Lecture 17
Second order linear ODEs with constant coefficients

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Outline

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Linear, 2nd order ODEs

- In this lecture we consider 2nd order linear ODEs (ordinary differential equation) with constant coefficients:

$$ay'' + by' + cy = 0,$$

  where a,b,c are real numbers.
- Since the equation is linear, it enjoys a property called "linear superposition principle". This means that a linear combination of two particular solutions is itself a solution.
- The linear super-position principle allows us to write the general solution of our ODE as a linear combination of two basic solutions. The coefficients of this linear combinations are the constants of integration.
- As the simplest example of such an ODE consder $y'' = 0$.
- There are two basic solutions: $y_0 = 1$ and $y_1 = x$.
- The general solution of $y'' = 0$ is therefore

$$y = C_0 y_0 + C_1 y_1 = C_0 + C_1 x,$$

  where C_0, C_1 are arbitrary constants.

The auxilliary equation

- We are discussing differential equations of the form

$$ay'' + by' + cy = 0.$$

- Let's look for a solution of the form $y = e^{rx}$. For such a function,

$$y' = ry, \quad y'' = r^2 y$$
$$ay'' + by' + cy = y(ar^2 + br + c)$$

- It follows that $y = e^{rx}$ is a solution if and only if r is a root of the quadratic equation

$$ar^2 + br + c = 0$$

- We can solve this auxilliary equation using the quadratic formula:

$$r_1, r_2 = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}.$$

- This gives us the particular solutions $y_1 = e^{r_1 x}$, $y_2 = e^{r_2 x}$.
- The general solution can now be written as the linear combination

$$y = C_1 e^{r_1 x} + C_2 e^{r_2 x},$$

where C_1, C_2 are arbitrary constants.

Example 1.

- Solve $y'' + y' - 6y = 0$
- The auxilliary equation is

$$r^2 + r - 6 = (r+3)(r-2) = 0$$

  The roots are $r_1 = -3,\ r_2 = 2$.
- Therefore, the basic solutions are

$$y_1 = e^{-3x}, \quad y_2 = e^{2x}.$$

- The general solution is

$$y = C_1 e^{-3x} + C_2 e^{2x},$$

  where C_1, C_2 are arbitrary constants.

The three classes

- Quadratic equations can be divided into one of 3 classes: distinct real roots, a double root (real), and conjugate complex roots.
- Each of these possibilities gives rise to a different class of ODEs, named case I,II, III in the text. Case I, where both roots of the auxilliary equation are real, was discussed above.
- Let us now discuss Case II, where the auxilliary equation has a repeated root. If the coefficients of the quadratic equation satisfy $b^2 - 4ac = 0$, then

$$ar^2 + br + c = a\left(r + \frac{b}{2a}\right)^2 \quad \Longrightarrow r = -\frac{b}{2a}.$$

- The corresponding particular solutions are $y_1 = e^{rx}$ and $y_2 = xe^{rx}$.
- Solution y_1 is obvious. Let's verify that y_2 is also a solution.

$$y_2' = (rx+1)e^{rx} \quad y_2'' = (r^2x + 2r)e^{rx}$$
$$ay_2'' + by_2' + cy_2 = xe^{rx}(ar^2 + br + c) + e^{rx}(2ar + b) = 0$$

- Therefore, the general solution of the Case II equation is

$$y = (C_1 + C_2 x)e^{rx},$$

where C_1, C_2 are arbitrary constants.

Example 2

- The ODE y''=0, discussed earlier, belongs to this class.
- The auxilliary equation is simply $r^2 = 0$. The double root is $r = 0$, so that

$$y = (C_1 + C_2 x)e^{0x} = C_1 + C_2 x$$

  matches the solution we obtained above.
- As a more complicated example, consider the ODE $4y'' + 12y' + 9y = 0$
- The auxilliary equation is

$$4r^2 + 12r + 9 = 4(r + 3/2)^2 = 0.$$

- The general solution is therefore

$$y = (C_1 + C_2 x)e^{-3/2x}.$$

Complex roots

- Case III of $ay'' + by' + cy = 0$ is the case where $b^2 - 4ac < 0$.
- This means that the roots of the indicial equation are complex:

$$r_1, r_2 = -\frac{b}{2a} \pm i\sqrt{\frac{c}{a} - \frac{b^2}{4a^2}}$$

- Writing $r_1 = s + it$ and $r_2 = s - it$ and applying Euler's formula,

$$y_1 = e^{r_1 x} = e^{sx}(\cos(tx) + I\sin(tx))$$
$$y_2 = e^{r_2 x} = e^{sx}(\cos(tx) - I\sin(tx))$$

