A COLLECTION OF MANUSCRIPTS RELATED TO THE FIBONACCI SEQUENCE

18th Anniversary Volume

THE FIBONACCI ASSOCIATION
U.S.A.

Edited by

VERNER E. HOGGATT, JR.

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MARJORIE BICKNELL-JOHNSON

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PREFACE

The Fibonacci Association celebrates the 18th anniversary of its founding with the publication of this collection of manuscripts. These manuscripts, published here for the first time, reflect the research efforts of an international range of mathematicians.

The primary vehicle for publication of Fibonacci-related material is *The Fibonacci Quarterly*, the official journal of The Fibonacci Association. However, the volume of research being done on topics related to the Fibonacci sequence has increased each year to the extent that the *Quarterly* is hard pressed to accommodate the timely publication of all worthwhile scholarly manuscripts being submitted to the Fibonacci Association for publication consideration.

To expedite the dissemination of the growing volume of Fibonacci research information to the worldwide mathematics community, the Fibonacci Association's Board of Directors has authorized publication of supplemental volumes such as this 18th anniversary issue to be published, when appropriate, and made available for separate purchase by Fibonacci Association members and nonmembers.

The editors hope these supplemental publications will benefit both the authors of manuscripts, by earliest possible publication of their material, and the readers interested in the Fibonacci sequence, by making more material available throughout the year.

Verner E. Hoggatt, Jr.
Marjorie Bicknell-Johnson

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SECONDARY FIBONACCI SEQUENCES

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1. PRELUDE: AN ENTERTAINMENT

DRAMATIS PERSONAE: Leonardo of Pisa and Édouard Lucas, well-known experts in the mathematics of deterministic modelling of the growth of animal populations.

SCENE: Circa fall 1974; lounge of a computing center, where they are whiling away the time as they wait for the number-cruncher to test their recent conjectures.

LEONARDO: I've made some new discoveries in the realm of our favorite common pastime, the Fibonacci numbers (as you so flatteringly refer to them), together with some conjectures I can't yet settle. Maybe we can go over it all together and see what we can come up with.

ÉDOUARD: Splendid! And what have you to reveal?

L: Let me fill you in on the background first. One fine day last summer I was hitchhiking. A ride with a boorish driver precluded good conversation; so, left to my own devices, I let my mind drift to mathematical games for amusement.

É: I suppose that is conclusive proof that you are mathematically inclined, for who else would choose such a pastime?

L: Who knows? At any rate, Fibonacci numbers are ideally suited for such sport, as you well know. By chance, I happened to add 13 and 987, and got 1000.

É: In other words, F_7 and F_{16} = 1000—OK, what's remarkable about that?

L: As you well know, Édouard, the journey to mathematical discovery often starts with noticing something unusual, however small and insignificant it may seem to be. I paused to consider the roundness of the sum, and then proceeded to wonder: if $F_7 + F_{16} = 1000$, then $F_8 + F_{17} = ?$

É: Well, 21 + 1597 = 1618. A fine coincidence! Those are the first four digits of the golden ratio $\phi = 1.618\ldots$.

L: I tried further pairs—let me summarize the results for you on the blackboard:

$$F_5 + F_{14} = 5 + 377 = 382 \qquad \phi^{-2} = .3819\ldots$$
$$F_6 + F_{15} = 8 + 610 = 618 \qquad \phi^{-1} = .618\ldots$$
$$F_7 + F_{16} = 13 + 987 = 1000 \qquad \phi^{0} = 1.$$
$$F_8 + F_{17} = 21 + 1597 = 1618 \qquad \phi^{1} = 1.618\ldots$$
$$F_9 + F_{18} = 34 + 2584 = 2618 \qquad \phi^{2} = 2.618\ldots$$

By forming $F_n + F_{n+9}$, I was getting 1000 times the three-decimal-place approximation of ϕ^{n-1}.

É: Not only that, but your new sequence was also a Fibonacci sequence.

L: Exactly! That was my next observation, and it seemed the more important property to investigate, since it seemed more susceptible of generalization. And generalize it does! Adding any Fibonacci sequence to itself at a constant index difference always produces another Fibonacci sequence. I decided to call the Fibonacci sequences generated in this fashion *secondary Fibonacci sequences*. I started investigating which Fibonacci sequences come out as secondary sequences. Let me show you.

É: Say, that's really interesting! But hold on a moment—let's use the blackboard to make a "formal" record of our brainstorming. After all, if this discussion amounts to anything, you should write a paper about it for the *Fibonacci Quarterly*.

L: I suppose you're right, but writing it all up in a paper isn't nearly as much fun as discovering it all in the first place. In fact, I hate writing papers—all that writing, rewriting, and rewriting again, and I'm still never satisfied with the final product. Besides, I expect you'll be making some contributions in the course of the discussion. If they prove valuable, you write the whole thing up. You're always dashing off mathematical notes and popular articles all over the place. For you this would be just another half-day's work.

É: It's hardly time to argue about that just yet! We'll cross that bridge if we come to it. Meanwhile, let's keep a record anyhow. Now tell me, which Fibonacci sequences turn up as secondary sequences, anyhow? . . .

2. BASIC PROPERTIES OF SECONDARY FIBONACCI SEQUENCES

Definition: A (positive)(Fibonacci) sequence $\{T_n\}_{n \in \mathbb{Z}}$, or T for short, is a doubly infinite sequence which satisfies the recursion relation

$$T_{n+1} = T_n + T_{n-1}$$

for all $n \in \mathbb{Z}$ and for which there is an $n_0 \in \mathbb{Z}$ such that all terms with index greater than n_0 are positive. We will be especially interested in integer sequences.

By definitional convention we are excluding from consideration the constant zero sequence, as well as sequences which are negative for every index exceeding a certain integer.

