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Sums of Fibonacci Numbers Indexed by Integer Parts,
Fibonacci Quart. **61** (2023), no. 2, 143–152.

Abstract

Consider the integer sequences $(F_{\lfloor \sqrt{n} \rfloor} : n \in \mathbb{N}_0)$ and $(F_{\lfloor \log_2 n \rfloor} : n \in \mathbb{N})$, letting $\lfloor x \rfloor$ denote the integer part of a nonnegative value x, and where F_n denotes the nth Fibonacci number for a nonnegative integer n. We apply an Abel-type summation lemma to prove explicit evaluations for $\sum_{n=1}^{m} F_{\lfloor \sqrt{n} \rfloor}$ and $\sum_{n=1}^{m} F_{\lfloor \log_2 n \rfloor}$ for a natural number m. We then apply this summation lemma to determine an analytical formula for $\sum_{n=1}^{m} F_{\lfloor \frac{n}{s} \rfloor}$, letting s denote a natural number parameter, and we demonstrate how our method may be applied to evaluate sums of the form $\sum_{n=1}^{m} F_{\left\lfloor \frac{\sqrt[r]{n}}{s} \right\rfloor}$ for integers $r \geq 2$ and $s \geq 1$. We also consider the problem of evaluating finite sums of expressions of the form $F_{\left\lfloor \log_2\left(\frac{n}{s}\right) \right\rfloor}$ for a natural number s. Much of our work is closely connected with evaluations for Fibonacci sums of the form $S(t, m) = \sum_{n=1}^{m} n^t F_n$, where t is a nonnegative integer.