- Thus, using the roots of the auxilliary equation directly gives complex-valued solutions. we can obtain real-valued basic solutions by taking the real and imaginary parts of these:

$$y_c = \mathbf{Re}\, y_1 = \mathbf{Re}\, y_2 = \frac{1}{2}(y_1 + y_2) = e^{sx}\cos(tx)$$
$$y_s = \mathbf{Im}\, y_1 = -\mathbf{Im}\, y_2 = \frac{1}{2i}(y_1 - y_2) = e^{sx}\sin(tx)$$

- Since y_c, y_s are linear combinations of y_1, y_2, they are also solutions by the princ. of superposition. The general real solution of a case III equation is

$$y = e^{sx}(C_1\cos(tx) + C_2\sin(tx)).$$

Example 3.

- Solve $y'' - 6y' + 13y = 0$.
- The auxilliary equation $r^2 - 6r + 13 = 0$ has complex roots

$$r_1, r_2 = 3 \pm \frac{1}{2}\sqrt{36 - 52} = 3 \pm 2i.$$

- The basic real solutions are $y_c = e^{3x} cos(2x)$ and $y_s = e^{3x} sin(2x)$.
- The general real solution is therefore

$$y = e^{3x}(C_1 \cos(2x) + C_2 \sin(2x)).$$

Initial value problems (IVP)

- A general solution to a 2nd order ODE has 2 constants of integrations. A particular solution corresponds to a particular value of these two constants based on two constraints. - One way to specialize to a particular solution is to impose initial conditions on y and y' at a particular point. A linear 2nd-order IVP is a the ODE plus two additional constraints

$$ay'' + by' + cy = 0, \quad y(x_0) = y_0 and y'(x_0) = y_1.$$

- After obtaining the general solution of the ODE we apply the constraints to obtain a linear system for the constants of integration. Solving this linear system we obtain a particular solution that satisfies the given initial conditions.

Example 4

- Solve the IVP

$$y'' + y' - 6y = 0, \quad y(0) = 1,\ y'(0) = 0.$$

- Earlier we obtained the general solution $y = C_1 e^{-3x} + C_2 e^{2x}$.
- Taking the derivative of the above gives $y' = -3C_1 e^{-3x} + 2C_2 e^{2x}$.
- Evaluating $y(0)$ and $y'(0)$ and applying the initial conditions gives

$$C_1 + C_2 = 1, \quad -3C - 1 + 2C_2 = 0.$$

- The solution of the initial condition system is $C_1 = 2/5,\ C_2 = 3/5$. Therefore, the solution of the IVP is

$$y = \frac{2}{5} e^{-3x} + \frac{3}{5} e^{2x}.$$

- Remember: while a general ODE possesses infinitely many solutions (solution with parameters), the solution to an IVP is a fixed function without parameters

Boundary value problems (BVP)

- A 2nd order linear BVP is an ODE $ay'' + by' + cy = 0$ together with constraints of the form $y(x_1) = y_1,\ y(x_2) = y_2$.
- Applying these contraints to a general solution gives a system of 2 linear equations in the constants of integration.
- Solving this linear system we obtain a particular solution that satisfies the given initial conditions.

Example 5

- Solve the BVP

$$y'' + 2y' + y = 0, \quad y(0) = 1, \quad y(1) = 3.$$

- The auxilliary equation is $r^2 + 2r + 1 = (r+1)^2 = 0$.
- Since there is a double root, the general solution is

$$y = (C_0 + C_1 x)e^{-x}.$$

- Applying the boundary constraints gives the system

$$C_0 = 1, \quad (C_0 + C_1)\mathrm{e}^{-1} = 3.$$

- This linear system has the solution $C_0 = 1, C_1 = 3\mathrm{e} - 1$. Therefore, the solution of the BVP is

$$y = (1 + (3\mathrm{e} - 1)x)e^{-x}.$$