SECONDARY FIBONACCI SEQUENCES

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1. PRELUDE: AN ENTERTAINMENT

DRAMATIS PERSONAE: Leonardo of Pisa and Édouard Lucas, well-known experts in the mathematics of deterministic modelling of the growth of animal populations.

SCENE: Circa fall 1974; lounge of a computing center, where they are whiling away the time as they wait for the number-cruncher to test their recent conjectures.

LEONARDO: I've made some new discoveries in the realm of our favorite common pastime, the Fibonacci numbers (as you so flatteringly refer to them), together with some conjectures I can't yet settle. Maybe we can go over it all together and see what we can come up with.

ÉDOUARD: Splendid! And what have you to reveal?

L: Let me fill you in on the background first. One fine day last summer I was hitchhiking. A ride with a boorish driver precluded good conversation; so, left to my own devices, I let my mind drift to mathematical games for amusement.

É: I suppose that is conclusive proof that you are mathematically inclined, for who else would choose such a pastime?

L: Who knows? At any rate, Fibonacci numbers are ideally suited for such sport, as you well know. By chance, I happened to add 13 and 987, and got 1000.

É: In other words, F_7 and F_{16} = 1000—OK, what's remarkable about that?

L: As you well know, Édouard, the journey to mathematical discovery often starts with noticing something unusual, however small and insignificant it may seem to be. I paused to consider the roundness of the sum, and then proceeded to wonder: if $F_7 + F_{16} = 1000$, then $F_8 + F_{17} = ?$

É: Well, 21 + 1597 = 1618. A fine coincidence! Those are the first four digits of the golden ratio $\phi = 1.618\ldots$.

L: I tried further pairs—let me summarize the results for you on the blackboard:

$$F_5 + F_{14} = 5 + 377 = 382 \qquad \phi^{-2} = .3819\ldots$$
$$F_6 + F_{15} = 8 + 610 = 618 \qquad \phi^{-1} = .618\ldots$$
$$F_7 + F_{16} = 13 + 987 = 1000 \qquad \phi^{0} = 1.$$
$$F_8 + F_{17} = 21 + 1597 = 1618 \qquad \phi^{1} = 1.618\ldots$$
$$F_9 + F_{18} = 34 + 2584 = 2618 \qquad \phi^{2} = 2.618\ldots$$

By forming $F_n + F_{n+9}$, I was getting 1000 times the three-decimal-place approximation of ϕ^{n-1}.

É: Not only that, but your new sequence was also a Fibonacci sequence.

L: Exactly! That was my next observation, and it seemed the more important property to investigate, since it seemed more susceptible of generalization. And generalize it does! Adding any Fibonacci sequence to itself at a constant index difference always produces another Fibonacci sequence. I decided to call the Fibonacci sequences generated in this fashion *secondary Fibonacci sequences*. I started investigating which Fibonacci sequences come out as secondary sequences. Let me show you.

É: Say, that's really interesting! But hold on a moment—let's use the blackboard to make a "formal" record of our brainstorming. After all, if this discussion amounts to anything, you should write a paper about it for the *Fibonacci Quarterly*.

L: I suppose you're right, but writing it all up in a paper isn't nearly as much fun as discovering it all in the first place. In fact, I hate writing papers—all that writing, rewriting, and rewriting again, and I'm still never satisfied with the final product. Besides, I expect you'll be making some contributions in the course of the discussion. If they prove valuable, you write the whole thing up. You're always dashing off mathematical notes and popular articles all over the place. For you this would be just another half-day's work.

É: It's hardly time to argue about that just yet! We'll cross that bridge if we come to it. Meanwhile, let's keep a record anyhow. Now tell me, which Fibonacci sequences turn up as secondary sequences, anyhow? . . .

2. BASIC PROPERTIES OF SECONDARY FIBONACCI SEQUENCES

Definition: A (positive)(Fibonacci) sequence $\{T_n\}_{n\in\mathbb{Z}}$, or T for short, is a doubly infinite sequence which satisfies the recursion relation

$$T_{n+1} = T_n + T_{n-1}$$

for all $n \in \mathbb{Z}$ and for which there is an $n_0 \in \mathbb{Z}$ such that all terms with index greater than n_0 are positive. We will be especially interested in integer sequences.

By definitional convention we are excluding from consideration the constant zero sequence, as well as sequences which are negative for every index exceeding a certain integer.

Proposition 1: Any Fibonacci sequence contains a unique pair of consecutive terms a and b, both positive, with either $a = b$ or $2a < b$.

Proof: See [1, p. 43].

Definition: A Fibonacci sequence T is in *standard format* if it is labelled so that $T_1 = a$, $T_2 = b$, with a and b as specified in Proposition 1. We will write $T = (a,b)$. A sequence for which a and b are relatively prime integers is said to be *primitive*. Two special sequences are distinguished: *the Fibonacci sequence* $F = (1,1)$ and the *Lucas sequence* $L = (1,3)$. As is well known, the terms of a sequence T in standard format are given by

$$T_n = aF_{n-2} + bF_{n-1}$$

for any $n \in \mathbb{Z}$.

Definition: Two sequences T, U are *equal*, written $T = U$, if $T_n = U_n$ for every $n \in \mathbb{Z}$. They are *equivalent*, written $T \equiv U$, if there is a k such that $T_n = U_{n+k}$ for all $n \in \mathbb{Z}$.

Definition: The *Lucas analogue* $V(T)$ of a Fibonacci sequence T is the sequence defined by

$$V(T)_n = T_{n+1} + T_{n-1},$$

and $V(T)$ may be denoted simply by V when no confusion would result. Note that V may fail to be in standard format.

Proposition 2: (i) $V(T)$ is a Fibonacci sequence; (ii) $V(V(T)) = 5T$; (iii) $V(F) = L$.

Proof: Left to the reader.

We now generalize the notion of the Lucas analogue of a sequence to embrace a whole family of sequences.

Definition: For $r > 0$, the rth *secondary sequence* of a sequence T, denoted ${}^rT$, is the sequence obtained by adding T to itself at a constant index difference r:

$${}^rT_n = T_{n+r} + T_n.$$

We will say that ${}^rT$ is *r-secondary from* T. Note that $V(T)$ is not, strictly speaking, a secondary sequence, though $V_n = {}^2T_{n-1}$ makes $V \equiv {}^2T$.

Proposition 3: A secondary sequence of a Fibonacci sequence is a Fibonacci sequence.

Proof: $${}^rT_n + {}^rT_{n-1} = (T_{n+r} + T_n) + (T_{n-1+r} + T_{n-1}) = (T_{n+r} + T_{n-1+r}) + (T_{n+r} + T_n)$$
$$= T_{n+1+r} + T_{n+1} = {}^rT_{n+1}.$$

We give here in table form the first twelve secondary sequences of F (taken from [11, p. 17]), in hopes of inspiring the reader to discover patterns before reading further.

F	$r =$ 1	2	3	4	5	6
0	1	1	2	3	5	8
1	2	3	4	6	9	14
1	3	4	6	9	14	22
2	5	7	10	15	23	36
3	8	11	16	24	37	58
5	13	18	26	39	60	94

F	r =	1	2	3	4	5	6
8		21	29	42	63	97	152
13		34	47	68	102	157	246

F	r =	7	8	9	10	11	12
-1		4	7	12	20	33	54
1		9	14	22	35	56	90
0		13	21	34	55	89	144
1		22	35	56	90	145	234
1		35	56	90	145	234	378
2		57	91	146	235	379	612
3		92	147	236	380	613	990
5		149	238	382	615	992	1602

The sequence $F \equiv {}^1F$ has often been cited as occurring in nature, and the occurrence of $L \equiv {}^2F$ is occasionally mentioned as well (see, e.g., [7, pp. 81-82]). What may perhaps be surprising is that $2F \equiv {}^3F$, $3F \equiv {}^4F$, and $(1,5) \equiv {}^5F$ have been observed as the parameters of sunflowers grown by Don Crowe, a geometer at the University of Wisconsin [5].

Our indexing of secondary sequences was arbitrary. Generally, a secondary sequence is not in standard format, and it is necessary to "backspace" by several index numbers to arrive at standard format. It turns out to be important for our purposes to keep trace of the indexing—if it were not, we could conveniently identify all equivalent sequences. In Section 4 we will specify exactly the amount of backspace for each secondary sequence.

Proposition 4:

$$ {}^{2t}T_n = \begin{cases} F_t \cdot V_{n+t}, & t \text{ odd} \\ L_t \cdot T_{n+t}, & t \text{ even} \end{cases} $$

Proof: The first proof of these well-known identities seems to be due to Tagiuri [12], according to Dickson [6, p. 404]. Both are cited by Horadam [8], who furnishes a more accessible proof. In any case, the proof is straightforward, and we leave it to the reader rather than reproduce it here.

It is not possible to find so simple an expression for ${}^rT$ when r is odd.

Definition: The *conjugate* $\overline{T}$ of a sequence T in standard format is the sequence defined by

$$ \overline{T}_n = \begin{cases} (-1)^n T_{-n}, & T \not\equiv F \\ T_n = (-1)^{n+1} T_{-n}, & T \equiv F \end{cases} $$

For a sequence T not in standard format, let $T_m = U_{n+k}$, U in standard format. Then define $\overline{T}_n = \overline{U}_{n+k}$. Note that $\overline{F} = F$, $\overline{L} = L$, and no other primitive sequence is self-conjugate.

