GENERALIZATION OF A PROBLEM OF GOULD AND ITS SOLUTION BY A CONTOUR INTEGRAL

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The following research problem was posed by H. W. Gould in [1].

Problem 1: If, for an arbitrary sequence $\{A_n\}_{n=0}^{\infty}$,

$$f(x) = \sum_{n=0}^{\infty} A_n x^n, \quad h(x) = \sum_{n=0}^{\infty} A_n^2 x^n,$$

how are functions f and h related?

The preceding problem is readily generalized as follows.

Problem 2: If, for arbitrary sequences $\{A_n\}_{n=0}^{\infty}$ and $\{B_n\}_{n=0}^{\infty}$,

$$f(x) = \sum_{n=0}^{\infty} A_n x^n, \quad g(x) = \sum_{n=0}^{\infty} B_n x^n, \quad \text{and} \quad h(x) = \sum_{n=0}^{\infty} A_n B_n x^n,$$

how is function h related to functions f and g?

Problem 2 was at least partially solved in a previous paper (viz. [2]), using the techniques of the umbral calculus. However, the "solution" obtained in [2] is expressed as a function of finite difference operators, thereby necessitating caution in its application. The aim of this paper is to obtain a rigorous solution to Problem 2 above, under the assumption that f and g are "sufficiently" analytic. We will find it slightly more tedious, but more far-reaching, to solve the even more general

Problem 3: If, for arbitrary sequences $\{A_n\}_{n=-\infty}^{\infty}$ and $\{B_n\}_{n=-\infty}^{\infty}$, and $z_0 \in C$,

$$f(z) = \sum_{n=-\infty}^{\infty} A_n(z - z_0)^n, \quad g(z) = \sum_{n=-\infty}^{\infty} B_n(z - z_0)^n, \quad h(z) = \sum_{n=-\infty}^{\infty} A_n B_n(z - z_0)^n,$$

how is function h related to functions f and g?

Before proceeding to the main theorem of this paper, which solves Problem 3, we will find it convenient to make a few preliminary definitions and remarks.

Definitions: Let z be an arbitrary point in the complex plane (i.e., the z-plane), and suppose the following Laurent series expansions valid in the annuli indicated:

(1) $$f(z) = \sum_{n=-\infty}^{\infty} A_n(z - z_0)^n, \quad \forall z \ni r_1 < |z - z_0| < R_1,$$

(2) $$g(z) = \sum_{n=-\infty}^{\infty} B_n(z - z_0)^n, \quad \forall z \ni r_2 < |z - z_0| < R_2,$$

where $\max(r_1, r_2) \equiv \rho_1 < \rho_2 \equiv \min(R_1, R_2)$.

We permit $\rho_1 = 0$, $\rho_2 = \infty$. Let

(3) $$D_1 = \{z : r_1 < |z - z_0| < R_1\},$$

(4) $$D_2 = \{z : r_2 < |z - z_0| < R_2\},$$

(5) $$D_3 = \{z : \rho_1^2 < |z - z_0| < \rho_2^2\},$$

(6) $$D_4 = \{z : r_1 r_2 < |z - z_0| < R_1 R_2\}.$$

Also, define the Laurent series

(7) $$h(z) = \sum_{n=-\infty}^{\infty} A_n B_n (z - z_0)^n,$$

which is necessarily valid $\forall z \in D_4$. Given $z \in D_3$, consider another complex plane (say, the w-plane), and define the annulus

(8) $$\Delta(z) = \{w : s_1(z) < |w - z_0| < s_2(z)\}, \text{ where}$$

(9) $$s_1(z) = \max(\rho_1, |z - z_0|/\rho_2), \quad s_2(z) = \min(\rho_2, |z - z_0|/\rho_1).$$

Let Γ be any simple closed contour contained in $\Delta(z)$ (in the w-plane), traversed in the positive direction, and containing the point $w = z_0$ in its interior.

Remarks: Note that $D_1 \cap D_2 \neq \emptyset$, and $D_3 \subseteq D_4$ (since $r_1 r_2 \leq \rho_1^2 < \rho_2^2 \leq R_1 R_2$). Also, if $z \in D_3$, then $\rho_1 < s_2(z)$ and $|z - z_0|/\rho_2 < s_2(z)$, so that $s_1(z) < s_2(z)$, which implies that $\Delta(z) \neq \emptyset$ for all $z \in D_3$.