Proposition 1: Any Fibonacci sequence contains a unique pair of consecutive terms a and b, both positive, with either $a = b$ or $2a < b$.

Proof: See [1, p. 43].

Definition: A Fibonacci sequence T is in *standard format* if it is labelled so that $T_1 = a$, $T_2 = b$, with a and b as specified in Proposition 1. We will write $T = (a,b)$. A sequence for which a and b are relatively prime integers is said to be *primitive*. Two special sequences are distinguished: *the Fibonacci sequence* $F = (1,1)$ and the *Lucas sequence* $L = (1,3)$. As is well known, the terms of a sequence T in standard format are given by

$$T_n = aF_{n-2} + bF_{n-1}$$

for any $n \in \mathbb{Z}$.

Definition: Two sequences T, U are *equal*, written $T = U$, if $T_n = U_n$ for every $n \in \mathbb{Z}$. They are *equivalent*, written $T \equiv U$, if there is a k such that $T_n = U_{n+k}$ for all $n \in \mathbb{Z}$.

Definition: The *Lucas analogue* $V(T)$ of a Fibonacci sequence T is the sequence defined by

$$V(T)_n = T_{n+1} + T_{n-1},$$

and $V(T)$ may be denoted simply by V when no confusion would result. Note that V may fail to be in standard format.

Proposition 2: (i) $V(T)$ is a Fibonacci sequence; (ii) $V(V(T)) = 5T$; (iii) $V(F) = L$.

Proof: Left to the reader.

We now generalize the notion of the Lucas analogue of a sequence to embrace a whole family of sequences.

Definition: For $r > 0$, the rth *secondary sequence* of a sequence T, denoted ${}^rT$, is the sequence obtained by adding T to itself at a constant index difference r:

$${}^rT_n = T_{n+r} + T_n.$$

We will say that ${}^rT$ is *r-secondary from T*. Note that $V(T)$ is not, strictly speaking, a secondary sequence, though $V_n = {}^2T_{n-1}$ makes $V \equiv {}^2T$.

Proposition 3: A secondary sequence of a Fibonacci sequence is a Fibonacci sequence.

Proof: ${}^rT_n + {}^rT_{n-1} = (T_{n+r} + T_n) + (T_{n-1+r} + T_{n-1}) = (T_{n+r} + T_{n-1+r}) + (T_{n+r} + T_n)$

$= T_{n+1+r} + T_{n+1} = {}^rT_{n+1}.$

We give here in table form the first twelve secondary sequences of F (taken from [11, p. 17]), in hopes of inspiring the reader to discover patterns before reading further.

F	$r =$ 1	2	3	4	5	6
0	1	1	2	3	5	8
1	2	3	4	6	9	14
1	3	4	6	9	14	22
2	5	7	10	15	23	36
3	8	11	16	24	37	58
5	13	18	26	39	60	94

F	$r =$	1	2	3	4	5	6
8		21	29	42	63	97	152
13		34	47	68	102	157	246

F	$r =$	7	8	9	10	11	12
-1		4	7	12	20	33	54
1		9	14	22	35	56	90
0		13	21	34	55	89	144
1		22	35	56	90	145	234
1		35	56	90	145	234	378
2		57	91	146	235	379	612
3		92	147	236	380	613	990
5		149	238	382	615	992	1602

The sequence $F \equiv {}^1F$ has often been cited as occurring in nature, and the occurrence of $L \equiv {}^2F$ is occasionally mentioned as well (see, e.g., [7, pp. 81-82]). What may perhaps be surprising is that $2F \equiv {}^3F$, $3F \equiv {}^4F$, and $(1,5) \equiv {}^5F$ have been observed as the parameters of sunflowers grown by Don Crowe, a geometer at the University of Wisconsin [5].

Our indexing of secondary sequences was arbitrary. Generally, a secondary sequence is not in standard format, and it is necessary to "backspace" by several index numbers to arrive at standard format. It turns out to be important for our purposes to keep trace of the indexing—if it were not, we could conveniently identify all equivalent sequences. In Section 4 we will specify exactly the amount of backspace for each secondary sequence.

Proposition 4:

$$ {}^{2t}T_n = \begin{cases} F_t \cdot V_{n+t}, & t \text{ odd} \\ L_t \cdot T_{n+t}, & t \text{ even} \end{cases} $$

Proof: The first proof of these well-known identities seems to be due to Tagiuri [12], according to Dickson [6, p. 404]. Both are cited by Horadam [8], who furnishes a more accessible proof. In any case, the proof is straightforward, and we leave it to the reader rather than reproduce it here.

It is not possible to find so simple an expression for ${}^rT$ when r is odd.

Definition: The *conjugate* $\overline{T}$ of a sequence T in standard format is the sequence defined by

$$ \overline{T}_n = \begin{cases} (-1)^n T_{-n}, & T \not\equiv F \\ T_n = (-1)^{n+1} T_{-n}, & T \equiv F \end{cases} $$

For a sequence T not in standard format, let $T_m = U_{n+k}$, U in standard format. Then define $\overline{T}_n = \overline{U}_{n+k}$. Note that $\overline{F} = F$, $\overline{L} = L$, and no other primitive sequence is self-conjugate.

Proposition 5: (i) $\overline{T}$ is a Fibonacci sequence; (ii) $\overline{\overline{T}} = T$; (iii) $\overline{V(T)} \equiv \overline{{}^2T} \equiv {}^2\overline{T} \equiv V(\overline{T})$;

(iv) $ {}^{2t}\overline{T} \equiv \begin{cases} F_t \cdot {}^2\overline{T}, & t \text{ odd} \\ L_t \cdot \overline{T}, & t \text{ even} \end{cases}$; (v) $\overline{{}^{2t}T} \equiv {}^{2t}\overline{T}$.

Proof: Left to the reader.