Proposition 5: (i) $\overline{T}$ is a Fibonacci sequence; (ii) $\overline{\overline{T}} = T$; (iii) $\overline{V(T)} \equiv \overline{{}^2T} \equiv {}^2\overline{T} \equiv V(\overline{T})$;

(iv) $$ {}^{2t}\overline{T} \equiv \begin{cases} F_t \cdot {}^2\overline{T}, & t \text{ odd} \\ L_t \cdot \overline{T}, & t \text{ even} \end{cases} $$; (v) $\overline{{}^{2t}T} \equiv {}^{2t}\overline{T}$.

Proof: Left to the reader.

Theorem 1: Let S and T be Fibonacci sequences. If $S \equiv {}^rT$, then

$$ \text{for } r \text{ odd: } {}^r\overline{S} \equiv L_r \cdot \overline{T} $$

$$ \text{for } r \text{ even, } r = 2t: {}^rS \equiv (L_r + 2)T \equiv \begin{cases} 5F_t^2 \cdot T, & t \text{ odd} \\ L_t^2 \cdot T, & t \text{ even} \end{cases} $$

Proof: r odd. We do the case $S \not\equiv F$, $T \not\equiv F$, S in standard format. For n even,

$$ \begin{aligned} {}^r\overline{S}_n &= \overline{S}_{n+r} + \overline{S}_n = (-1)^{n+r} S_{-n-r} + (-1)^n S_{-n} \\ &= -S_{-n-r} + S_{-n} = -(T_{-n+r} + T_{-n}) + (T_{-n} + T_{-n-r}) \\ &= T_{-n+r} - T_{-n-r} + L_r T_{-n}, \text{ by Proposition 4} \\ &= L_r \cdot (-1)^n T_{-n} = L_r \cdot \overline{T}_n. \end{aligned} $$

The proof for n odd is analogous, as are the proofs for the other cases.

$r = 2t$.

$$ {}^{2t}S = {}^{2t}({}^{2t}T) \equiv \begin{cases} {}^{2t}(F_t \cdot V) \equiv F_t^2 V(V(T)) \equiv 5F_t^2 \cdot T, & t \text{ odd} \\ {}^{2t}L_t \cdot T \equiv L_t^2 \cdot T, & t \text{ even} \end{cases} $$

Example: $T = (1,7)$, $S = {}^9T \equiv 2(11,36)$. Then $\overline{T} = (5,11)$, $\overline{S} = 2(14,39)$, ${}^9\overline{S} \equiv 76(5,11) = L_9 \cdot \overline{T}$.

Proposition 6: ${}^3T \equiv 2T$.

Proof: $T_1 + T_4 = a + (a + 2b) = 2(a + b)$, $T_2 + T_5 = b + (2a + 3b) = 2(a + 2b)$.

The results of the theorem suggest the definition of an inverse to the operation ${}^r(\ )$ of taking the rth secondary sequence of a sequence T.

Definition: ${}^{1/r}T = \overline{{}^r\overline{T}}/[L_r + 1 + (-1)^r]$, with the terms of ${}^{1/r}T$ being allowed to be fractional. Note that ${}^{1/2t}T = T/(L_{2t} + 2)$, by Proposition 5.

Proposition 7: (i) ${}^{1/r}T$ is a Fibonacci sequence; (ii) ${}^{1/r}({}^rT) \equiv {}^r({}^{1/r}T) \equiv T$; (iii) Up to equivalence, ${}^{1/r}T$ is the only sequence whose rth secondary sequence is T.

Proof: (i) Neither $(\overline{\ })$ nor ${}^r(\ )$ disturbs the recursion relation.

(ii) r odd.

$$ {}^{1/r}({}^rT) \equiv \frac{\overline{{}^r\overline{T}}}{L_r} \quad \frac{\overline{L_r\overline{T}}}{L_r} \equiv T, \text{ by Theorem 1.} $$

$$ {}^r({}^{1/r}T) \equiv {}^r\left(\frac{\overline{{}^r\overline{T}}}{L_n}\right) \equiv \overline{{}^r\left(\frac{\overline{{}^r\overline{T}}}{L_n}\right)}, \text{ which by the line above is just } T. $$

$r = 2t$, t odd.

$$ {}^{1/2t}({}^{2t}T) \equiv \overline{{}^{2t}\overline{{}^{2t}T}}/(L_{2t} + 2) \equiv \overline{{}^{2t}\overline{F_t \cdot V}}/(L_{2t} + 2) \equiv \frac{\overline{5F_t^2 \cdot \overline{T}}}{F_t^2} \equiv T $$

$$ {}^{2t}({}^{1/2t}T) \equiv {}^{2t}\left(\overline{{}^{2t}\overline{T}}/(L_{2t} + 2)\right) \equiv {}^{2t}(F_t\overline{V})/(L_{2t} + 2) \equiv {}^{2t}(F_t V)/(5F_t^2) $$

$$ \equiv F_t^2 \cdot 5T/5F_t^2 \equiv T $$

$r = 2t$, t even.

$$ {}^{1/2t}({}^{2t}T) \equiv \overline{{}^{2t}\overline{{}^{2t}T}}/(L_{2t} + 2) \equiv \overline{{}^{2t}\overline{L_t \cdot T}}/L_t^2 = \overline{L_t \cdot L_t \cdot \overline{T}}/L_t^2 \equiv T $$

$$ {}^{2t}({}^{1/2t}T) \equiv {}^{2t}(\overline{{}^{2t}\overline{T}})/(L_{2t} + 2) \equiv L(\overline{L_t \cdot \overline{T}})/L_t^2 \equiv T $$

(iii) Suppose ${}^rS \equiv {}^rS' \equiv T$. Then ${}^{1/r}({}^rS) \equiv {}^{1/r}({}^rS') \equiv {}^{1/r}T$, so that $S \equiv S' \equiv {}^{1/r}T$.

Example: ${}^5(1,7) \equiv (10,29)$, ${}^{1/5}(10,29) \equiv (1,7)$, ${}^{1/7}(10,29) \equiv \left(\frac{79}{29}, \frac{184}{29}\right)$.

A major effort of the remainder of the paper is to determine exactly what integer sequences are secondary from other integer sequences.

3. STANDARD-FORMATTING SECONDARY SEQUENCES

Definition: Let I be the 2×2 identity matrix and let

$$ P = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} $$

Note that if T is a Fibonacci sequence, then

$$ (T_{n-1}, T_n) \cdot P = (T_n, T_{n+1}), $$

where the ordered pairs are considered as 1×2 matrices. Also,

$$ P^m = \begin{pmatrix} F_{m-1} & F_m \\ F_m & F_{m+1} \end{pmatrix}. $$

Theorem 2: Let $S = (c,d)$ and $T = (a,b)$ be in standard format. Then some multiple of an equivalent of S is secondary from T if and only if there exist positive integers λ, r and a nonnegative $m \le r + 1$ such that one of the following equivalent conditions holds:

(i) $\lambda(c,d)P^m = (a,b)(I + P^r)$

(ii) $\lambda c = (-1)^m[(-F_m T_{r+2} + F_{m+1}T_{r+1}) + (-F_m b + F_{m+1}a)]$

$\lambda d = (-1)^m[(-F_m T_{r+1} + F_{m-1}T_{r+2}) + (-F_m a + F_{m-1}b)]$

(iii) $\lambda c = T_{r-m+1} + (-1)^m (aF_{m+1} - bF_m)$

$\lambda d = F_{r-m+2} + (-1)^m (-aF_m + bF_{m-1})$

(iv) $\lambda c = a[F_{r-m-1} + (-1)^m F_{m+1}] + b[F_{r-m} - (-1)^m F_m]$

$\lambda d = a[F_{r-m} - (-1)^m F_m] + b[F_{r-m+1} + (-1)^m F_{m-1}]$

Proof: If the relation (i) holds, it exhibits a λ-multiple of an equivalent of S as a secondary sequence of T.

Conversely, suppose some multiple, say by λ, of an equivalent of S arises as a secondary sequence, say the rth, of T. Then $({}^rT_1, {}^rT_2) = \lambda \cdot (S_{m+1}, S_{m+2})$ for some m. But

$$(a,b)(I + P^r) = ({}^rT_1, {}^rT_2) = \lambda(S_{m+1}, S_{m+2}) = (c,d)P^m.$$

The quantity m represents the number of places it is necessary to backspace $({}^rT_1, {}^rT_2)$ to arrive at standard format. We must show that $0 \le m \le r + 1$.

Since T is in standard format, $0 < T_1 \le T_2 < T_n < T_{n+1}$, for $n > 2$, so $0 < {}^rT_1 = T_1 + T_{r+1} \le {}^rT_2 = T_2 + T_{r+2}$, for $r > 0$. Hence, $m \ge 0$.

To see that $m \le r + 1$, backspace $(r+2)$ places: $({}^rT_1, {}^rT_2)P^{-r-2} = (T_{-r-1} + T_{-1}, T_{-r} + T_0)$. Since T is in standard format and $r > 0$, exactly one of T_{-r-1} and T_{-r} is negative. If $T \not\equiv F$: r even makes $T_{-r-1} + T_{-1}$ negative, while r odd and $r \ge 3$ forces $T_{-r} + T_0$ negative; $r = 1$ yields $T_{-r-1} + T_{-1} > T_{-r} + T_0 > 0$. In any case, we have certainly backspaced too far. The case for $T \equiv F$ is analogous.

