Lecture 19
Oscillatory systems

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Outline

- Text: 17.3
- The forces in a vibrating system
- Example: harmonic oscillator
- Phase-Amplitude form
- Periodic driving force
- Resonance

Mechanical vibrations

- A vibrating mechanical system acted on by an external force obeys the differential equation

$$mx''(t) + c\,x'(t) + kx(t) = F(t),$$

where m is the mass of the object, k the spring constant, $c > 0$ the damping constant and $F(t)$ an external force expressed as a function of time.

- Since force = mass $\times$ acceleration, we can read the above equation as the acceleration produced by a superposition of forces:

$$x'' = -\frac{c}{m}x' - \frac{k}{m}x + \frac{F}{m}.$$

- The term proportional to the velocity x' is a damping force directed so as to oppose the current velocity.
- The term proportional to the position x is the restoring force as per Hooke's law.
- The damping force and the restoring force are added to the external force. The result is divided by the mass to describe the overall acceleration.

Harmonic oscillator

- A simple, but fundamental subclass of vibrating systems is the case of free vibrations, that is vibrations free from both damping and external forces. The resulting system is called a harmonic oscillator for reasons we will describe below.
- Let us take $c = 0$, $F = 0$ and seek the general solution of

$$mx''(t) + kx(t) = 0.$$

- The auxilliary equation is $r^2 + k/m = 0$ with two imaginary roots $r = \pm i\omega$ where $\omega = \sqrt{k/m}$.
- Thus, the general solution is

$$x(t) = C_1 \cos(\omega t) + C_2 \sin(\omega t).$$

Phase-amplitude Form

- It is instructive to express the solution

$$x(t) = C_1 \cos(\omega t) + C_2 \sin(\omega t).$$

  of the harmonic oscillator equation using phase-amplitude form

$$x(t) = A\cos(\omega t - \delta) = A\cos(\omega t)\cos\delta + A\sin(\omega t)\sin\delta.$$

- Here

$$A = \sqrt{C_1^2 + C_2^2}, \quad \delta = \tan^{-1}(C_2/C_1)$$

  are called, respectively, the amplitude and the phase shift.
- We recognize the above expressesions from the unit on polar coordinates. Indeed, (A, δ) are the polar coords of (C_1, C_2).
- In other words,

$$A\mathrm{e}^{i\delta} = A\cos\delta + iA\sin\delta = C_1 + iC_2.$$

Example 1.

- Consider a frictionless spring system with a mass of 2kg and a natural length of 0.5 m. Suppose that a force of 25.6 Newtons stretchs the spring to 0.7. Determine the frequency and amplitude of the corresponding vibration.
- We calculate the value of the spring constant k by observing that

$$mk(0.7 - 0.5) = 25.6$$

  Solving the above equation gives $k = 128$.
- Our spring is released with an initial velocity of 0. Therfore, we must consider the initial value problem (IVP)

$$2x'' + 128x = 0, \quad x(0) = 0.2, \; x'(0) = 0$$

  where we have assumed that the origin $x = 0$ corresponds to the end of the spring in its uncompressed form.
- The aux. equation $r^2 + 64 = 0$ gives $r = \pm 8i$, $\omega = 8$. The general solution is $x = C_1 \cos(8t) + C_2 \sin(8t)$. Applying the initial conditions we obtain

$$x(0) = C_1 = 0.2, \quad x'(0) = 8C_2 = 0 \Rightarrow C_2 = 0.$$

- The answer is already in phase-amplitdue form, with $A = 0.2, \delta = 0$

Oscillatory forcing

- Consider an external force that is, itself, oscillatory
$$F(t) = F_0 \cos(\omega_0 t),$$
with a single frequency ω_0 and amplitude F_0.
- If an external force acts on an undamped system, the energy content of the system oscillates. The corresponding differential equation is
$$mx'' + kx = F_0 \cos(\omega_0 t) \tag{1}$$
- Using undetermined coefficiants, we seek a particular solution
$$x_p = a\cos(\omega_0 t) \qquad mx_p'' + kx_p = a(-m\omega_0^2 + k)\cos(\omega_0 t)$$
- Letting $k/m = \omega^2$, we see that a must satisfiy
$$F_0 = a(k - m\omega_0^2) = am(\omega^2 - \omega_0^2) \quad \Longrightarrow a = \frac{F_0/m}{\omega^2 - \omega_0^2}.$$
- If $\omega \neq \omega_0$, the general solution of (1) is therefore,
$$x(t) = A_{\text{ext}}\cos(\omega_0 t) + A_{\text{int}}\cos(\omega t + \delta), \quad \text{where } A_{\text{ext}} = \left(\frac{F_0/m}{\omega^2 - \omega_0^2}\right).$$
and A_{int}, δ are constants determined by initial conditions.

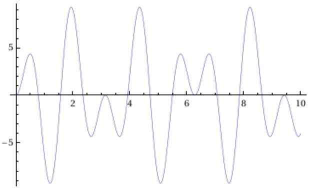
Example 1

- Consider an undampled oscillator with $m = 1, k = 9$ subject to a forced oscillation $F_0 = 80,\ \omega_0 = 5$. Solve the corresponding IVP with initial conditions $x(0) = x'(0) = 0$.
- The corresponding differential equation is $x'' + 9x = 80\cos 5t$.
- Following the procedure in the previous slide we obtain

$$x(t) = A_{\text{int}}\cos(\omega t + \delta) + A_{\text{ext}}\cos(\omega_0 t),$$

where $\omega = \sqrt{\dfrac{k}{m}} = 3, \quad A_{\text{ext}} = \dfrac{80/1}{3^2 - 5^2} = -5.$

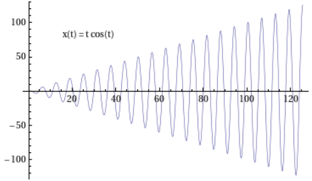
- Applying the initial conditions gives the equations
  $0 = A_{\text{int}}\cos(\delta) - 5,$
  $0 = -A_{\text{int}}\sin(\delta).$
  The solution is $\delta = 0,\ A_{\text{int}} = 5.$
- The solution of the IVP is a superposition of oscillations at two frequencies:
  $x(t) = 5\cos 3t - 5\cos 5t$



Resonance

- On the preceding slide we considered the equation for undamped oscillation with periodic forcing: $mx'' + kx = F_0 \cos(\omega_0 t)$.
- Letting $\omega^2 = k/m$ be the characteristic frequency of the system we may rewrite the above as $x'' + \omega^2 x = (F_0/m)\cos(\omega_0 t)$.
- Above we assumed that $\omega \neq \omega_0$. If $\omega = \omega_0$, we have a resonant system; the external force continuously adds energy to the system.
- If $\omega_0 = \omega$ the solution form is

$$
\begin{aligned}
x_p &= at\sin(\omega t)\\
x_p' &= a\sin(\omega t) + at\omega\cos(\omega t)\\
x_p'' &= 2a\omega\cos(\omega t) - at\omega^2\sin(\omega t)\\
x_p'' + \omega^2 x_p &= 2a\omega\cos(\omega t) \Longrightarrow a = \frac{F_0}{2m\omega}.
\end{aligned}
$$



- The amplitude of this solution grows without bound, and is therefore unphysical. An actual physical system cannot contain an unbounded amount of energy. In an actual resonance results in some kind of catastrophe: a structure collapses, or a circuit burns out (positive feedback between a microphone and a speaker).