Theorem 1: Let S and T be Fibonacci sequences. If $S \equiv {}^rT$, then

$$ \text{for } r \text{ odd: } {}^r\overline{S} \equiv L_r \cdot \overline{T} $$

$$ \text{for } r \text{ even, } r = 2t: {}^rS \equiv (L_r + 2)T \equiv \begin{cases} 5F_t^2 \cdot T, & t \text{ odd} \\ L_t^2 \cdot T, & t \text{ even} \end{cases} $$

Proof: r odd. We do the case $S \not\equiv F$, $T \not\equiv F$, S in standard format. For n even,

$$ \begin{aligned} {}^r\overline{S}_n &= \overline{S}_{n+r} + \overline{S}_n = (-1)^{n+r}S_{-n-r} + (-1)^n S_{-n} \\ &= -S_{-n-r} + S_{-n} = -(T_{-n+r} + T_{-n}) + (T_{-n} + T_{-n-r}) \\ &= T_{-n+r} - T_{-n-r} + L_r T_{-n}, \text{ by Proposition 4} \\ &= L_r \cdot (-1)^n T_{-n} = L_r \cdot \overline{T}_n. \end{aligned} $$

The proof for n odd is analogous, as are the proofs for the other cases.

$r = 2t$.

$$ {}^{2t}S = {}^{2t}({}^{2t}T) \equiv \begin{cases} {}^{2t}(F_t \cdot V) \equiv F_t^2 V(V(T)) \equiv 5F_t^2 \cdot T, & t \text{ odd} \\ {}^{2t}L_t \cdot T \equiv L_t^2 \cdot T, & t \text{ even} \end{cases} $$

Example: $T = (1,7)$, $S = {}^{9}T \equiv 2(11,36)$. Then $\overline{T} = (5,11)$, $\overline{S} = 2(14,39)$, ${}^{9}\overline{S} \equiv 76(5,11) = L_9 \cdot \overline{T}$.

Proposition 6: ${}^{3}T \equiv 2T$.

Proof: $T_1 + T_4 = a + (a + 2b) = 2(a + b)$, $T_2 + T_5 = b + (2a + 3b) = 2(a + 2b)$.

The results of the theorem suggest the definition of an inverse to the operation ${}^{r}(\ )$ of taking the rth secondary sequence of a sequence T.

Definition: ${}^{1/r}T = \overline{{}^{r}\overline{T}}/[L_r + 1 + (-1)^r]$, with the terms of ${}^{1/r}T$ being allowed to be fractional. Note that ${}^{1/2t}T = T/(L_{2t} + 2)$, by Proposition 5.

Proposition 7: (i) ${}^{1/r}T$ is a Fibonacci sequence; (ii) ${}^{1/r}({}^{r}T) \equiv {}^{r}({}^{1/r}T) \equiv T$; (iii) Up to equivalence, ${}^{1/r}T$ is the only sequence whose rth secondary sequence is T.

Proof: (i) Neither $(\overline{\ })$ nor ${}^{r}(\ )$ disturbs the recursion relation.

(ii) r odd.

$$ {}^{1/r}({}^{r}T) \equiv \frac{\overline{{}^{r}\overline{T}}}{L_r} \quad \frac{\overline{L_r\overline{T}}}{L_r} \equiv T, \text{ by Theorem 1.} $$

$$ {}^{r}({}^{1/r}T) \equiv {}^{r}\left(\frac{\overline{{}^{r}\overline{T}}}{L_n}\right) \equiv \overline{{}^{r}\left(\frac{{}^{r}\overline{T}}{L_n}\right)}, \text{ which by the line above is just } T. $$

$r = 2t$, t odd.

$$ {}^{1/2t}({}^{2t}T) \equiv \overline{{}^{2t}\overline{{}^{2t}T}}/(L_{2t} + 2) \equiv \overline{{}^{2t}\overline{F_t \cdot V}}/(L_{2t} + 2) \equiv \frac{\overline{5F_t^2 \cdot \overline{T}}}{F_t^2} \equiv T $$

$$ {}^{2t}({}^{1/2t}T) \equiv {}^{2t}\left(\overline{{}^{2t}\overline{T}}/(L_{2t} + 2)\right) \equiv {}^{2t}(F_t\overline{V})/(L_{2t} + 2) \equiv {}^{2t}(F_t V)/(5F_t^2) $$

$$ \equiv F_t^2 \cdot 5T/5F_t^2 \equiv T $$

$r = 2t$, t even.

$$ {}^{1/2t}({}^{2t}T) \equiv \overline{{}^{2t}\overline{{}^{2t}T}}/(L_{2t} + 2) \equiv \overline{{}^{2t}\overline{L_t \cdot \overline{T}}}/L_t^2 = \overline{L_t \cdot L_t \cdot \overline{T}}/L_t^2 \equiv T $$

$$ {}^{2t}({}^{1/2t}T) \equiv {}^{2t}(\overline{{}^{2t}\overline{T}})/(L_{2t} + 2) \equiv L(\overline{L_t \cdot \overline{T}})/L_t^2 \equiv T $$

(iii) Suppose ${}^{r}S \equiv {}^{r}S' \equiv T$. Then ${}^{1/r}({}^{r}S) \equiv {}^{1/r}({}^{r}S') \equiv {}^{1/r}T$, so that $S \equiv S' \equiv {}^{1/r}T$.

Example: ${}^{5}(1,7) \equiv (10,29)$, ${}^{1/5}(10,29) \equiv (1,7)$, ${}^{1/7}(10,29) \equiv \left(\frac{79}{29}, \frac{184}{29}\right)$.

A major effort of the remainder of the paper is to determine exactly what integer sequences are secondary from other integer sequences.