(i) ⇒ (ii). $\lambda(c,d)P^m = (\lambda c, \lambda d)\begin{pmatrix} F_{m-1} & F_m \\ F_m & F_{m+1} \end{pmatrix}$, while $(a,b)(I + P^r)$

$$= (a,b)\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} F_{r-1} & F_r \\ F_r & F_{r+1} \end{pmatrix}\right] = [(1 + F_{r-1})a + bF_r,\ aF_r + b(1 + F_{r-1})].$$

Now, (i) asserts that these quantities are equal, so Cramer's Rule yields

$$\lambda c = \begin{vmatrix} T_{r+1} + a & F_m \\ T_{r+2} + b & F_{m+1} \end{vmatrix} \Big/ \Delta \qquad \lambda d = \begin{vmatrix} F_{m-1} & T_{r+1} + a \\ F_m & T_{r+2} + b \end{vmatrix} \Big/ \Delta$$

$$\Delta = \begin{vmatrix} F_{m-1} & F_m \\ F_m & F_{m+1} \end{vmatrix} = F_{m-1}F_{m+1} - F_m^2 = (-1)^m$$

and (ii) follows.

(ii) ⇒ (iii). This implication is based on the reduction

$$\begin{aligned} -F_m T_{r+2} + F_{m+1} T_{m+2} &= -F_m(T_{r+1} + T_r) + (F_m + F_{m-1})\, T_{r+1} \\ &= -F_m T_{r+1} - F_m T_r + F_m T_{r+1} + F_{m-1} T_{r+1} \\ &= F_{m-1} T_{r+1} - F_m T_{r-1} \\ &= (-1)(-F_{m-1} T_{r+1} + F_m T_r). \end{aligned}$$

Repetition for a total of m times yields

$$-F_m T_{r+2} + F_{m+1} T_{r+1} = (-1)^m (-F_0 T_{r+2-m} + F_1 T_{r+1-m}) = (-1)^m T_{r+1-m}.$$

Thus,

$$\lambda c = (-1)^m [(-1)^m T_{r+1-m} + aF_{m+1} - bF_m] = T_{r+1-m} + (-1)^m (aF_{m+1} - bF_m).$$

A similar argument gives the corresponding expression for λd.

(iii) ⇒ (iv). The equations in (iv) can be obtained directly from (iii) by use of the identity

$$T_{n+2} = aT_n + bT_{n+1}.$$

(iv) ⇒ (i). We verify the first coordinate of the matrix equation (i) using the substitution (iv). The first coordinate of $\lambda(c,d)P^m$ is

$$
\begin{aligned}
&\{a[F_{r-m-1} + (-1)^m F_{m+1}] + b[F_{r-m} - (-1)^m F_m\} \cdot F_{m-1} \\
&\qquad + \{a[F_{r-m} - (-1)^m F_m] + b[F_{r-m+1} + (-1)^m F_{m-1}\} \cdot F_m \\
&= a[F_{r-m}F_{m-1} + (-1)^m F_{m+1}F_{m-1} + F_{r-m}F_m - (-1)^m F_m^2] \\
&\qquad + b[F_{r-m}F_{m-1} - (-1)^m F_m F_{m-1} + F_{r-m+1}F_m + (-1)^m F_{m-1}F_m] \\
&= a(F_{r-m-1}F_{m-1} + F_{r-m}F_m + 1) + b(F_{r-m}F_{m-1} + F_{r-m+1}F_m) \\
&= a + F_m(aF_{r-m} + bF_{r-m+1}) + F_{m-1}(aF_{r-m-1} + bF_{r-m}) \\
&= T_1 + F_m T_{r-m+2} + F_{m-1}T_{r-m+1} \\
&= T_1 + (F_{m-1} + F_{m-2})T_{r-m+2} + F_{m-1}T_{r-m+1} \\
&= T_1 + F_{m-1}T_{r-m+3} + F_{m-2}T_{r-m+2}.
\end{aligned}
$$

The last three lines comprise a reduction, which iterated for a total of $(m-1)$ times yields

$$T_1 + F_1 T_{r-m+2+m-1} + F_0 T_{r-m+1+m-1} = T_1 + T_{r+1},$$

the first coordinate of $(a,b)(I + P^r)$.

Corollary: Some multiple of an equivalent of a sequence $S = (c,d)$ in standard format is secondary from F if and only if there exist positive integers λ, r and nonnegative $m \le r + 1$ such that

$$\lambda c = F_{r+1-m} + (-1)^m F_{m-1}$$

$$\lambda d = F_{r+2-m} - (-1)^m F_{m-2}$$

Examination of the equations of the corollary makes it clear that stronger constraints operate on m than just $0 \le m \le r + 1$. In the next section we pin m down precisely.

4. BACKSPACE OF A SECONDARY SEQUENCE

Throughout this section, m will denote the backspace necessary to bring the sequence ${}^rT$ into standard format, where $T = (a,b)$ is in standard format and primitive.

Definition: The *eccentricity* of a sequence $T = (a,b)$ in standard format is the quantity $\varepsilon = b/a$.

Proposition 8: For $r = 2$, $m = \begin{cases} 0, & \varepsilon > 3 \\ 1, & T = F \text{ or } T = L \\ 2, & 2 < \varepsilon < 3 \end{cases}$

For $r = 3$, $m = 2$.

Proof: The terms ${}^2T_{-2}, \ldots, {}^2T_2$ are given, respectively, by $3b - 4a$, $3a - b$, $2b - a$, $2a + b$, and $a + 3b$.

If $3a < b$, $2(2a + b) = 4a + 2b < a + 3b$, so $m = 0$.

If $2a < b < 3a$, $3a - b > 0$ and $2(3a - b) = 6a - 2b < 3b - 2b = b < b + (b - 2a) = 2b - 2a < 2b - a$, so $m = 2$.

The reader may confirm that $m = 1$ for $T = F$ and $T = L$. The case $r = 3$ has already been settled (implicitly) by Proposition 6.

Theorem 3: For $r = 2t$,

$$m = \begin{cases} t - 1, & \varepsilon > 3 \text{ and } t \text{ odd} \\ t, & t \text{ even or } T = F \text{ or } T = L \\ t + 1, & 2 < \varepsilon < 3 \text{ and } t \text{ odd} \end{cases}$$

Proof: From Proposition 4, Proposition 8, and the fact that $V_n = {}^2T_{n-1}$.

As we would expect by now, the case of r odd offers greater challenge and, as it turns out, some surprises.

Theorem 4: For $r > 3$, r odd, $r = 4k + 1$ or $r = 4k + 3$, $k \ge 1$. Define $A_r = F_{2k+2} - F_{r-2k-2}$, $B_r = F_{r-2k-3} + F_{2k+3}$, and $\alpha_r = B_r/A_r$. Then

$$m = \begin{cases} 2k, & \varepsilon < \alpha_r \\ 2k + 1, & \varepsilon = 1 \text{ or } \varepsilon = \alpha_r \\ 2k + 2, & 2 < \varepsilon < \alpha_r \end{cases}$$

Proof: Exclude at first the possibilities $T = F$ or ${}^rT \equiv \lambda F$. We examine the case of m even, arriving at the results of the theorem; then we show that m cannot be odd. Finally, we readmit F to the arena and distinguish cases to arrive at the remaining clause of the theorem, which allows for odd m.

Case I: $T \neq F$, ${}^rT \not\equiv F$. The equations (iv) of Theorem 2 give an exact expression for $T = (\lambda c, \lambda d)$ in standard format. Proposition 1 reminds us of the conditions λc and λd must satisfy in the event that ${}^rT$ is not equivalent to F:

$$\lambda c > 0, \text{ or}$$

(*) $$a[F_{r-m-1} + (-1)^m F_{m+1}] + b[F_{r-m} - (-1)^m F_m] > 0$$

$$2\lambda c < \lambda d, \text{ or}$$

(**) $$2\Big[a(F_{r-m-1} - (-1)^m F_{m+1}) + b(F_{r-m} - (-1)^m F_m)\Big] < a[F_{r-m} - (-1)^m F_m] + b[F_{r-m+1} + (-1)^m F_{m-1}].$$

Subcase, m even. Let $i = m - 2k$, so that $m = 2k + i$, i even. Since $0 \le m \le r + 1$, $-2k \le i \le r - 2k + 1$. Equations (*) and (**) now take the forms

(*e) $$a(F_{r-2k-i-1} + F_{2k+i+1}) > b(F_{2k+i} - F_{r-2k-i})$$

(**e) $$a(F_{2k+i+3} + F_{r-2k-i-3}) < b(F_{2k+i+2} - F_{r-2k-i-2})$$

If $r - 2k + 1 \ge i \ge 2$, then $-1 \le r - 2h - i \le 4h + 3 - 2k - i \le 2k - i + 3 \le 2k + 1 < 2k + i$, and $2k + i \ge 4$, so the R.H.S. of (*e) is positive. Also, $T \neq F$ implies $2a < b$. Consequently,

$$a(F_{r-2k-i} + F_{2k+i+1}) > b(F_{2k+1} - F_{r-2k-i}) > 2a(F_{2k+i} - F_{r-2k-i})$$

and hence

$$F_{r-2k-i-1} + F_{2k+i+1} > 2(F_{2k+i} - F_{r-2k-i})$$

or, after simplification and use of the recurrence relation, $F_{r-2k-i+2} > F_{2k+i-2}$.

The subscripts are positive, so we must have $r - 2k - i + 2 > 2k + i - 2$ or $2i(r - 4k) + 4 < 7$, or $i < 7/2$. By hypothesis, i is positive and even, so $i = 2$ and $m = 2k + 2$.

