Lecture 16
Complex Numbers

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Outline

- Text: appendix G
- Algebraic definition
- The polar form of a complex number
- DeMoivre's Formula
- Euler's Formula

Complex numbers

- Multiplication by -1 represents a 180-degree turn. Two such multiplications represent multiplication by 1, a 360-degree rotation that produces no net change.
- Let's introduce a special number called i which represents a rotation by $\pi/2$ or 90 degrees. Two such rotations represent a rotation by 180 degrees. In other words $i \times i = -1$ or $i = \sqrt{-1}$.
- A complex number is an expression of the form $a + bi$ where a, b are real numbers. A complex number $a + bi$ represents the point (a, b).
- Complex numbers addition: $(a + bi) + (c + di) = (a + c) + (b + d)i$.
- The relation $i^2 = -1$ allows us to multiply complex numbers. Thus,

$$(a + bi)(c + di) = ac + adi + bci + (bi)(di) = (ac - bd) + (ad + bc)i.$$

- Complex conjugation $\overline{a + bi} = a - bi$.
- Complex numbers can also be divided once we note that

$$(a + bi)\overline{(a + bi)} = (a + bi)(a - bi) = a^2 + b^2.$$

- The rule for complex division. Set $z = a + bi, w = c + di$. Get

$$\frac{w}{z} = \frac{w\bar{z}}{z\bar{z}} = \frac{(c + di)(a - bi)}{a^2 + b^2} = \frac{ac + bd}{a^2 + b^2} + \frac{ad - bc}{a^2 + b^2}\, i.$$

Complex roots

- Complex numbers arise naturally when we consider roots of equation.
- For example, the solution of a quadratic equation $ax^2 + bx + c = 0$ is

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

  If the number inside the square root is negative, the equation has complex solutions.
- Example: solve $x^2 + x + 1 = 0$. The solutions are

$$x = -\frac{1}{2} \pm \frac{\sqrt{-3}}{2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$$

- Example: solve $x^4 = 1$. We factor

$$x^4 - 1 = (x^2 - 1)(x^2 + 1) = (x - 1)(x + 1)(x^2 + 1).$$

  Therefore, there are two real solutions, $x = 1, -1$ and two imaginary solutions $x = i, -i$.

The polar form of a complex number

- Let $x + iy = r(\cos\theta + i\sin\theta)$ be a complex number, where r, θ are the polar coordinates of the point (x, y). Recall that $r = \sqrt{x^2 + y^2}$ is the magnitude and $\theta = \tan^{-1}(y/x)$ is the argument.
- Multiplication using the polar form. Complex multiplication multiplies the magnitudes and adds the arguments.
- Let $z = x + iy = r(\cos\theta + i\sin\theta)$ and $w = u + iv = \rho(\cos\phi + i\sin\phi)$.
  Claim: $(x + iy)(u + iv) = r\rho(\cos(\theta + \phi) + i\sin(\theta + \phi))$.
  Proof: By the angle addition formulas,

$$\begin{aligned} & r\rho(\cos(\theta + \phi) + i\sin(\theta + \phi)) \\ &\quad = r\rho(\cos\theta\cos\phi - \sin\theta\sin\phi) + ir\rho(\sin\theta\cos\phi + \cos\theta\sin\phi) \\ &\quad = (xu - yv) + i(yu + xv) \\ &\quad = (x + iy)(u + iv) \end{aligned}$$