Theorem: Given (1)-(9), then for all $z \in D_3$,

$$(10) \qquad h(z) = \frac{1}{2i\pi} \oint_\Gamma \frac{f(w) g\left\{z_0 + \dfrac{z - z_0}{w - z_0}\right\}}{w - z_0} dw = \frac{1}{2i\pi} \oint_\Gamma \frac{f\left\{z_0 + \dfrac{z - z_0}{w - z_0}\right\} g(w)}{w - z_0} dw$$

Proof: Because of the symmetry between functions f and g, it suffices to prove only the first relation in (10). Let

$$G(w) = g\left\{z_0 + \frac{z - z_0}{w - z_0}\right\}.$$

Then, by (2), the Laurent series for G, namely

$$(11) \qquad G(w) = \sum_{n=-\infty}^{\infty} B_n \left(\frac{z - z_0}{w - z_0}\right)^n,$$

represents an analytic function in the annulus $\Delta_2(z)$ given by:

$$(12) \qquad \Delta_2(z) = \{w : |z - z_0|/R_2 < |w - z_0| < |z - z_0|/r_2\}.$$

Also, the Laurent series for $f(w)$, given in (1), but replacing z by w, represents an analytic function in the annulus $\Delta_1(z)$ of the w-plane, given by:

$$(13) \qquad \Delta_1(z) = \{w : r_1 < |w - z_0| < R_1\}.$$

For all $z \in D_3$, $r_1 r_2 \leq \rho_1^2 < |z - z_0| < \rho_2^2 \leq R_1 R_2$; hence, $|z - z_0|/R_2 < R_1$ and $|z - z_0|/r_2 > r_1$. Also, $r_1 < R_1$, and $|z - z_0|/R_2 < |z - z_0|/r_2$ (since $r_2 < R_2$). It follows that $\max(r_1, |z - z_0|/R_2) < \min(R_1, |z - z_0|/r_2)$. This, in turn, implies that

$$\Delta_1(z) \cap \Delta_2(z) \neq \emptyset.$$

Next, observe that, for all $z \in D_3$, $s_1(z) = \max(\rho_1, |z - z_0|/\rho_2) = \max\{\max(r_1, r_2), \max(|z - z_0|/R_1, |z - z_0|/R_2)\} = \max(r_1, r_2, |z - z_0|/R_1, |z - z_0|/R_2) \geq \max(r_1, |z - z_0|/R_2)$. Also, $s_2(z) = \min(\rho_2, |z - z_0|/\rho_1) = \min\{\min(R_1, R_2), \min(|z - z_0|/r_1, |z - z_0|/r_2)\} = \min(R_1, R_2, |z - z_0|/r_1, |z - z_0|/r_2) \leq \min(R_1, |z - z_0|/r_2)$. This implies that, for all $z \in D_3$,

$$\Delta(z) \subseteq \{\Delta_1(z) \cap \Delta_2(z)\}.$$

Since $\Gamma \subset \Delta(z)$ and z lies in the interior of Γ, thus the function $f(w)/(w - z_0)$ is continuous on $\Gamma \subset \Delta_1(z)$; moreover, G is analytic on $\Delta(z) \subseteq \Delta_2(z)$. By a well-known theorem of complex analysis, it is therefore legitimate to interchange the integral and summation signs in the following expression.

$$(14) \qquad \frac{1}{2i\pi} \oint_\Gamma \frac{f(w)}{w - z_0} \sum_{n=-\infty}^{\infty} B_n \left(\frac{z - z_0}{w - z_0}\right)^n dw = \sum_{n=-\infty}^{\infty} B_n (z - z_0)^n \frac{1}{2i\pi} \oint_\Gamma \frac{f(w) dw}{(w - z_0)^{n+1}}.$$

But, since $\Gamma \subset \Delta(z) \subseteq \Delta_1(z)$, we may apply the formula for the coefficients of a Laurent series, namely:

$$A_n = \frac{1}{2i\pi} \oint_\Gamma f(w)/(w - z_0)^{n+1} dw.$$

Hence, the right member of (14) is the restriction of $h(z)$ to D_3, a subset of D_4. The left member of (14), by (11), is equal to the first integral expression in (10). This establishes the first equation of (10), and therefore the theorem.

Additional Remarks: Although the result of the theorem has been proven valid for all $z \in D_3$, as given in (5), the series defining $h(z)$ represents an analytic function in the larger domain D_4, as given in (6). Hence, the series in (7) is the *analytic continuation* of the integral expression for h in (10), from D_3 to D_4. If the latter expression yields a "closed" formula for $h(z)$, then this must be the closed form "sum-function" of h as given by (7), and holds for all $z \in D_4$.