If $-2k \le i \le 0$, then $2k + i + 3 \ge 3$ and $r - 2k - i - 3 \ge r - 2k - 3 = (4k + 1) - 2k - 3 = 2k - 2 \ge 0$, so the L.H.S. of (**e) is positive. As a result, the R.H.S. must also be positive, yielding $F_{2k+i+2} > F_{r-2k-i-2}$. The subscripts are positive, so we must have $2k + i + 2 > r - 2k - i - 2$, or $2i > (r - 4k) - 4 > -3$, or $i > -3/2$. By hypothesis, i is nonpositive and even, of $i = 0$ and $m = 2k$.

The upshot so far is that if m is even, it can only take on the values stated in the theorem.

In case $m = 2k + 2$, the R.H.S. of (*e) is positive, so dividing both sides by $a(F_{2k+2} - F_{r-2k-2})$ retains the sense of the inequality and yields

$$\varepsilon = b/a < (F_{r-2k-3} + F_{2k+3})/(F_{2k+2} - F_{r-2k-2}) = \alpha_r.$$

In case $m = 2$, the L.H.S. of (**e) is positive, so dividing both sides by $a(F_{2k+2} - F_{r-2k-2})$ retains the sense of the inequality and yields

$$\varepsilon = b/a > (F_{r-2k-3} + F_{2k+3})/(F_{2k+2} - F_{r-2k-2}) = \alpha_r.$$

Subcase, m odd. The equation (**) becomes

(**o) $$2[a(F_{r-m-1} - F_{m+1}) + b(F_{r-m} + F_m)] < a(F_{r-m} + F_m) + b(F_{r-m+1} - F_{m-1}).$$

After simplification and use of the recurrence relation, we have

$$a(F_{m+3} - F_{r-m-3}) > b(F_{r-m-2} + F_{m+2}).$$

Since $r + 1 \ge m \ge 0$ and $r \ge 5$, the R.H.S. is positive. Since $T \neq F$, $b > 2a$, and so

$$a(F_{m+3} - F_{r-m-3}) > 2a(F_{r-m-2} + F_{m+2}) > 0$$

and

$$F_{m+3} - F_{r-m-3} > 2(F_{r-m-2} + F_{m+2}).$$

So after simplification and use of the recurrence, $-F_{m+1} - F_{r-m} > 0$, which is impossible for such positive subscripts.

Case II: $T = F$, ${}^rT \not\equiv \lambda F$. We have $a = b = 1$.

Subcase, m even. Equation (**) becomes $F_{m+3} + F_{r-m-3} < F_{m+2} - F_{r-m-2}$, which gives $F_{m+1} + F_{r-m-1} < 0$, which is impossible for such positive subscripts.

Subcase, m odd. Equations (*) and (**) become $F_{r-m-1} - F_{m+1} + F_{r-m} + F_m > 0$, so $F_{r-m-1} - F_{m-1} > 0$; $F_{m+3} - F_{r-m-3} > F_{r-m-2} + F_{m+2}$, so $F_{m+1} - F_{r-m-1} > 0$. The subscripts being nonnegative, these inequalities require that $m + 1 > r - m - 1$ and $r - m + 1 > m - 1$, or $r/2 - 1 < m < r/2 + 1$. The only integers between the bounds are $(r-1)/2$ and $(r+1)/2$, only one of which is odd. If $r = 4k + 1$, $(r+1)/2 = 2k + 1$ is odd; if $r = 4k + 3$, $(r-1)/2 = 2k + 1$ is odd. In either case, $m = 2k + 1$.

Case III: $T \neq F$, ${}^rT \equiv \lambda F$ for some λ.

Subcase, m even. Here we have now $\lambda c = \lambda d > 0$ and the corresponding substitute for (*) and (**):

$$a(F_{r-m-1} + F_{m+1}) + b(F_{r-m} - F_m) = a(F_{r-m} - F_m) + b(F_{r-m+1} + F_{m-1}) > 0.$$

Simplification gives $a(-F_{r-m-2} + F_{m+2}) = b(F_{r-m-1} + F_{m+1})$, which is positive since the subscripts on the R.H.S. are positive. Using the fact $b > 2a$, and dividing by a, we get $F_{m+2} - F_{r-m-2} > 2(F_{r-m-1} + F_{m+1})$, which leads to the contradiction $-F_{m-1} - F_{r-m+1} > 0$.

Subcase, m odd. The equations of (iv) of Theorem 2 become

$$a(F_{r-m-1} - F_{m+1}) + b(F_{r-m} + F_m) = a(F_{r-m-2} + F_m) + b(F_{r-m-1} + F_{m+1}) > 0.$$

Simplification gives $a(F_{r-m-2} + F_{m+2}) = b(-F_{r-m-1} + F_{m+1})$. The subscripts on the L.H.S. are, respectively, nonnegative (m odd implies $m \leq 2k - 1$) and positive, so that $-F_{r-m-1} + F_{m+1} > 0$; and using the familiar $b > 2a$ and dividing by a in the original inequality gives $F_{r-m-2} + F_{m+2} > 2(F_{m+1} - F_{r-m-1})$. Simplification reduces this to $F_{r-m+1} > F_{m-1}$. We are now in the situation of Case II, m odd, so we may conclude $m = 2k + 1$. Here, $b/a = B_r/A_r$ follows without difficulty.

Case IV: $T = F$, ${}^rT \equiv \lambda F$. We may follow Case III to the points

m even: $a(F_{r-m-2} - F_{m+2}) = b(F_{r-m-1} + F_{m+1})$;

m odd: $a(F_{r-m-2} + F_{m+2}) = b(-F_{r-m-1} + F_{m+1})$.

Here in Case IV we have $a = b = 1$:

m even: $F_{r-m-2} - F_{m+2} = F_{r-m-1} + F_{m+1}$, so $F_m = F_{r-m}$ and either $r = 2m$ (impossible: r is odd); $m = 1$, $r = 3$ (impossible: m is even); or $m = 2$, $r = 3$ (excluded by hypothesis).

m odd: $F_{r-m-2} + F_{m+2} = -F_{r-m-1} + F_{m+1}$, so $F_{r-m} + F_m = 0$, and the restriction $0 \leq m \leq r + 1$ forces the contradiction $m = r = 0$.

Corollary: For $r = 4k + 1$, $k \geq 1$:

$$A_r = 2F_{2k},\ B_r = 2F_{2k} + F_{2k+2},\ \lim_{\substack{r=4k+1\\k\to\infty}} \alpha_r = \frac{\phi + 3}{2} \approx 2.309.$$

For $r = 4k + 3$, $k \geq 1$: $A_r = F_{2k}$, $B_r = F_{2k} + F_{2k+3}$, $\lim_{\substack{r=4k+3\\k\to\infty}} \alpha_r = 2(\phi + 1) \approx 5.236.$

(The number ϕ is the golden ratio.) Moreover, because of the recurrence relation for F, each of the sequences $\{\alpha_{4k+1}\}$, $\{\alpha_{4k+3}\}$ consists of every other term of the respective Farey sequences $\{(2F_n + F_{n+2})/2F_n\}$, $\{(F_n + F_{n+3})/F_n\}$.

Proof:

$$A_{4k+1} = F_{2k+2} - F_{4k+1-2k-2} = F_{2k+2} - F_{2k-1} = F_{2k+1} + F_{2k} - F_{2k-1} = 2F_{2k}.$$

$$\begin{aligned} B_{4k+1} &= F_{4k+1-2k-3} + F_{2k+3} = F_{2k-2} + F_{2k+2} + F_{2k+1} = 2F_{2k+1} + F_{2k} + F_{2k-2} \\ &= 3F_{2k} + 2F_{2k-1} + F_{2k-2} = 2F_{2k} + 2F_{2k} + F_{2k-1} = 2F_{2k} + F_{2k+2}. \end{aligned}$$

$$\lim_{k\to\infty} \alpha_{4k+1} = \lim_{k\to\infty}(2F_{2k} + F_{2k+2})/2F_{2k} = 1 + \tfrac{1}{2}\phi^2 = (\phi + 3)/2.$$

$$A_{4k+3} = F_{2k+2} - F_{4k+3-2k-2} = F_{2k+2} - F_{2k+1} = F_{2k}.$$

$$B_{4k+3} = F_{4k+3-2k-3} + F_{2k+3} = F_{2k} + F_{2k+3}.$$

$$\lim_{k\to\infty} \alpha_{4k+1} = \lim_{k\to\infty}(F_{2k} + F_{2k+3})/F_{2k} = 1 + \phi^3 = 2(\phi + 1).$$

(In each case the existence of the limit is guaranteed because the sequence is monotone and bounded.)

We present below a table of the Farey sequences which contain the values α_r. The parenthetical entries, consisting of the values of the Farey sequences intermediate between values α_r, form their own sequence which we shall call β_r:

Definition: $\beta_{4k+1} = (F_{2k-1} + F_{2k+2})/F_{2k-1}$;

$\beta_{4k+3} = (2F_{2k+1} + F_{2k+3})/2F_{2k+1}$.

We even examine what the calculated values of α_r and β_r would be for $r = 3$ and $r = 1$, even though the theorem above does not extend to these.