3. STANDARD-FORMATTING SECONDARY SEQUENCES

Definition: Let I be the 2 x 2 identity matrix and let

$$ P = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} $$

Note that if T is a Fibonacci sequence, then

$$ (T_{n-1}, T_n) \cdot P = (T_n, T_{n+1}), $$

where the ordered pairs are considered as 1 x 2 matrices. Also,

$$ P^m = \begin{pmatrix} F_{m-1} & F_m \\ F_m & F_{m+1} \end{pmatrix}. $$

Theorem 2: Let $S = (c,d)$ and $T = (a,b)$ be in standard format. Then some multiple of an equivalent of S is secondary from T if and only if there exist positive integers λ, r and a nonnegative $m \le r + 1$ such that one of the following equivalent conditions holds:

(i) $\lambda(c,d)P^m = (a,b)(I + P^r)$

(ii) $\lambda c = (-1)^m[(-F_m T_{r+2} + F_{m+1}T_{r+1}) + (-F_m b + F_{m+1}a)]$

$\lambda d = (-1)^m[(-F_m T_{r+1} + F_{m-1}T_{r+2}) + (-F_m a + F_{m-1}b)]$

(iii) $\lambda c = T_{r-m+1} + (-1)^m (aF_{m+1} - bF_m)$

$\lambda d = F_{r-m+2} + (-1)^m (-aF_m + bF_{m-1})$

(iv) $\lambda c = a[F_{r-m-1} + (-1)^m F_{m+1}] + b[F_{r-m} - (-1)^m F_m]$

$\lambda d = a[F_{r-m} - (-1)^m F_m] + b[F_{r-m+1} + (-1)^m F_{m-1}]$

Proof: If the relation (i) holds, it exhibits a λ-multiple of an equivalent of S as a secondary sequence of T.

Conversely, suppose some multiple, say by λ, of an equivalent of S arises as a secondary sequence, say the rth, of T. Then $({}^rT_1, {}^rT_2) = \lambda \cdot (S_{m+1}, S_{m+2})$ for some m. But

$$(a,b)(I + P^r) = ({}^rT_1, {}^rT_2) = \lambda(S_{m+1}, S_{m+2}) = (c,d)P^m.$$

The quantity m represents the number of places it is necessary to backspace $({}^rT_1, {}^rT_2)$ to arrive at standard format. We must show that $0 \le m \le r + 1$.

Since T is in standard format, $0 < T_1 \le T_2 < T_n < T_{n+1}$, for $n > 2$, so $0 < {}^rT_1 = T_1 + T_{r+1} \le {}^rT_2 = T_2 + T_{r+2}$, for $r > 0$. Hence, $m \ge 0$.

To see that $m \le r + 1$, backspace $(r+2)$ places: $({}^rT_1, {}^rT_2)P^{-r-2} = (T_{-r-1} + T_{-1}, T_{-r} + T_0)$. Since T is in standard format and $r > 0$, exactly one of T_{-r-1} and T_{-r} is negative. If $T \not\equiv F$: r even makes $T_{-r-1} + T_{-1}$ negative, while r odd and $r \ge 3$ forces $T_{-r} + T_0$ negative; $r = 1$ yields $T_{-r-1} + T_{-1} > T_{-r} + T_0 > 0$. In any case, we have certainly backspaced too far. The case for $T \equiv F$ is analogous.

(i) $\Rightarrow$ (ii). $\lambda(c,d)P^m = (\lambda c, \lambda d)\begin{pmatrix} F_{m-1} & F_m \\ F_m & F_{m+1} \end{pmatrix}$, while $(a,b)(I + P^r)$

$$= (a,b)\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} F_{r-1} & F_r \\ F_r & F_{r+1} \end{pmatrix}\right] = [(1 + F_{r-1})a + bF_r,\ aF_r + b(1 + F_{r-1})].$$

Now, (i) asserts that these quantities are equal, so Cramer's Rule yields

$$\lambda c = \begin{vmatrix} T_{r+1} + a & F_m \\ T_{r+2} + b & F_{m+1} \end{vmatrix} \Big/ \Delta \qquad \lambda d = \begin{vmatrix} F_{m-1} & T_{r+1} + a \\ F_m & T_{r+2} + b \end{vmatrix} \Big/ \Delta$$

$$\Delta = \begin{vmatrix} F_{m-1} & F_m \\ F_m & F_{m+1} \end{vmatrix} = F_{m-1}F_{m+1} - F_m^2 = (-1)^m$$

and (ii) follows.

(ii) $\Rightarrow$ (iii). This implication is based on the reduction

$$\begin{aligned} -F_m T_{r+2} + F_{m+1} T_{m+2} &= -F_m (T_{r+1} + T_r) + (F_m + F_{m-1})\, T_{r+1} \\ &= -F_m T_{r+1} - F_m T_r + F_m T_{r+1} + F_{m-1} T_{r+1} \\ &= F_{m-1} T_{r+1} - F_m T_{r-1} \\ &= (-1)(-F_{m-1} T_{r+1} + F_m T_r). \end{aligned}$$

Repetition for a total of m times yields

$$-F_m T_{r+2} + F_{m+1} T_{r+1} = (-1)^m (-F_0 T_{r+2-m} + F_1 T_{r+1-m}) = (-1)^m T_{r+1-m}.$$

Thus,

$$\lambda c = (-1)^m [(-1)^m T_{r+1-m} + aF_{m+1} - bF_m] = T_{r+1-m} + (-1)^m (aF_{m+1} - bF_m).$$

A similar argument gives the corresponding expression for λd.