The argument proving the preceding theorem may be slightly modified, and is somewhat simplified, if $r_1 = r_2 = 0$, thereby leading to a corresponding result involving Taylor, instead of Laurent series.

Corollary: Suppose f and g are as given in (1) and (2), with $A_{-n} = B_{-n} = 0$ ($n = 1, 2, \ldots$), i.e.,

$$f(z) = \sum_{n=0}^{\infty} A_n (z - z_0)^n, \quad z \in D_1 = \{z : |z - z_0| < R_1\}, \tag{15}$$

$$g(z) = \sum_{n=0}^{\infty} B_n (z - z_0)^n, \quad z \in D_2 = \{z : |z - z_0| < R_2\}. \tag{16}$$

Then, for all $z \in D_3 = \{z : |z - z_0| < \rho_2^2\}$ (where ρ_2 has been previously defined):

$$h(z) \equiv \sum_{n=0}^{\infty} A_n B_n (z - z_0)^n = \frac{1}{2i\pi} \oint_\Gamma \frac{f(w) g\left\{z_0 + \dfrac{z - z_0}{w - z_0}\right\}}{w - z_0} dw = \frac{1}{2i\pi} \oint_\Gamma \frac{f\left\{z_0 + \dfrac{z - z_0}{w - z_0}\right\} g(w)}{w - z_0} dw, \tag{17}$$

where Γ is as described in the theorem, except that $s_1(z) = |z - z_0|/\rho_2$, $s_2(z) = \rho_2$.

We illustrate the theorem and its corollary with several examples, the first few of which are trivial (but serve to corroborate the results), the last few a bit more interesting.

Example 1: Let $z_0 = 0$, $B_n = 1$ ($n = 0, 1, 2, \ldots$), $A_{-n} = B_{-n} = 0$ ($n = 1, 2, \ldots$). Then $g(z) = (1 - z)^{-1}$ for all $z \in D_2$, the open unit disk. For all $z \in D_3$, where $D_3 = \{z : |z| < \rho_2^2\}$, let $\Delta(z) = \{w : |z|/\rho_2 < |w| < \rho_2\}$. We see that $h(z) = f(z)$, trivially, and expect that the corollary yields this result. Naively applying the corollary, we then obtain, $\forall z \in D_3$:

$$h(z) = \frac{1}{2i\pi} \oint_\Gamma \frac{f(w) g(z/w)}{w} dw = \frac{1}{2i\pi} \oint_\Gamma \frac{f(w) dw}{w - z} = f(w)\Big|_{w=z} = f(z),$$

as expected. By the "additional remark" preceding, the foregoing result is still true for all $z \in D_4 = D_1 = \{z : |z| < R_1\}$.

Example 2: Let $z_0 = 0$, $A_n = a^n$, $B_n = b^n$ ($n = 0, 1, 2, \ldots$), $A_{-n} = B_{-n} = 0$ ($n = 1, 2, \ldots$), where $a \neq 0$, $b \neq 0$; without loss of generality, we may assume $|a| \leq |b|$. Then $D_1 = \{z : |z| < 1/|a|\}$, $D_2 = \{z : |z| < 1/|b|\}$, $D_3 = \{z : |z| < |b|^{-2}\}$, $D_4 = \{z : |z| < 1/|ab|\}$, and $\Delta(z) = \{w : |bz| < |w| < 1/|b|\}$. It is clear that $f(z) = (1 - az)^{-1}$, $g(z) = (1 - bz)^{-1}$, and $h(z) = (1 - abz)^{-1}$, for all $z \in D_1$, D_2, and D_4, respectively; nevertheless, it will be instructive to derive this from the corollary. Applying the first formula in (17), we have:

$$h(z) = \frac{1}{2i\pi} \oint_\Gamma \frac{dw}{(1 - aw)(w - bz)}, \quad \forall z \in D_3.$$

Hence, since the points $w = bz$ and $w = 1/a$ are interior and exterior to Γ, respectively, we find upon applying the Cauchy integral theorem, that $h(z) = (1 - aw)^{-1}\Big|_{w=bz} = (1 - abz)^{-1}$, $\forall z \in D_3$. Again using the remark on analytic continuation, we obtain the anticipated result, namely $h(z) = (1 - abz)^{-1}\ \forall z \in D_4$.