In fact, we can extend the definition of the α's and β's as follows:

Definition: $\alpha_{2t} = \beta_{2t} = 3$, t odd;

$\alpha_{2t} = 3$, $\beta_{2t} = 2$, t even;

$\alpha_3 = 3$, $\beta_3 = 2$, $\alpha_1 = 2$, $\beta_1 = 2$.

r ...	1	3	5	7	9	11	13	15	17	19	...	
...	$\frac{4}{2}$	$\frac{1}{0}$	$\frac{5}{2}$	$\frac{9}{4}$	$\frac{14}{6}$	$\frac{23}{10}$	$\frac{37}{16}$	$\frac{60}{26}$	$\frac{97}{42}$	$\frac{157}{68}$	...	$\to (\phi + 3)/2$
...	$\frac{2}{1}$	$\frac{2}{0}$	$\frac{4}{1}$	$\frac{6}{1}$	$\frac{10}{2}$	$\frac{16}{3}$	$\frac{26}{5}$	$\frac{42}{8}$	$\frac{68}{13}$	$\frac{110}{21}$	...	$\to 2(\phi + 1)$

Thus, we have the sequences formed as follows, from first element on:

$$\alpha:\ 2,\ 3,\ 3,\ 3,\ \tfrac{5}{2},\ 3,\ \tfrac{6}{1},\ 3,\ \tfrac{14}{6},\ 3,\ \tfrac{16}{3},\ 3,\ \tfrac{37}{16},\ 3,\ \ldots$$

$$\beta:\ 2,\ 3,\ 2,\ 2,\ \tfrac{4}{1},\ 3,\ \tfrac{9}{4},\ 2,\ \tfrac{10}{2},\ 3,\ \tfrac{23}{10},\ 2,\ \tfrac{26}{5},\ 3,\ \ldots$$

The results of this section so far may be summed up in saying that m depends only on r and ε and is uniquely determined once they are specified. The same is true for the quantity d/c. Easy algebra applied to the equations (iv) of Theorem 2 yields a general formula for d/c; we rename this quantity $\delta_r(\varepsilon)$ to indicate the independent variables on which it depends. It is convenient, however, to express it in terms of the variable m also, which itself depends on r and ε.

Proposition 9: The eccentricity $\delta_r(\varepsilon)$ of T^r, where ε is the eccentricity of T, is given by

$$\delta_r(\varepsilon) = \frac{[F_{r-m} - (-1)^m F_m] + [F_{r-m+1} + (-1)^m F_{m-1}]}{[F_{r-m} + (-1)^m F_{m+1}] + [F_{r-m} - (-1)^m F_m]}$$

Conversely,

$$\varepsilon = \frac{\delta_r(t)[F_{r-m} + (-1)^m F_{m+1}] - [F_{r-m} - (-1)^m F_m]}{[F_{r-m+1} + (-1)^m F_{m-1}] - \delta_r(\varepsilon)[F_{r-m} - (-1)^m F_m]}$$

The function δ_r is one-to-one, so that ε in turn is uniquely determined by r and δ_r; in other words, we may speak of the inverse function δ_r^{-1}.

Proof: If $\delta_r(\varepsilon_1) = \delta_r(\varepsilon_2)$, then the corresponding secondary sequences (using left subscripts to distinguish) ${}^r_1T$, ${}^r_2T$ must be equivalent to multiples of the same primitive sequence U, so ${}^r_1T \equiv k_1U$, ${}^r_2T \equiv k_2U$. By Proposition 7(iii), for $i \in \{1,2\}$, ${}^{1/r}({}_iT) \equiv {}^{1/r}(k_iU) = k_i^{1/r}\,U$ is the only sequence, up to equivalence, whose rth secondary sequence is T. But the upshot is that ${}^{1/r}_{\ \ 1}T$ and ${}^{1/r}_{\ \ 2}T$ must be equivalent to multiples of the same primitive sequence T. Hence $\varepsilon_1 = \varepsilon_2$.

Proposition 10: $\varepsilon_{\overline{T}} = 2 + 1/(\varepsilon - 2)$.

Proof: $T_0 = b - a$, $T_{-1} = 2a - b$, $T_{-2} = 2b - 3a$, so $\varepsilon_{\overline{T}} = (2b - 3a)/(b - 2a) = (2\varepsilon - 3)/(\varepsilon - 2) = 2 + 1/(\varepsilon - 2)$.

Theorem 5: For $r = 1$, $r = 3$, or $r \equiv 0 \pmod 4$, $\delta_r(\varepsilon) = \varepsilon$. Otherwise, δ_r maps

$$1 \to \beta_r$$

$$(2,\alpha_r) \to (\beta_r,\infty),\ \text{order-preserving}$$

$$\alpha_r \to 1$$

$$(\alpha_r, \infty) \to (2, \beta_r), \text{ order-preserving}$$

and δ_r is a bijection from $\{1\} \cup (2,\infty)$ into itself.

Proof: For $r \neq 3$, $r \not\equiv 0 \pmod 4$, and $\varepsilon \neq 1$, $\varepsilon \neq \delta_r$, we have m even, so that the first equation of Proposition 9 holds with the $(-1)^m$ deleted.

$\varepsilon < \delta_r$ implies $m = 2k + 2$, if r is odd, and $m = t + 1$, if $r = 2t$, t odd.

$$\lim_{\varepsilon \to 2^+} \delta_r(t) = \frac{[F_{r-m} - F_m] + 2[F_{r-m+1} + F_{m-1}]}{[F_{r-m-1} + F_{m+1}] + 2[F_{r-m} - F_m]} = \frac{F_{r-m-3} + F_{m-3}}{F_{r-m-2} - F_{m-2}}$$

since δ_r is clearly continuous in ε on $(2,\alpha_r)$. Treatment by cases gives

$$\lim_{\varepsilon \to 2^+} \delta_r(\varepsilon) = \begin{cases} (F_{2k+4} + F_{2k-1})/(F_{2k+3} - F_{2k}) = (F_{2k+3} + 2F_{2k+1}) = \beta_{4k+3}, & \text{for } r = 4k + 3; \\ (F_{2k+2} + F_{2k-1})/(F_{2k+1} - F_{2k}) = \beta_{4k+1}, & \text{for } r = 4k + 1; \\ (F_{t+2} + F_{t-2})/(F_{t+1} - F_{t-1}) = 3F_t/F_t = 3 = \beta_{2t}, & \text{for } r = 2t,\ t \text{ odd.} \end{cases}$$

In short, $\lim_{\varepsilon \to 2^+} \delta_r(\varepsilon) = \beta_r$. Similarly, $\lim_{\varepsilon \to \alpha_r^-} \delta_r(t) = \infty$. The numerator of $\delta_r(\varepsilon)$ is of the form $e + \varepsilon f$, while the denominator is of the form $g + \varepsilon h$. Now, with r given, the fact that ε is in $(2,\alpha_r)$ determines m, so that in this interval e, f, g, and h are constant.

$$\frac{d}{d\varepsilon}\delta_r(\varepsilon) = \frac{d}{d\varepsilon}\frac{e + f}{g + f} = \frac{f(g + h) - h(e + f)}{(g + h)} = \frac{fg - he}{(g + h)^2}.$$

So the sign of the derivative of δ_r is constant in $(2,\alpha_r)$. From the limits established above, we realize that δ_r is increasing throughout $(2,\alpha_r)$.

The same argument may be applied to the behavior of δ_r on (α_r,∞).

The cases $r = 1$, $r = 3$, $r \equiv 0 \pmod 4$ offer no challenge.

Example:

$$\delta_5(\varepsilon) = \begin{cases} 4, & \varepsilon = 1 \\ (3\varepsilon - 2)/(5 - 2\varepsilon), & 2 < \varepsilon < 2\tfrac{1}{2} \\ 1, & \varepsilon = 2\tfrac{1}{2} \\ (1 + 4\varepsilon)/(3 + \varepsilon), & \varepsilon > 2\tfrac{1}{2} \end{cases}$$

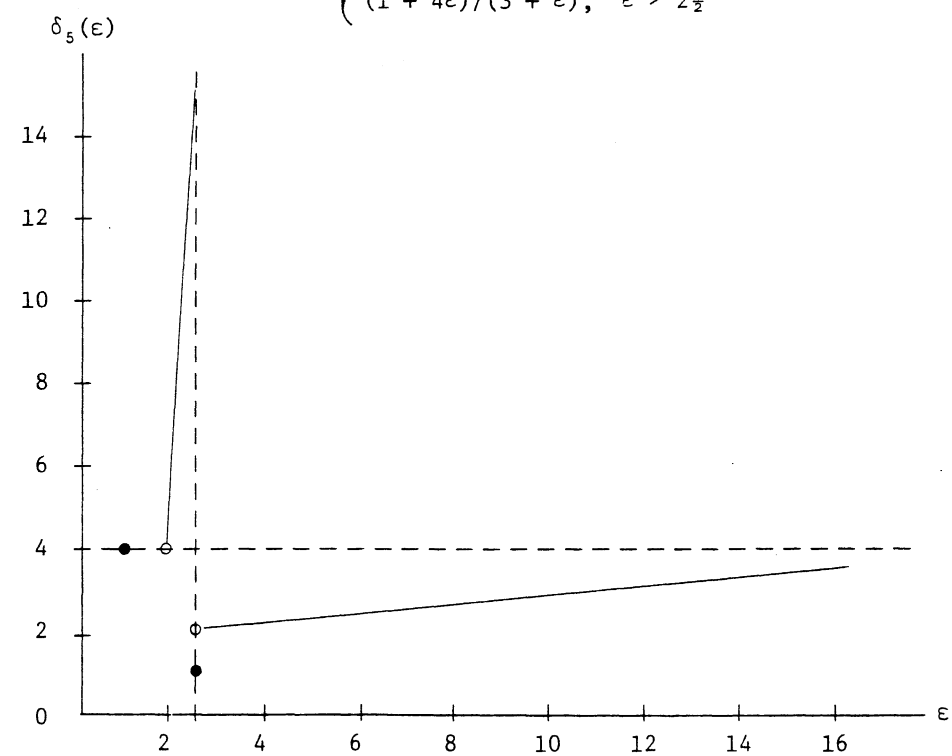

FIGURE 1

5. CHARACTERISTIC NUMBERS OF SECONDARY SEQUENCES

The concept of characteristic number of a Fibonacci sequence was introduced in [1] to structure the collection of Fibonacci sequences.