(iii) $\Rightarrow$ (iv). The equations in (iv) can be obtained directly from (iii) by use of the identity

$$T_{n+2} = aT_n + bT_{n+1}.$$

(iv) $\Rightarrow$ (i). We verify the first coordinate of the matrix equation (i) using the substitution (iv). The first coordinate of $\lambda(c,d)P^m$ is

$$\{a[F_{r-m-1} + (-1)^m F_{m+1}] + b[F_{r-m} - (-1)^m F_m\} \cdot F_{m-1}$$
$$+ \{a[F_{r-m} - (-1)^m F_m] + b[F_{r-m+1} + (-1)^m F_{m-1}\} \cdot F_m$$
$$= a[F_{r-m}F_{m-1} + (-1)^m F_{m+1}F_{m-1} + F_{r-m}F_m - (-1)^m F_m^2]$$
$$+ b[F_{r-m}F_{m-1} - (-1)^m F_m F_{m-1} + F_{r-m+1}F_m + (-1)^m F_{m-1}F_m]$$
$$= a(F_{r-m-1}F_{m-1} + F_{r-m}F_m + 1) + b(F_{r-m}F_{m-1} + F_{r-m+1}F_m)$$
$$= a + F_m(aF_{r-m} + bF_{r-m+1}) + F_{m-1}(aF_{r-m-1} + bF_{r-m})$$
$$= T_1 + F_m T_{r-m+2} + F_{m-1}T_{r-m+1}$$
$$= T_1 + (F_{m-1} + F_{m-2})T_{r-m+2} + F_{m-1}T_{r-m+1}$$
$$= T_1 + F_{m-1}T_{r-m+3} + F_{m-2}T_{r-m+2}.$$

The last three lines comprise a reduction, which iterated for a total of $(m-1)$ times yields

$$T_1 + F_1 T_{r-m+2+m-1} + F_0 T_{r-m+1+m-1} = T_1 + T_{r+1},$$

the first coordinate of $(a,b)(I + P^r)$.

Corollary: Some multiple of an equivalent of a sequence $S = (c,d)$ in standard format is secondary from F if and only if there exist positive integers λ, r and nonnegative $m \le r + 1$ such that

$$\lambda c = F_{r+1-m} + (-1)^m F_{m-1}$$

$$\lambda d = F_{r+2-m} - (-1)^m F_{m-2}$$

Examination of the equations of the corollary makes it clear that stronger constraints operate on m than just $0 \le m \le r + 1$. In the next section we pin m down precisely.

4. BACKSPACE OF A SECONDARY SEQUENCE

Throughout this section, m will denote the backspace necessary to bring the sequence ${}^rT$ into standard format, where $T = (a,b)$ is in standard format and primitive.

Definition: The *eccentricity* of a sequence $T = (a,b)$ in standard format is the quantity $\varepsilon = b/a$.

Proposition 8: For $r = 2$, $m = \begin{cases} 0, & \varepsilon > 3 \\ 1, & T = F \text{ or } T = L \\ 2, & 2 < \varepsilon < 3 \end{cases}$

For $r = 3$, $m = 2$.

Proof: The terms ${}^2T_{-2}, \ldots, {}^2T_2$ are given, respectively, by $3b - 4a$, $3a - b$, $2b - a$, $2a + b$, and $a + 3b$.

If $3a < b$, $2(2a + b) = 4a + 2b < a + 3b$, so $m = 0$.

If $2a < b < 3a$, $3a - b > 0$ and $2(3a - b) = 6a - 2b < 3b - 2b = b < b + (b - 2a)$
$= 2b - 2a < 2b - a$, so $m = 2$.

The reader may confirm that $m = 1$ for $T = F$ and $T = L$. The case $r = 3$ has already been settled (implicitly) by Proposition 6.

Theorem 3: For $r = 2t$,

$$m = \begin{cases} t - 1, & \varepsilon > 3 \text{ and } t \text{ odd} \\ t, & t \text{ even or } T = F \text{ or } T = L \\ t + 1, & 2 < \varepsilon < 3 \text{ and } t \text{ odd} \end{cases}$$

Proof: From Proposition 4, Proposition 8, and the fact that $V_n = {}^2T_{n-1}$.

As we would expect by now, the case of r odd offers greater challenge and, as it turns out, some surprises.

Theorem 4: For $r > 3$, r odd, $r = 4k + 1$ or $r = 4k + 3$, $k \ge 1$. Define $A_r = F_{2k+2} - F_{r-2k-2}$, $B_r = F_{r-2k-3} + F_{2k+3}$, and $\alpha_r = B_r/A_r$. Then

$$m = \begin{cases} 2k, & \varepsilon < \alpha_r \\ 2k + 1, & \varepsilon = 1 \text{ or } \varepsilon = \alpha_r \\ 2k + 2, & 2 < \varepsilon < \alpha_r \end{cases}$$

Proof: Exclude at first the possibilities $T = F$ or ${}^rT \equiv \lambda F$. We examine the case of m even, arriving at the results of the theorem; then we show that m cannot be odd. Finally, we readmit F to the arena and distinguish cases to arrive at the remaining clause of the theorem, which allows for odd m.

Case I: $T \neq F$, ${}^rT \not\equiv F$. The equations (iv) of Theorem 2 give an exact expression for $T = (\lambda c, \lambda d)$ in standard format. Proposition 1 reminds us of the conditions λc and λd must satisfy in the event that ${}^rT$ is not equivalent to F:

$$\lambda c > 0, \text{ or}$$

$$(*) \qquad a[F_{r-m-1} + (-1)^m F_{m+1}] + b[F_{r-m} - (-1)^m F_m] > 0$$

$$2\lambda c < \lambda d, \text{ or}$$

$$(**) \qquad 2\left[a(F_{r-m-1} - (-1)^m F_{m+1}) + b(F_{r-m} - (-1)^m F_m)\right] < a[F_{r-m} - (-1)^m F_m] + b[F_{r-m+1} + (-1)^m F_{m-1}].$$

