbf{k}, \quad (x, y) \in D$$
$$\mathbf{r}_3(x, y) = x\mathbf{i} + y\mathbf{j} + f(x, y)\mathbf{k}, \quad (x, y) \in D$$

  The corresponding normal vectors are

$$\mathbf{N}_1 = -g_x\mathbf{i} - g_y\mathbf{j} + \mathbf{k}$$
$$\mathbf{N}_3 = -f_x\mathbf{i} - f_y\mathbf{j} + \mathbf{k}$$

- Our flux integral is therefore (remember that $\mathbf{F} = R\,\mathbf{k}$),

$$\iint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \iint_{S_1} \mathbf{F} \cdot \mathrm{d}\mathbf{S} - \iint_{S_3} \mathbf{F} \cdot \mathrm{d}\mathbf{S}$$
$$= \iint_D \left(R(x, y, g(x, y)) - R(x, y, f(x, y))\right) dA$$

Proof part 4.

- Recall again that $E = \{(x, y, z) : (x, y) \in D,\ f(x, y) \leq z \leq g(x, y)\}$
- Hence, for the RHS of the divergence theorem formula, we have

$$\begin{aligned}\iiint_E \operatorname{div} \mathbf{F} dV &= \iiint_E R_z(x, y, z) dV \\ &= \iint_D \left(\int_{z=f(x,y)}^{z=g(x,y)} R_z(x, y, z) dz \right) dA\end{aligned}$$

- We now apply the fundamental theorem of calculus to the inner integral to conclude that

$$\iiint_E \operatorname{div} \mathbf{F} dV = \iint_D \left(R(x, y, g(x, y)) - R(x, y, f(x, y))\right) dA,$$

- One the previous slide we derived that

$$\iint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \iint_D \left(R(x, y, g(x, y)) - R(x, y, f(x, y))\right) dA$$

- Therefore, the LHS of the divergence theorem matches the RHS.

Geometric definition of divergence

- The divergence theorem can be used to give a geometric characterization of the divergence operator.
- Let B_a be a ball of radius a about a fixed point $\mathbf{p}_0$, and let S_a be the corresponding boundary sphere of radius a. Let $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$ be a vector field that represents the flow of a fluid. By the divergence theorem

$$\iint_{S_a} \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \iiint_{B_a} \operatorname{div} \mathbf{F} dV.$$

- For small values of a, $\operatorname{div}(\mathbf{F})(\mathbf{p}) \approx \operatorname{div}(\mathbf{F})(\mathbf{p}_a)$ for $\mathbf{p} \in B_a$. Hence,

$$\iiint_{B_a} \operatorname{div} \mathbf{F} dV \approx \operatorname{div}(\mathbf{F})(\mathbf{p}_0) \times \iiint_{B_a} dV = \frac{4}{3}\pi a^3 (\operatorname{div} \mathbf{F})(\mathbf{p}_0).$$

- Combining the above approximation with the divergence theorem,

$$\operatorname{div}(\mathbf{F})(\mathbf{p}_0) \approx \frac{1}{\mathrm{Vol}(B_a)} \iint_{S_a} \mathbf{F} \cdot \mathrm{d}\mathbf{S},$$

  with equality attained in the limit as $a \to 0$.
- This means that, intuitively, divergence is net outward flux per unit volume. Positive divergence means fluid is expanding; negative divergence means that the fluid is being compressed.

Divergence theorem in 2D

- It is possible to interpret Green's Theorem ans the 2D version of the divergence theorem. However, in order to do so, one has to introduce a new type of line integral, one that calculates flux across a boundary rather than a circulation.
- To that end, we measure a 2D flux using an integral of the form

$$\int_C (\mathbf{F} \cdot \mathbf{N}) ds = \int_C (-Q\mathbf{i} + P\mathbf{j}) \cdot \mathbf{T}\, ds = \int_C -Qdx + Pdy,$$

  where $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$ is a 2D vector field, where C is a 2D curve, where $\mathbf{T}$ is the unit tangent to C, and where $\mathbf{N}$ is the unit normal.
- To make sense of the above equation we must note that $-Q\mathbf{i} + P\mathbf{j}$ is the vector $\mathbf{F}$ rotated by 90 degrees counter-clockwise. Assuming that $\mathbf{T}$ (the unit tangent) is obtained by rotating $\mathbf{N}$ in the same way, it follows that

$$\mathbf{F} \cdot \mathbf{N} = (-Q\mathbf{i} + P\mathbf{j}) \cdot \mathbf{T}.$$

- Finally, recall that $\mathbf{T}\mathrm{d}s = \mathrm{d}\mathbf{r}$, and hence that

$$\int_C (-Q\mathbf{i} + P\mathbf{j}) \cdot \mathbf{T}\, ds = \int_C (-Q\mathbf{i} + P\mathbf{j})\mathrm{d}\mathbf{r} = \int_C -Qdx + Pdy.$$

2D divergence cont.

- Next, let us introduce 2D divergence as

$$\operatorname{div} \mathbf{F} = P_x + Q_y.$$

- Finally, we can state the 2D divergence theorem as

$$\oint_C (\mathbf{F} \cdot \mathbf{N}) ds = \iint_D \operatorname{div} \mathbf{F} \mathrm{d}A,$$

  where D is the interior of a closed curve C and where $\mathbf{N}$ is chosen to point outward.
- The outward orientation of $\mathbf{N}$ means that C is being traversed in a counter-clockwise fashion. This means that we may apply Green's Theorem to assert that

$$\oint_C -Qdx + Pdy = \iint_D (Px - (-Q_y))dA.$$

- We recognize the above relation as a restatement of the 2D divergence theorem.
- In conclusion, by interpreting a line integral $\int -Qdx + Pdy$ as a 2D flux, rather than as a circulation integral we may interpret Green's theorem as the 2D version of the divergence theorem.