Definition: The *characteristic number* D_T of a Fibonacci sequence T is $D_T = |T_n^2 - T_{n-1}T_{n+1}|$.

Readers familiar with the elementary properties of Fibonacci sequences will recognize that the value of D_T is independent of the choice of n, so that D_T is well defined.

A table of characteristic numbers of primitive Fibonacci sequences for $D < 2000$ can be found in [4, pp. 42-44].

We summarize some useful properties of characteristic numbers in the following proposition.

Proposition 11: (i) $D_{kT} = k^2 D_T$; (ii) $D_V = 5D_T$; (iii) $D_{\bar{T}} = D_T$.

Proof: Left to the reader.

Proposition 12: (i) A natural number $n = a^2 b$, b square-free, is the characteristic number of a [primitive] Fibonacci sequence if and only if all prime factors of b are of the forms $10k \pm 1$ and 5 [and additionally, all prime factors of a are of the forms $10k \pm 1$]. (ii) Let D have n distinct-prime factors of the forms $10k \pm 1$. Then there are exactly 2^n primitive sequences with characteristic number D.

Proof: (i) Cf. Theorem 2 of [9, p. 78]. The same source gives an expression $r(D)$ for the number of inequivalent Fibonacci sequences having characteristic D. The only difference here is the observation that the only primes of the forms $5k \pm 1$ are indeed of the forms $10k \pm 1$.

Note that D_T may have square factors even for primitive T; for example, $D_{(3,13)} = 121 = 11^2$. (ii) See [3] and [9].

Theorem 6: Let $S \equiv {}^rT$. Then $D_S = D_T[L_r + 1 + (-1)^r]$.

Proof:

$$\begin{aligned}
D_{{}^rT} &= |({}^rT_n)^2 - {}^rT_{n+1}\,{}^rT_{n-1}| = |({}^rT_2)^2 - {}^rT_3\,{}^rT_1| \\
&= |(T_2 + T_{r+2})^2 - (T_3 + T_{r+3})(T_1 + T_{r+1})| \\
&= |T_2^2 + 2T_2T_{r+2} + T_{r+2}^2 - T_3T_1 - T_3T_{r+1} - T_1T_{r+3} - T_{r+3}T_{r+1}| \\
&= |(T_2^2 - T_3T_1) + (T_{r+2}^2 + T_{r+3}T_{r+1}) + 2T_2T_{r+2} - T_1T_{r+3} - T_3T_{r+1}| \\
&= |(T_2^2 - T_3T_1)[1 + (-1)^r] + 2T_2(F_rT_1 + F_{r+1}T_2) \\
&\qquad - T_1(F_{r+1}T_1 + F_{r+2}T_2) - (T_1 + T_2)(F_{r-1}T_1 + F_rT_2)| \\
&= |(T_2^2 - T_3T_1)[1 + (-1)^r] + T_2^2(2F_{r+1} - F_r) - T^2(F_{r+1} + F_{r-1}) \\
&\qquad - T_1T_2(F_{r+2} + F_{r+1} - 2F_r)| \\
&= |(T_2^2 - T_3T_1)[1 + (-1)^r] + T_2^2(F_{r+1} + F_{r-1}) - T_1^2(F_{r+1} + F_{r-1}) - T_1T_2(F_{r+1} + F_{r-1})| \\
&= |(T_2^2 - T_3T_1)[1 + (-1)^r] + L_i(T_2^2 - T_1^2 - T_1T_2)| \\
&= |(T_2^2 - T_3T_1)[1 + (-1)^r + L_i]| \\
&= D_T[1 + (-1)^r + L_i].
\end{aligned}$$

Corollary: Let $T \equiv {}^{1/r}S$. Then $D_T = D_S/[L_i + 1 + (-1)^r]$.

Corollary: D_TL_r square-free, r odd $\Rightarrow {}^rT$ primitive.

Proof: Immediate from Proposition 7(ii) and Theorem 6.

Corollary: Let $S \equiv {}^rT$. Then

$$D_S = \begin{cases} D_T \cdot (L_r + 2),\ r \text{ even} = \begin{cases} D_TL_t^2,\ r = 2t,\ t \text{ even} \\ D_T \cdot 5 \cdot F_t^2,\ r = 2t,\ t \text{ odd} \end{cases} \\ D_T \cdot L_r,\ r \text{ odd} \end{cases}$$

The question of which Fibonacci sequences occur as secondary sequences is completely settled by the work of Section 4, but only if we are willing to identify multiples of equivalent sequences; the answer then is that every sequence is, for every r, r-secondary. If, however, we decline to make the identification, our curiosity may be piqued by examples like the following.

Example: An examination of the table of characteristic numbers of primitive sequences provides the information:

Characteristic Number	Corresponding Sequences (in conjugate pairs)
11	(1,4)(2.5)
19	(1,5)(3,7)
$209 = 11 \cdot 19$	$\begin{cases}(1,15)(13,27)\\(5,18)(8,21)\end{cases}$

We note the following relations:

$^9(1,4) \equiv 2(8,21)$
$^9(2,5) \equiv 2(13,27)$ $\quad$ $^5(13,27) \equiv (1,5)$ $\quad$ $^9(5,18) \equiv (2,5)$
$^5(5,18) \equiv (3,7)$

$^5(1,5) \equiv (8,21)$
$^5(3,7) \equiv (1,15)$ $\quad$ $^9(1,15) \equiv (1,4)$

We may abstract this information into the table below, where a + represents that a secondary sequence of the sequence in the left column is equivalent to a multiple of the sequence in the top row; and a - represents the reverse.

	(1,15)	(13,27)	(5,18)	(8,21)
(1,4)	-			+
(2,5)		+	-	
(1,5)		-		+
(3,7)	+		-	

What is strange is that although one multiple each of (1,15) and (13,27) is equivalent to a secondary sequence, and (8,21) has this happen twice, it fails to happen at all for (5,18). At least, no multiple of (5,18) is secondary from an equivalent of what seem the most likely candidates: that is, the four primitive sequences with characteristic number dividing 209, the characteristic number of (5,18). It may come as a surprise that the characteristic number of a secondary sequence need not be a multiple of that of the sequence it is secondary from, and even that a sequence can be secondary from another of much larger characteristic number. The exact conditions are given in the theorem below.

Definition: Let a sequence T be a multiple of an equivalent of the primitive sequence U; we will refer to U as the *base* of T.

Proposition 13: Let $D_S = x^2y$, $x,y \in \mathbb{Q}^+$ (so S not necessarily integral), and let $\lambda S \equiv {}^rT$. Then D_T cannot be of the form u^2y, unless $r = 1$, $r = 3$, or $r \equiv 0 \pmod 4$.

Proof: Suppose $D_T = u^2y$. Then if r is odd, $\lambda^2 D_S = \lambda^2 x^2 y = L_r \cdot u^2 y = L_r \cdot D_T$, which implies L_r is a square. By [2], the only square Lucas numbers are $L_1 = 1$, $L_3 = 4$. If $r \equiv 2 \pmod 4$, then $\lambda^2 D_S = \lambda^2 x^2 y = F_{r/2}^2 \cdot 5 \cdot u^2 y$, which is impossible.

Corollary: Let S be a primitive sequence with $D_S = m^2 > 1$. Then no multiple of S is secondary from an equivalent of F. That is, no secondary sequence of F has a base whose characteristic number is a perfect square greater than 1.

Proof: Secondary sequences of F of even order have either F or L as their base, and $D_F = 1$, $D_L = 5$. Suppose ${}^rF \equiv \lambda S$, r odd. Since $D_S = m^2 > 1$, but $D_F = 1$, then by the proposition we must have $r = 1$ or $r = 3$. But ${}^1F \equiv F$, which has $D = 1$, while ${}^3F \equiv 2F$, which is not primitive.

Example: $S = (7,17)$, $D_S = 11^2$. S is not secondary from any equivalent of F, nor from any sequence T with $D_T < 11^2$.

Theorem 7: Let r and S be given, $r \in \mathbb{N}$ and S a primitive sequence. Then the only solutions to $T \equiv \lambda S$ with T primitive are:

	λ	T
$r = 2t$, t odd:	F_t	$V(S)$
$r = 2t$, t even:	L_t	S
$r = 1$	1	S
$r = 3$	2	S
r odd, $r \geq 5$:	ij	$ij \cdot {}^{1/r}S$

where i and j are determined as follows:

Let $G = \text{GCD}(D_S, L_r)$, with $d = D_S/G$, $\ell = L_r/G$, and write ℓ as $\ell = i^2 j$, j square-free.

Proof: It suffices to direct our attention to the last case listed, the others being straightforward consequences of earlier theorems.

