H. W. Gould and Jocelyn Quaintance
Generalizations of Vosmansky's Identity,
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Abstract

We first use the Fundamental Theorem of Algebra to give an almost immediate proof of the identity

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\binom{x+k}{2n}\binom{x+2n-k}{2n}=(-1)^n\binom{x}{n}\binom{x+n}{n} \tag{1}$$

valid for all complex values of x and all non-negative integers n. The identity was found by J. Vosmansky when x is a non-negative integer, and proved, in this case, by L. Carlitz. We then generalize, and prove, that for any integer r, and any complex x.

(2)

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\binom{x+k}{2n+r}\binom{x+2n-k}{2n+r}=(-1)^n\binom{2n}{n}\binom{x+n}{2n+r}\frac{\binom{x+n}{n+r}}{\binom{x+n}{n}}.$$

In fact we prove more generally that

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{x+k}{n+r}\binom{y+n-k}{n+r} \tag{3}$$

$$=(-1)^n\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{x}{k+r}\binom{y}{n-k+r} \tag{4}$$

$$=(-1)^n\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{x+k}{k+r}\binom{y+n-k}{n-k+r} \tag{5}$$

valid for all complex x and y and any integer r and for any integer $n\geq 0$.