Example 3: Let $f(z) = f(z,t) = \exp\{\frac{1}{2}t(z - z^{-1})\} = \sum_{n=-\infty}^{\infty} J_n(t) z^n$, the generating function of the Bessel functions of integral order. Similarly, let $g(z) = f(z,u)$. It is known that both series converge and represent analytic functions of z in the domain $D_1 = D_2 = D_3 = D_4 = \{z : 0 < |z| < \infty\}$, i.e., for all finite z except $z = 0$. In the nomenclature of the theorem's conditions, $z_0 = 0$, $r_1 = r_2 = 0$, $R_1 = R_2 = \infty$; hence, $\Delta(z) = \{w : 0 < |w| < \infty\}$. Thus, taking Γ as in the theorem, by the formula for the coefficients of a Laurent series:

$$(19) \qquad J_n(t) = \frac{1}{2i\pi} \oint_\Gamma \frac{\exp\{\frac{1}{2}t(w - w^{-1})\}}{w^{n+1}}\, dw, \quad n = 0, \pm 1, \pm 2, \ldots,$$

with a similar formula for $J_n(u)$, valid for all complex t (or u). Applying the theorem, we then have, for all $z \in D_3 = D_4$,

$$h(z) \equiv \sum_{n=-\infty}^{\infty} J_n(t)J_n(u)z^n = \frac{1}{2i\pi}\oint_\Gamma \exp\{\tfrac{1}{2}t(w - w^{-1})\} \exp\{\tfrac{1}{2}u(z/w - w/z)\} \cdot \frac{dw}{w},$$

or

$$(20) \qquad h(z) = \frac{1}{2i\pi}\oint_\Gamma \exp\{\tfrac{1}{2}(t - u/z)w\} \exp\{\tfrac{1}{2}(uz - t)w^{-1}\} \cdot \frac{dw}{w}.$$

We now make the substitution

$$w = a\xi, \text{ where } a = (t - uz)^{\frac{1}{2}}(t - u/z)^{-\frac{1}{2}},$$

and restrict z further so that $z \neq t/u$, $z \neq u/t$, which implies $a \in D_1$. Since, in the w-plane and ξ-plane, a is a constant, the above substitution transforms Γ into a topologically equivalent simple closed contour Γ' in the ξ-plane, which is still oriented in the positive direction. Therefore,

$$(21) \qquad h(z) = \frac{1}{2i\pi}\oint_{\Gamma'} \exp\{\tfrac{1}{2}b(\xi - \xi^{-1})\} \cdot \frac{d\xi}{\xi}, \text{ where } b = (t - uz)^{\frac{1}{2}}(t - u/z)^{\frac{1}{2}}.$$

Comparing this last expression with (19), we see that

$$(22) \qquad h(z) = J_0(b).$$

Thus, we have proved the interesting identity

$$(23) \qquad \sum_{n=-\infty}^{\infty} J_n(t)J_n(u)z^n = J_0\{(t - uz)^{\frac{1}{2}}(t - u/z)^{\frac{1}{2}}\}, \quad \forall z \neq 0.$$

Note that (23) is valid also for the previously excluded values $z = t/u$ and $z = u/t$, provided $t \neq 0$, $u \neq 0$ (by analytic continuation). Therefore, we obtain the following formulas, as special cases of (23):

$$(24) \qquad \sum_{n=-\infty}^{\infty} J_n(t)J_n(u)(t/u)^n = 1,$$

$$(25) \qquad \sum_{n=-\infty}^{\infty} J_n(t)J_n(u) = J_0(t - u), \quad \forall t, u \neq 0.$$

The identity given in (23) is not in itself new, appearing (in variant form), e.g., in [3].

Example 4: Let $f_m(z)$ be the generating function for the mth powers of the Fibonacci numbers ($m = 1, 2, \ldots$), i.e.,

$$(26) \qquad f_m(z) = \sum_{n=0}^{\infty} F_n^m z^n, \text{ valid for all } z \in D_2 = \{z : |z| < \alpha^{-m}\}$$