If r is odd, $r \geq 5$, then λ must satisfy $L_r \cdot D_T = \lambda^2 D_S$ in such a fashion that D_T is integral. For such a λ, $T = \lambda^{1/r}S$ is guaranteed to be the unique solution of ${}^rT \equiv \lambda S$ by Proposition 7(iii). So, the only question is what values are admissible for λ.

Using the notation of the theorem, we have

$$D_T = \frac{\lambda^2 D_S}{L_r} = \frac{\lambda^2 dG}{\ell G} = \frac{\lambda^2 d}{\ell}.$$

Since GCD(d,ℓ), $\ell = i^2 j$ must divide λ^2. Any λ satisfying this requirement yields a solution; the smallest such λ is ij, and for some multiple of $\lambda = ij$, the sequence T is primitive. Larger values of λ lead to multiples of that sequence.

Definition: If a prime p divides some member of the Lucas sequence, then the first member L_n of L which p divides is known as the *entry point of* p *in* L, and p is called a *primitive prime divisor* of L_n. We say p *enters* L at index n.

Proposition 14: (i) If a prime p enters L at L_n, then $p|L_{n(2k-1)}$, $k \in \mathbb{N}$, and p divides no other members of L. (ii) a) The primes which enter $\{L_{2n}\}$ include all primes of the forms $20k + 3$, $20k + 7$, and some primes of the forms $20k + 1$, $20k + 9$; b) for $\{L_{2n+1}\}$, all primes of the forms $20k + 11$, $20k + 19$, and 2, and a different collection of primes of the forms $20k + 1$, $20k + 9$; c) for $\{F_{2n+1}\}$, all primes of the forms $20k + 13$, $20k + 17$, and 5, plus the remaining primes of the forms $20k + 1$, $20k + 9$; d) all primes enter $\{F_{2n}\}$.

Proof: Lucas was the first to prove (i) [10, p. 35]; he also proved most of (ii) [10, pp. 22-23], though Zeckendorf [13] was the first to prove it in the version given (it is usually called Zeckendorf's Theorem).

Corollary to the Theorem: Let S and T be primitives ${}^rT \equiv \lambda S$. If r is odd and not less than 5, and D_S has no prime factors which enter L at odd index, then D_T is a multiple of D_S.

Proof: Apart possibly from 2, the prime factors of L_r all enter L at odd index. Since $L_r \cdot D_T = \lambda^2 D_S$, and GCD$(D_S, L_r) = 1$ (S is primitive, so $2 \nmid D_S$), $D_S | D_T$.

Corollary: Given primitive sequences S and T, and given r, a necessary (but not sufficient) condition for ${}^rT \equiv \lambda S$ to hold is that sqf$(D_T D_S)$ = sqf(L_r), where for $n = e^2 f$, f square-free, sqf$(n) = f$.

Proof: If ${}^rT \equiv \lambda S$, sqf$(D_T D_S)$ = sqf$(\lambda^2 D_S D_T)$ = sqf$(L_r D_T)$ = sqf(L_r).

The sequence $S = (5,18)$ has $D_S = 209 = 11 \cdot 19$, and the sequence $T = (3,7)$ has $D_T = 19$, so that sqf$(D_S D_T) = 11 = L_5$. But we have seen that ${}^5(3,7) \equiv (1,15)$, which is not a multiple of $(5,18)$.

The result of the second corollary tells us that characteristic number alone cannot give us a complete criterion for judging if one sequence is secondary from another. Of course, this was to be expected, since there are always at least two sequences with the same characteristic number (unless it is 1).

In the example in the proof of the corollary, everything would work out nicely if we were to identify conjugate sequences, for $\overline{S} = (8,21)$, $\overline{T} = (1,5)$, and ${}^5(1,5) = (8,21)$. This will not work in general, however. Consider any D_S, D_T, each with at least two prime factors apart from possibly 5. Then to each of D_S, D_T, there correspond at least two pairs of conjugate sequences, and it is easy to envision a "switch" that allows sqf$(D_S D_T)$ to be equal to sqf(L_r) without any of ${}^rT \equiv \lambda S$, ${}^r\overline{T} \equiv \lambda S$, ${}^r\overline{T} \equiv \lambda \overline{S}$, ${}^rT \equiv \lambda \overline{S}$ holding.

For concreteness, take $D_S = 589 = 19 \cdot 31$, $S = (7,29)$. $D_T = 209 = 11 \cdot 19$, $T = (5,18)$, $r = 15$, $L_r = 1364 = 4 \cdot 11 \cdot 31$. Then sqf$(D_T D_S) = 11 \cdot 31$ = sqf(L_r), but ${}^{15}T \equiv 22(3,26)$, ${}^{15}\overline{T} = {}^{15}(8,21) \equiv 2(84,325)$, while $\overline{S} = (15,37)$.

From among the four items r, λ, S, and T (S, T primitive), specification of any two either determines what the other two must be for there to be a solution to ${}^rT \equiv \lambda S$, or else determines that no solution exists.

Example: $L_{25} = 167761 = 11 \cdot 101 \cdot 151$.

Suppose $D_S = 101$, $D_T = 151$. Then $\lambda S \equiv {}^{25}T$ is impossible since $101 \cdot 151$ = sqf$(D_S D_T) \neq$ sqf $L_{25} = 11 \cdot 101 \cdot 151$.

However, since $L_5 = 11$, we are led to wonder if perhaps λS could be reached from T in two stages; for example, $151 \cdot S \stackrel{?}{=} {}^{1/5}({}^{25}T)$. This will be our next topic of investigation.

6. CHAIN-SECONDARY SEQUENCES

Definitions: ${}^-T = \overline{T}$

$\mathbb{E} = \{2n | n \in \mathbb{N}\} \cup \{1/2n | n \in \mathbb{N}\}$

$\mathbb{O} = \{2n - 1 | n \in \mathbb{N}\} \cup \{1/(2n - 1) | n \in \mathbb{N}\}$

$$B = E \cup \mathbf{0} = \{n | n \in N\} \cup \{1/n | n \in N\}$$
$$X^- = X \cup \{\overline{X}\},\ X = E,\ \mathbf{0},\ \text{or } B$$
$$^{r_k \cdots r_1}T = {}^{r_k}({}^{r_{k-1}} \ldots (\ldots {}^{r_2}({}^{r_1}T) \ldots)),\ r_i \in B^-.$$

Definition: A primitive sequence S is a *chain-secondary sequence* of a primitive sequence T if and only if there is a chain $\{_iT\}_{i=0}^k$ of (not necessarily integral) sequences such that

(i) S is the base of $_kT$, with $_kT \equiv \lambda_0 S$, $\lambda_0 \in Q^+$
(ii) $T \equiv {_0T}$
(iii) for each i between 1 and k inclusive, there are $\lambda_i \in B$ and $r_i \in B^-$ such that $^{r_i}_{i-1}T \equiv \lambda_i\, {_iT}$.

When such a chain exists, we say that S is *derivable* from T, writing $S \leftarrow T$.

Notice that allowing $\lambda_i \in Q^+$ would not achieve any greater generality, since we are free to have as many "links" in the chain with $r_i = 1$ as we like.

The definition in effect allows free substitution of a sequence for its conjugate in pursuing a derivation from T to S, without going so far as to identify the two conjugate sequences. We have already seen, following Theorem 6, an example of $^{-,r}T \not\equiv {}^{r,-}T$ when r is odd and $r \geq 5$; the introduction of conjugates in fact banishes us from the complete commutativity we would otherwise enjoy in conjugate-free chains:

Proposition 15: $^{s,r}T \equiv {}^{r,s}T$, for $r,s \in B$.

Proof: For $r,s \in N$:

$$\begin{aligned}({}^{s,r}T)_n &= {}^rT_{n+s} + {}^rT_n = (T_{n+s+r} + T_{n+s}) + (T_{n+r} + T_n) \\ &= (T_{n+r+s} + T_{n+r}) + (T_{n+s} + T_n) \\ &= {}^sT_{n+r} + {}^sT_n = ({}^{r,s}T)_n\end{aligned}$$
$$^{s,1/r}T \equiv {}^{1/r,\,r,\,s,\,1/r}T \equiv {}^{1/r,\,s,\,r,\,1/r}T \equiv {}^{1/r,\,s}T$$

since we now know we are allowed to pass s all the way to the left.

The condition $S \leftarrow T$ is equivalent to the existence of k, and some $\lambda_i \in B$, $r_i \in B$, for $i = 1, \ldots, k$, $\lambda_0 \in Q^+$, such that

$$^{r_k \cdots r_1}T \equiv \left(\prod_{i=0}^{k} \lambda_i\right) S. \tag{1}$$

Proposition 16: $S \leftarrow T$ if and only if there is a chain $\{_iT'\}_{i=0}^{k'}$ of (not necessarily integral) sequences such that

(i) S is the base of $_kT'$, with $_kT' \equiv \lambda_0' S$, $\lambda_0' \in Q^+$
(ii) $T \equiv {_0T'}$
(iii) for each i between 1 and k' inclusive, there are $\lambda_i' \in B$ and $r_i' \in Q^+$ such that $^{r_i'}_{i-1}T' \equiv \lambda_i'\, {_iT'}$.
(iv) $r_1' = 1$ or $r_1' = 2$.