  The latter is exactly the value of $(x + iy)(u + iv)$

DeMoivre's Formula: powers and roots of Cx numbers

- Let $x + iy = r(\cos\theta + i\sin\theta)$ be the polar form of a complex number. Then,
$$(x + iy)^p = r^p(\cos(p\theta) + \sin(p\theta)).$$
- For $n = 1, 2, 3, \ldots$ the formula is proved by repeated multiplication.
- For $n = -1, -2, -3, \ldots$ the formula is proved by observing that
$$(x + iy)^{-1} = \frac{x - iy}{x^2 + y^2} = r^{-1}(\cos(-\theta) + i\sin(-\theta))$$
- The formula remains correct when p is a fractional power, but we have to take into account the multivaluedness of the argument:
$$x + iy = r(\cos\theta + i\sin\theta) = r(\cos(\theta + 2\pi j) + \sin(\theta + 2\pi j)),$$
where $j = 0, \pm1, \pm2, \ldots$. Therefore,
$$(x + iy)^{\frac{1}{n}} = r^{\frac{1}{n}}\left(\cos\left(\frac{\theta + 2\pi j}{n}\right) + i\sin\left(\frac{\theta + 2\pi j}{n}\right)\right),$$
where $j = 0, 1, \ldots, n - 1$. Note that for $j = -1, -2, \ldots$ and for $j = n, n + 1, n + 2, \ldots$, the values of the root repeat. Therefore, the nth root of a complex number also has n different values.

Euler's formula

- Euler's formula is the relation

$$e^{it} = \cos t + i \sin t.$$

One proves this formula, by considering the Taylor series for the functions on the left and on the right. Set $x = it$ to obtain,

$$\begin{aligned} e^{it} &= 1 + it - \frac{t^2}{2!} - i\frac{t^3}{3!} + \frac{t^4}{4!} + i\frac{t^5}{5!} - \frac{t^6}{6!} - i\frac{t^7}{7!} + \cdots \\ &= \left(1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \cdots\right) + i\left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots\right) \\ &= cost + i \sin t \end{aligned}$$

- An important application of Euler's is a concise polar formalism for working with complex numbers:

$$x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}.$$

- Complex multiplication: $(r_1 e^{i\theta_1}) \cdot (r_2 e^{i\theta_2}) = (r_1 r_2) e^{i(\theta_1 + \theta_2)}$
- DeMoivre's formula: $(x + iy)^p = (re^{i\theta})^p = r^p e^{ip\theta}$.
- Euler's identity: $e^{\pi i} + 1 = 0$

Sample calculations

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\begin{aligned}
1+i &= \sqrt{2}e^{\pi i/4} = \sqrt{2}(\cos(\pi/4) + i\sin(\pi/4)),\\
1-i &= \sqrt{2}e^{-\pi i/4} = \sqrt{2}(\cos(\pi/4) - i\sin(\pi/4))\\
(1+i)(1-i) &= (\sqrt{2}\cdot\sqrt{2})e^{\pi i/4-\pi i/4} = 2(\cos(0) + i\sin(0)) = 2\\
\frac{1+i}{1-i} &= \frac{\sqrt{2}}{\sqrt{2}}e^{(\pi i/4-(-\pi i/4))} = \cos(\pi/2) + i\sin(\pi/2) = i\\
(1+i)^2 &= (\sqrt{2})^2 e^{2\pi i/4} = 2(\cos(\pi/2) + i\sin(\pi/2)) = 2i\\
(1+i)^3 &= 2^{3/2}e^{3\pi i/4} = 2\sqrt{2}(\cos(3\pi/4) + i\sin(3\pi/4)) = -2+2i\\
i &= e^{\pi i/2} = \cos(\pi/2) + i\sin(\pi/2)\\
&= e^{5\pi i/2} = \cos(5\pi/2) + i\sin(5\pi/2)\\
\sqrt{4i} = (4i)^{1/2} &= 4^{1/2}e^{\frac{1}{2}\frac{\pi i}{2}} = 4^{1/2}(\cos(\pi/4) + i\sin(\pi/4)) = \sqrt{2} + i\sqrt{2}\\
&= 4^{1/2}e^{\frac{1}{2}\frac{5\pi i}{2}} = 4^{1/2}(\cos(5\pi/4) + i\sin(5\pi/4)) = -\sqrt{2} - i\sqrt{2}\\
(-1)^{1/3} &= e^{\pi i/3} = \cos\pi/3 + i\sin\pi/3 = 1/2 + i\sqrt{3}/2\\
&= e^{(\pi+2\pi)i/3} = \cos\pi + i\sin\pi = -1\\
&= e^{(\pi+4\pi)i/3} = \cos 5\pi/3 + i\sin 5\pi/3 = 1/2 - i\sqrt{3}/2\\
(x-e^{\pi i/3})(x-e^{3\pi i/3})(x-e^{5\pi i/3}) &= x^3+1
\end{aligned}
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