[in this example, $\alpha = \frac{1}{2}(1 + \sqrt{5})$, $\beta = \frac{1}{2}(1 - \sqrt{5})$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$]. We let $f(z) = f_1(z) = z(1 - z - z^2)^{-1} = 5^{-\frac{1}{2}}\{(1 - \alpha z)^{-1} - (1 - \beta z)^{-1}\}$, and $g = f_m$ in the corollary, with $R_1 = \alpha^{-1}$, $R_2 = \alpha^{-m} = \rho_2$; then $D_3 = \{z : |z| < \alpha^{-2m}\}$, $D_4 = \{z : |z| < \alpha^{-m-1}\}$, and $\Delta(z) = \{w : |z|\alpha^m < |w| < \alpha^{-m}\}\ \forall z \in D_3$. We see readily, from (7), that $h(z) = f_{m+1}(z)$. Choosing Γ in $\Delta(z)$, we note that it contains the points $w = \alpha z$ and $w = \beta z$ in its interior, since $|\beta z| = \alpha^{-1}|z| < \alpha|z| \leq \alpha^m|z| < |w|\ \forall w \in \Gamma$. Applying the corollary, we thus have, for $m = 1, 2, \ldots$:

$$f_{m+1}(z) = \frac{1}{2i\pi}\oint_\Gamma f_m(w)f_1(z/w)\cdot\frac{dw}{w} = \frac{1}{2i\pi}\oint_\Gamma 5^{-\frac{1}{2}}f_m(w)\{(w-\alpha z)^{-1} - (w-\beta z)^{-1}\}dw,$$

which reduces to the elegant recursion

$$f_{m+1}(z) = 1/\sqrt{5}\{f_m(\alpha z) - f_m(\beta z)\}, \tag{27}$$

which is actually valid for all $z \in D_4$, $m = 0, 1, 2, \ldots$.

Of course, (27) may readily be derived using more elementary techniques, but the item of interest here is the method by which it was derived. Without too much difficulty, induction may be used on (27) to derive the partial fraction decomposition of $f_m(z)$, which is given by:

$$f_m(z) = 5^{-\frac{1}{2}m}\sum_{k=0}^{m}(-1)^k\binom{m}{k}(1-\alpha^{m-k}\beta^k z)^{-1}. \tag{28}$$

This is a variant of a result in [4].

Example 5: Recall the generating function of the Legendre polynomials,

$$f(z) = f(z,t) = (1 - 2tz + z^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} P_n(t)z^n. \tag{29}$$

The radius of convergence of this series depends on t and complicates the subsequent computations. Therefore, we will assume that the indicated operations, throughout this example, are legitimate and we will not attempt to justify them. It may be shown that, for appropriately chosen t and z, full rigor may be obtained. Let

$$g(z) = e^z = \sum_{n=0}^{\infty}\frac{z^n}{n!}. \tag{30}$$

Then, for an appropriately chosen contour Γ, the corollary implies that

$$h(z) \equiv \sum_{n=0}^{\infty} P_n(t)\frac{z^n}{n!} = \frac{1}{2i\pi}\oint_\Gamma \frac{e^w dw}{(w^2 - 2tzw + z^2)^{\frac{1}{2}}}. \tag{31}$$

In order to evaluate the last integral, we make the substitution $w = tz + \frac{1}{2}z(1-t^2)^{\frac{1}{2}}(\xi - \xi^{-1})$. For suitably chosen t, z, and Γ, this mapping transforms Γ into another simple closed contour Γ' with sufficiently desirable properties. Proceeding formally, we obtain, after some simplification,

$$h(z) = \frac{1}{2i\pi}\oint_{\Gamma'} e^{tz}\exp\{\tfrac{1}{2}z(1-t^2)^{\frac{1}{2}}(\xi - \xi^{-1})\}\frac{d\xi}{\xi}. \tag{32}$$

The quantity e^{tz} may be factored out of the integrand and the remaining expression, compared (again) with (19), yields the following identity:

$$\sum_{n=0}^{\infty} P_n(t)\frac{z^n}{n!} = e^{tz}J_0(z\sqrt{1-t^2}). \tag{33}$$

The last identity is actually valid for all finite z, since the right-hand member of (33) is clearly an entire function. The identity in (33) is indicated in [5], along with the comment that it is of unknown origin.

The foregoing examples adequately illustrate the applicability of the theorem and its corollary, to obtain a closed form expression for $h(z)$. This may be immediately obvious, or may require an appropriate transformation and/or recognition of known relations, as the previous examples illustrate. If this is possible (and this may not always be the case), a certain degree of ingenuity is required to hit upon the proper transformation. With sufficient imagination and industry, the interested reader will discover other relations of the types illustrated above. The aim of this paper was to obtain a solution of Gould's problem, in closed form or otherwise, and this has been accomplished by the theorem and its corollary.

REFERENCES

1. H. W. Gould. "Some Combinatorial Identities of Bruckman—A Systematic Treatment with Relation to the Older Literature." *The Fibonacci Quarterly* 10, No. 6 (1972):625-626.
2. Paul S. Bruckman. "On Generating Functions with Composite Coefficients." *The Fibonacci Quarterly* 15, No. 3 (1977):269-275.