Proof: The operation $^r(\ )$ commutes with $(^-)$ for $r \in E$, by Proposition 5 and the definition of $^{1/r}(\ )$ for $r \in N$; and we have just seen in Proposition 15 that $^r(\ )$ commutes with $^s(\ )$, up to equivalence, for $r,s \in B$. The net effect of our remarks is that any "link" in the chain for which $r_i \in E$—call it an "even link"—may be repositioned elsewhere in the chain while preserving $S \leftarrow T$. In particular, we may permute the links of the chain so that all even links occur first, still preserving $S \leftarrow T$, provided we do not alter the order of succession of the remaining links. Even links are trivial, in that apart from altering $_{i-1}T$ by a factor F_t/λ_i or L_t/λ_i they do not affect it at all, except possibly to transform it to its Lucas dual. We conveniently absorb all of the multiplicative effect of the even links into λ_0'. We may then eliminate all of them except possibly for a single link with $r = 2$, since as an operation the Lucas dual has order 2.

Proposition 16: The relation $\leftarrow$ is an equivalence relation (and henceforth we will write it as $\leftrightarrow$).

Proof: Reflexivity and transitivity offer no difficulty. If $S \leftarrow T$, so that (1) holds, then $^{1/r_1,\ \ldots,\ 1/r_k}S = \left[\prod_{i=0}^{k} \left(\frac{1}{\lambda_i}\right)\right] T$ with the symbol $1/^-$ defined to be $^-$; and $T \leftarrow S$.

Definition: The equivalence classes into which $\leftrightarrow$ divides the set of all primitive sequences we will refer to as *families*. The *Brousseau number* of a family is the smallest of the characteristic numbers associated with members of the family; the corresponding sequence and its conjugate are the *founders* of the family. We will represent the set of Brousseau numbers by $\mathcal{B}$. The set $\mathcal{L}$ of *$\mathcal{L}$-factors* is the set

$$\mathcal{M} \cup \{5m \mid m \in \mathcal{M}\}$$

where $\mathcal{M}$ is the smallest subset of Q^+ containing all odd-index Lucas numbers which is closed under multiplication, division, and powers.

Examples: $L_{45} = 2537720636 = 4 \cdot 11 \cdot 19 \cdot 31 \cdot 97921$ gives rise to the following L-factors: $19 \cdot 97921$ (since $4 \cdot 11 \cdot 31 = L_{15}$), $31 \cdot 97921$ (since $4 \cdot 11 \cdot 19 = L_5 L_9$), $31 \cdot 19 \cdot 97921$ (since $4 \cdot 11 = L_3 L_5$), and L_{45} itself.

In light of Proposition 16, the condition that $S \leftarrow T$ is equivalent to (1) holding for some k and some $\lambda_0 \in Q^+$, $\lambda_i \in B$, $r_i \in O$, $i = 1, \ldots, k$, with $r_1 = 1$ or 2. Converting to the corresponding necessary condition on characteristic numbers gives

$$(2) \qquad D_T 5^\alpha \frac{\prod_{r_i \in N} L_{r_i}}{\prod_{1/r_i \in N} L_{1/r_i}} = \frac{q_1^2 \prod_{r_i \in N} \lambda_i^2}{q_2^2 \prod_{1/\lambda_i \in N} (1/\lambda_i)^2} D_S ,$$

or

$$D_T \cdot 5^\alpha \cdot \prod_{r_i \in N} L_{r_i} \cdot q_2^2 \cdot \prod_{1/\lambda_i \in N} (1/\lambda_i)^2 = D_S \cdot \prod_{1/r_i \in N} L_{1/r_i} \cdot q_1^2 \cdot \prod_{\lambda_i \in N} \lambda_1^2$$

with $\alpha = 0$ or 1; and $\lambda = q_1/q_2$ in lowest terms, $q_i \in N$.

Proposition 17: Let S,T be primitive. Let p be a prime which is not an odd-index-entry Lucas prime. If $p^t | D_S$, then $p^t | D_T$, for $t \in N$.

Proof: The only other possibility is that p^t is "absorbed" by the denominator of the fraction on the R.H.S. of (2). Denote that denominator by B^2, and the corresponding numerator by A^2, and suppose that $p^t | B^2$, $p^t | A^2$. From (1) we know that $B^{r_k \cdots r_1} T \equiv A \cdot S$. Now, the fact that p does not divide any L_r, r odd, means that no term $({}^{r_k \cdots r_1} T)$, written in lowest terms, can have p as a factor of its denominator, since $({}^{r_k \cdots r_1} T)$ can incur only $\mathcal{L}$-factors there. Hence, $p|B$ implies $p|A$, because S is primitive; but this leads to the conclusion that $p^t | D_T$.

Consequently, a prime of the form $10k \pm 1$ which has odd-index entry in F or even-index entry in L is a Brousseau number, for some family of sequences. The product of powers of such primes is also a Brousseau number, and we will call such numbers *Brousseau numbers of the first kind*. Every sequence whose characteristic number is a Brousseau number of the first kind is the founder of a family.

The remaining Brousseau numbers are either products of powers of primes of odd-index Lucas entry (*the second kind*), or mixed products of Brousseau numbers of the first and second kinds (*the third kind*).

Example: $D_F = 1$, $D_{(1,7)} = 41$, $D_F \cdot D_{(1,7)} = 41$. But $41 | L_{10+20k}$ and no other Lucas numbers; hence $41 \notin \mathcal{L}$, so F and (1,7) must be in different equivalence classes.

Example: $L_{25} = 11 \cdot 101 \cdot 151 = L_5 \cdot 101 \cdot 151$. The primes 101 and 151 are both primitive prime divisors of L_{25}, and both have period 50. Each of them is a Brousseau number, but their product is an $\mathcal{L}$-factor.

Corollary: Two sequences with relatively prime Brousseau numbers belong to different families.

Theorem 8: If S and T are in the same family, then $D_T D_S$ is an $\mathcal{L}$-factor times a rational square. If S and T are both primitive, then $\text{sqf}(D_T D_S) = \text{sqf}(\ell)$, ℓ an integral $\mathcal{L}$-factor.

Proof: Algebraic manipulation of (2) easily leads to the first conclusion, with, say, $D_T D_S = \frac{\ell_1}{\ell_2} \cdot \frac{s_1^2}{s_2^2}$, ℓ_1, ℓ_2 products of odd-index Lucas numbers, $s_1, s_2 \in N$, s_1/s_2 in lowest terms. If S and T are both primitive, $D_T D_S \in N$. Since $\text{GCD}(s_1, s_2) = 1$, we must have $\ell_2 | s_1^2$. Writing ℓ_2 as $a^2 b$, b square-free, we obtain $s_1 = abc$ for some c, and

$$a^2 s_2^2 D_T D_S = \ell_1 s_1^2 a^2 / \ell_2 = \ell_1 a^2 b^2 c^2 a^2 / a^2 b = \ell_1 a^2 bc = \ell_1 \ell_2 c^2$$

and $\text{sqf}(D_T D_S) = \text{sqf}(a^2 s_2^2 D_T D_S) = \text{sqf}(\ell_1 \ell_2 c^2) = \text{sqf}(\ell_1 \ell_2)$, with $\ell_1 \ell_2$ clearly an $\mathcal{L}$-factor.

We would like to find a criterion involving characteristic numbers which would enable us to determine if two sequences belong to the same family or not. We conclude with conjectures in this direction:

Conjecture 1: $D_S = D_T \Rightarrow S \leftrightarrow T$

Conjecture 2: $S \leftrightarrow T \Longleftrightarrow D_S D_T$ is an $\mathcal{L}$-factor times a rational square. It would also be desirable to have an algorithm to produce the derivation given the $\mathcal{L}$-factor.

Conjecture 3: p is a Brousseau number $\Rightarrow$ each of the powers of p corresponds to a distinct family of sequences.

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AN ESTIMATE FOR THE LENGTH OF A FINITE JACOBI ALGORITHM

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There are many papers concerning the length of the continued fraction expansion of a rational number (see, e.g., M. Mendès-France [2]). Following a method given by J. D. Dixon [1] in an elementary way, an estimate can be given for the length of the Jacobi algorithm of a rational point.

The Jacobi algorithm may be described in the following way: Let

$$B = \{x = (x_1, \ldots, x\ )\,|\,0 \le x_j < 1,\ 1 \le j \le n\}.$$

If $x = (0, \ldots, 0)$, then $Tx = x$. If $x_1 = \cdots = x_t = 0$, $x_{t+1} > 0$ for $0 \le t < n$, then,

$$T(0, \ldots, 0, x_{t+1}, \ldots, x_n) = (0, \ldots, 0, x_{t+2}/x_{t+1} - [x_{t+2}/x_{t+1}], \ldots, 1/x_{t+1} - [1/x_{t+1}]).$$

We define $x^{(g)} = T^g x$. We say that the algorithm of x has length $L(x) = G$ if

$$G = \min\{g \ge 0\ x^{(g)} = (0, \ldots, 0)\}.$$

Let $x^{(s)} = (0, \ldots, 0, x_{t+1}^{(s)}, \ldots, x_n^{(s)})$, then we define

$$k_0^{(s+1)} = \cdots = k_{t-1}^{(s+1)} = 0$$
$$k_t^{(s+1)} = 1 \text{ (if } t = 0, \text{ then } k_0^{(s+1)} = 1)$$
$$k_{t+1}^{(s+1)} = [x_{t+2}^{(s)}/x_{t+1}^{(s)}],\ \ldots,\ k_n^{(s+1)} = [1/x_{t+1}^{(s)}]$$
$$A_i^{(j)} = \delta_{ij} \text{ for } 0 \le i,\ j \le n$$