Subcase, m even. Let $i = m - 2k$, so that $m = 2k + i$, i even. Since $0 \leq m \leq r + 1$, $-2k \leq i \leq r - 2k + 1$. Equations (*) and (**) now take the forms

$$(*\text{e}) \qquad a(F_{r-2k-i-1} + F_{2k+i+1}) > b(F_{2k+i} - F_{r-2k-i})$$

$$(**\text{e}) \qquad a(F_{2k+i+3} + F_{r-2k-i-3}) < b(F_{2k+i+2} - F_{r-2k-i-2})$$

If $r - 2k + 1 \geq i \geq 2$, then $-1 \leq r - 2h - i \leq 4h + 3 - 2k - i \leq 2k - i + 3 \leq 2k + 1 < 2k + i$, and $2k + i \geq 4$, so the R.H.S. of (*e) is positive. Also, $T \neq F$ implies $2a < b$. Consequently,

$$a(F_{r-2k-i} + F_{2k+i+1}) > b(F_{2k+1} - F_{r-2k-i}) > 2a(F_{2k+i} - F_{r-2k-i})$$

and hence

$$F_{r-2k-i-1} + F_{2k+i+1} > 2(F_{2k+i} - F_{r-2k-i})$$

or, after simplification and use of the recurrence relation, $F_{r-2k-i+2} > F_{2k+i-2}$.

The subscripts are positive, so we must have $r - 2k - i + 2 > 2k + i - 2$ or $2i(r - 4k) + 4 < 7$, or $i < 7/2$. By hypothesis, i is positive and even, so $i = 2$ and $m = 2k + 2$.

If $-2k \leq i \leq 0$, then $2k + i + 3 \geq 3$ and $r - 2k - i - 3 \geq r - 2k - 3 = (4k + 1) - 2k - 3 = 2k - 2 \geq 0$, so the L.H.S. of (**e) is positive. As a result, the R.H.S. must also be positive, yielding $F_{2k+i+2} > F_{r-2k-i-2}$. The subscripts are positive, so we must have $2k + i + 2 > r - 2k - i - 2$, or $2i > (r - 4k) - 4 > -3$, or $i > -3/2$. By hypothesis, i is nonpositive and even, of $i = 0$ and $m = 2k$.

The upshot so far is that if m is even, it can only take on the values stated in the theorem.

In case $m = 2k + 2$, the R.H.S. of (*e) is positive, so dividing both sides by $a(F_{2k+2} - F_{r-2k-2})$ retains the sense of the inequality and yields

$$\varepsilon = b/a < (F_{r-2k-3} + F_{2k+3})/(F_{2k+2} - F_{r-2k-2}) = \alpha_r.$$

In case $m = 2$, the L.H.S. of (**e) is positive, so dividing both sides by $a(F_{2k+2} - F_{r-2k-2})$ retains the sense of the inequality and yields

$$\varepsilon = b/a > (F_{r-2k-3} + F_{2k+3})/(F_{2k+2} - F_{r-2k-2}) = \alpha_r.$$

Subcase, m odd. The equation (**) becomes

$$(**\text{o}) \qquad 2[a(F_{r-m-1} - F_{m+1}) + b(F_{r-m} + F_m)] < a(F_{r-m} + F_m) + b(F_{r-m+1} - F_{m-1}).$$

After simplification and use of the recurrence relation, we have

$$a(F_{m+3} - F_{r-m-3}) > b(F_{r-m-2} + F_{m+2}).$$

Since $r + 1 \geq m \geq 0$ and $r \geq 5$, the R.H.S. is positive. Since $T \neq F$, $b > 2a$, and so

$$a(F_{m+3} - F_{r-m-3}) > 2a(F_{r-m-2} + F_{m+2}) > 0$$

and

$$F_{m+3} - F_{r-m-3} > 2(F_{r-m-2} + F_{m+2}).$$

So after simplification and use of the recurrence, $-F_{m+1} - F_{r-m} > 0$, which is impossible for such positive subscripts.

Case II: $T = F$, ${}^rT \not\equiv \lambda F$. We have $a = b = 1$.

Subcase, m even. Equation (**) becomes $F_{m+3} + F_{r-m-3} < F_{m+2} - F_{r-m-2}$, which gives $F_{m+1} + F_{r-m-1} < 0$, which is impossible for such positive subscripts.

Subcase, m odd. Equations (*) and (**) become $F_{r-m-1} - F_{m+1} + F_{r-m} + F_m > 0$, so $F_{r-m-1} - F_{m-1} > 0$; $F_{m+3} - F_{r-m-3} > F_{r-m-2} + F_{m+2}$, so $F_{m+1} - F_{r-m-1} > 0$. The subscripts being nonnegative, these inequalities require that $m + 1 > r - m - 1$ and $r - m + 1 > m - 1$, or $r/2 - 1 < m < r/2 + 1$. The only integers between the bounds are $(r-1)/2$ and $(r+1)/2$, only one of which is odd. If $r = 4k + 1$, $(r+1)/2 = 2k + 1$ is odd; if $r = 4k + 3$, $(r-1)/2 = 2k + 1$ is odd. In either case, $m = 2k + 1$.

Case III: $T \neq F$, ${}^rT \equiv \lambda F$ for some λ.

Subcase, m even. Here we have now $\lambda c = \lambda d > 0$ and the corresponding substitute for (*) and (**):

$$a(F_{r-m-1} + F_{m+1}) + b(F_{r-m} - F_m) = a(F_{r-m} - F_m) + b(F_{r-m+1} + F_{m-1}) > 0.$$

Simplification gives $a(-F_{r-m-2} + F_{m+2}) = b(F_{r-m-1} + F_{m+1})$, which is positive since the subscripts on the R.H.S. are positive. Using the fact $b > 2a$, and dividing by a, we get $F_{m+2} - F_{r-m-2} > 2(F_{r-m-1} + F_{m+1})$, which leads to the contradiction $-F_{m-1} - F_{r-m+1} > 0$.