3. E. T. Whittaker and G. N. Watson. *A Course of Modern Analysis.* Cambridge, 1948, p. 380.
4. John Riordan. "Generating Functions for Powers of Fibonacci Numbers." *Duke Math. J.* 29 (1962):5-12.
5. Earl D. Rainville. *Special Functions.* Chelsea, 1960, p. 165.

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A MISCELLANY OF 1979 CURIOSA

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(A) The digital root of 1979 is 8, which also is the sum of the two absent odd digits, 3 and 5. Otherwise, $F_4 + F_5 = F_6$.

$1 \cdot 9 \cdot 7 \cdot 9 = 567$, three consecutive digits in ascending order.

$1^9 \cdot 7^9 = 40353607$, which contains five consecutive digits.

1979 is a cyclic compression of two palindromes—the composite 979 (= 11 • 89) and the prime 919.

(B) $1979_{10} = 118E_{12} = 153X_{11} = 2638_9 = 3673_8 = 5525_7 = 13055_6$
$= 3044_5 = 132323_4 = 2201022_3 = 11110111011_2.$

In base four, the integer is almost smoothly undulating. In base three, the palindromic integer contains the three distinct digits in that base. In base two, the groups of 1's form a decreasing sequence.

(C)
$$\begin{aligned}1979 &= (11)(11)(11) + [(111 - 1)/(1 + 1) - 1](11 + 1)\\ &= 2222 - 222 - 22 + 2/2\\ &= (333 - 3)(3!) - 3/3\\ &= 4(444 + 44 + 4 + 4) - 4 - 4/4\\ &= 5 \cdot 5 \cdot 5 \cdot 5 + 5 \cdot 5 \cdot 5 \cdot 5 + 555 + 5 \cdot 5 \cdot 5 + 55 - 5 - 5/5\\ &= 6 \cdot 6 \cdot 6 \cdot 6 + 666 + 6 + 6 + 6 - 6/6\\ &= 7 \cdot 7 \cdot 7 \cdot 7 - 7 \cdot 7 \cdot 7 - 77 - 7/7 - 7/7\\ &= 888 + 888 + 88 + 88 + 8 + 8 + 8 + 8/8 + 8/8 + 8/8\\ &= (9999 - 999)/9 + 999 - 9 - 9 - 9/9 - 9/9\end{aligned}$$

(D) $1 + 9 + 7 + 9 = 26$
$19 + 97 + 79 + 91 = 286$
$197 + 979 + 791 + 919 = 2886$
$1979 + 9791 + 7919 + 9197 = 28886$

(E) Here are several of the ways that 1979 can be written using conventional mathematical symbols and one 1, nine 9's, seven 7's, and nine 9's.

$$\begin{aligned}1979 &= 1(999 + 9997/9997) + 9(99 + 779/779) + 7(9 + 9/9) + 9\\ &= 1(999 + 9/9) + 9(99 + 9/9) + 7(9 + 9/9) + 9(99777/99777)\\ &= 19(99 + 99999/99999) + 7(\sqrt{9}\sqrt{9} + 7779/7779) + 9\\ &= 197(9 + 777/777) + \sqrt{9}\sqrt{9}(9999999/9999999)\\ &= 1(999 + 9/9) + \sqrt{9}\sqrt{9}(99 + 7/7) + 7(77/77 + 9) + 9 + 9(999 - 999)\end{aligned}$$

In the last expression, the digit groups are intact and in the order of occurrence in 1979.

(F) 19 • 79 = 1501 is one of eleven composite integers between the primes 1499 and 1511. Consequently, it is the corner element of the following third-order magic square composed of composite elements and having a magic constant of $4512 = 2 \cdot 47 \cdot 48 = 2^5 \cdot 3 \cdot 47$.

1501	1506	1505		$19 \cdot 79$	$2 \cdot 3 \cdot 251$	$5 \cdot 7 \cdot 43$
1508	1504	1500	or	$2^2 \cdot 13 \cdot 29$	$2^5 \cdot 47$	$2^2 \cdot 3 \cdot 5^3$
1503	1502	1507		$3^2 \cdot 167$	$2 \cdot 751$	$11 \cdot 137$

(G)
$$\begin{aligned}1979 &= 1979 + 1 + \sqrt{9} - 7 + \sqrt{9}\\ &= 1979(-1\sqrt{9} + 7 - \sqrt{9})\end{aligned}$$

(continued)