Subcase, m odd. The equations of (iv) of Theorem 2 become

$$a(F_{r-m-1} - F_{m+1}) + b(F_{r-m} + F_m) = a(F_{r-m-2} + F_m) + b(F_{r-m-1} + F_{m+1}) > 0.$$

Simplification gives $a(F_{r-m-2} + F_{m+2}) = b(-F_{r-m-1} + F_{m+1})$. The subscripts on the L.H.S. are, respectively, nonnegative (m odd implies $m \leq 2k - 1$) and positive, so that $-F_{r-m-1} + F_{m+1} > 0$; and using the familiar $b > 2a$ and dividing by a in the original inequality gives $F_{r-m-2} + F_{m+2} > 2(F_{m+1} - F_{r-m-1})$. Simplification reduces this to $F_{r-m+1} > F_{m-1}$. We are now in the situation of Case II, m odd, so we may conclude $m = 2k + 1$. Here, $b/a = B_r/A_r$ follows without difficulty.

Case IV: $T = F$, ${}^rT \equiv \lambda F$. We may follow Case III to the points

m even: $a(F_{r-m-2} - F_{m+2}) = b(F_{r-m-1} + F_{m+1})$;

m odd: $a(F_{r-m-2} + F_{m+2}) = b(-F_{r-m-1} + F_{m+1})$.

Here in Case IV we have $a = b = 1$:

m even: $F_{r-m-2} - F_{m+2} = F_{r-m-1} + F_{m+1}$, so $F_m = F_{r-m}$ and either $r = 2m$ (impossible: r is odd); $m = 1$, $r = 3$ (impossible: m is even); or $m = 2$, $r = 3$ (excluded by hypothesis).

m odd: $F_{r-m-2} + F_{m+2} = -F_{r-m-1} + F_{m+1}$, so $F_{r-m} + F_m = 0$, and the restriction $0 \leq m \leq r + 1$ forces the contradiction $m = r = 0$.

Corollary: For $r = 4k + 1$, $k \geq 1$:

$$A_r = 2F_{2k}, \quad B_r = 2F_{2k} + F_{2k+2}, \quad \lim_{\substack{r=4k+1 \\ k\to\infty}} \alpha_r = \frac{\phi + 3}{2} \approx 2.309.$$

For $r = 4k + 3$, $k \geq 1$: $A_r = F_{2k}$, $B_r = F_{2k} + F_{2k+3}$, $\lim_{\substack{r=4k+3 \\ k\to\infty}} \alpha_r = 2(\phi + 1) \approx 5.236$.

(The number ϕ is the golden ratio.) Moreover, because of the recurrence relation for F, each of the sequences $\{\alpha_{4k+1}\}$, $\{\alpha_{4k+3}\}$ consists of every other term of the respective Farey sequences $\{(2F_n + F_{n+2})/2F_n\}$, $\{(F_n + F_{n+3})/F_n\}$.

Proof:

$$A_{4k+1} = F_{2k+2} - F_{4k+1-2k-2} = F_{2k+2} - F_{2k-1} = F_{2k+1} + F_{2k} - F_{2k-1} = 2F_{2k}.$$

$$B_{4k+1} = F_{4k+1-2k-3} + F_{2k+3} = F_{2k-2} + F_{2k+2} + F_{2k+1} = 2F_{2k+1} + F_{2k} + F_{2k-2}$$

$$= 3F_{2k} + 2F_{2k-1} + F_{2k-2} = 2F_{2k} + 2F_{2k} + F_{2k-1} = 2F_{2k} + F_{2k+2}.$$

$$\lim_{k\to\infty} \alpha_{4k+1} = \lim_{k\to\infty}(2F_{2k} + F_{2k+2})/2F_{2k} = 1 + \tfrac{1}{2}\phi^2 = (\phi + 3)/2.$$

$$A_{4k+3} = F_{2k+2} - F_{4k+3-2k-2} = F_{2k+2} - F_{2k+1} = F_{2k}.$$

$$B_{4k+3} = F_{4k+3-2k-3} + F_{2k+3} = F_{2k} + F_{2k+3}.$$

$$\lim_{k\to\infty} \alpha_{4k+1} = \lim_{k\to\infty}(F_{2k} + F_{2k+3})/F_{2k} = 1 + \phi^3 = 2(\phi + 1).$$

(In each case the existence of the limit is guaranteed because the sequence is monotone and bounded.)

We present below a table of the Farey sequences which contain the values α_r. The parenthetical entries, consisting of the values of the Farey sequences intermediate between values α_r, form their own sequence which we shall call β_r:

Definition: $\beta_{4k+1} = (F_{2k-1} + F_{2k+2})/F_{2k-1}$;

$\beta_{4k+3} = (2F_{2k+1} + F_{2k+3})/2F_{2k+1}$.

We even examine what the calculated values of α_r and β_r would be for $r = 3$ and $r = 1$, even though the theorem above does not extend to these.

In fact, we can extend the definition of the α's and β's as follows:

Definition: $\alpha_{2t} = \beta_{2t} = 3$, t odd;

$\alpha_{2t} = 3$, $\beta_{2t} = 2$, t even;

$\alpha_3 = 3$, $\beta_3 = 2$, $\alpha_1 = 2$, $\beta_1 = 2$.

r ...	1	3	5	7	9	11	13	15	17	19	...	
...	$\frac{4}{2}$	$\frac{1}{0}$	$\frac{5}{2}$	$\frac{9}{4}$	$\frac{14}{6}$	$\frac{23}{10}$	$\frac{37}{16}$	$\frac{60}{26}$	$\frac{97}{42}$	$\frac{157}{68}$	...	$\to (\phi + 3)/2$
...	$\frac{2}{1}$	$\frac{2}{0}$	$\frac{4}{1}$	$\frac{6}{1}$	$\frac{10}{2}$	$\frac{16}{3}$	$\frac{26}{5}$	$\frac{42}{8}$	$\frac{68}{13}$	$\frac{110}{21}$	...	$\to 2(\phi + 1)$

Thus, we have the sequences formed as follows, from first element on:

$$\alpha:\ 2,\ 3,\ 3,\ 3,\ \frac{5}{2},\ 3,\ \frac{6}{1},\ 3,\ \frac{14}{6},\ 3,\ \frac{16}{3},\ 3,\ \frac{37}{16},\ 3,\ \ldots$$

$$\beta:\ 2,\ 3,\ 2,\ 2,\ \frac{4}{1},\ 3,\ \frac{9}{4},\ 2,\ \frac{10}{2},\ 3,\ \frac{23}{10},\ 2,\ \frac{26}{5},\ 3,\ \ldots$$

The results of this section so far may be summed up in saying that m depends only on r and ε and is uniquely determined once they are specified. The same is true for the quantity d/c. Easy algebra applied to the equations (iv) of Theorem 2 yields a general formula for d/c; we rename this quantity $\delta_r(\varepsilon)$ to indicate the independent variables on which it depends. It is convenient, however, to express it in terms of the variable m also, which itself depends on r and ε.

Proposition 9: The eccentricity $\delta_r(\varepsilon)$ of T^r, where ε is the eccentricity of T, is given by

$$\delta_r(\varepsilon) = \frac{[F_{r-m} - (-1)^m F_m] + [F_{r-m+1} + (-1)^m F_{m-1}]}{[F_{r-m} + (-1)^m F_{m+1}] + [F_{r-m} - (-1)^m F_m]}$$

Conversely,

$$\varepsilon = \frac{\delta_r(t)[F_{r-m} + (-1)^m F_{m+1}] - [F_{r-m} - (-1)^m F_m]}{[F_{r-m+1} + (-1)^m F_{m-1}] - \delta_r(\varepsilon)[F_{r-m} - (-1)^m F_m]}$$

The function δ_r is one-to-one, so that ε in turn is uniquely determined by r and δ_r; in other words, we may speak of the inverse function δ_r^{-1}.

Proof: If $\delta_r(\varepsilon_1) = \delta_r(\varepsilon_2)$, then the corresponding secondary sequences (using left subscripts to distinguish) ${}^r_1T$, ${}^r_2T$ must be equivalent to multiples of the same primitive sequence U, so ${}^r_1T \equiv k_1U$, ${}^r_2T \equiv k_2U$. By Proposition 7(iii), for $i \in \{1,2\}$, ${}^{1/r}({}_iT) \equiv {}^{1/r}(k_iU) = k_i^{1/r}U$ is the only sequence, up to equivalence, whose rth secondary sequence is T. But the upshot is that ${}^{1/r}_{\ 1}T$ and ${}^{1/r}_{\ 2}T$ must be equivalent to multiples of the same primitive sequence T. Hence $\varepsilon_1 = \varepsilon_2$.

Proposition 10: $\varepsilon_{\overline{T}} = 2 + 1/(\varepsilon - 2)$.

Proof: $T_0 = b - a$, $T_{-1} = 2a - b$, $T_{-2} = 2b - 3a$, so $\varepsilon_{\overline{T}} = (2b - 3a)/(b - 2a) = (2\varepsilon - 3)/(\varepsilon - 2) = 2 + 1/(\varepsilon - 2)$.

Theorem 5: For $r = 1$, $r = 3$, or $r \equiv 0 \pmod 4$, $\delta_r(\varepsilon) = \varepsilon$. Otherwise, δ_r maps

$$1 \to \beta_r$$

$$(2,\alpha_r) \to (\beta_r,\infty),\ \text{order-preserving}$$

$$\alpha_r \to 1$$

$$(\alpha_r,\infty) \to (2,\beta_r), \text{ order-preserving}$$

and δ_r is a bijection from $\{1\} \cup (2,\infty)$ into itself.

Proof: For $r \neq 3$, $r \not\equiv 0 \pmod 4$, and $\varepsilon \neq 1$, $\varepsilon \neq \delta_r$, we have m even, so that the first equation of Proposition 9 holds with the $(-1)^m$ deleted.

$\varepsilon < \delta_r$ implies $m = 2k + 2$, if r is odd, and $m = t + 1$, if $r = 2t$, t odd.

$$\lim_{\varepsilon \to 2^+} \delta_r(t) = \frac{[F_{r-m} - F_m] + 2[F_{r-m+1} + F_{m-1}]}{[F_{r-m-1} + F_{m+1}] + 2[F_{r-m} - F_m]} = \frac{F_{r-m-3} + F_{m-3}}{F_{r-m-2} - F_{m-2}}$$

since δ_r is clearly continuous in ε on $(2,\alpha_r)$. Treatment by cases gives

$$\lim_{\varepsilon \to 2^+} \delta_r(\varepsilon) = \begin{cases} (F_{2k+4} + F_{2k-1})/(F_{2k+3} - F_{2k}) = (F_{2k+3} + 2F_{2k+1}) = \beta_{4k+3}, & \text{for } r = 4k+3; \\ (F_{2k+2} + F_{2k-1})/(F_{2k+1} - F_{2k}) = \beta_{4k+1}, & \text{for } r = 4k+1; \\ (F_{t+2} + F_{t-2})/(F_{t+1} - F_{t-1}) = 3F_t/F_t = 3 = \beta_{2t}, & \text{for } r = 2t,\ t \text{ odd.} \end{cases}$$

In short, $\lim_{\varepsilon \to 2^+} \delta_r(\varepsilon) = \beta_r$. Similarly, $\lim_{\varepsilon \to \alpha_r^-} \delta_r(t) = \infty$. The numerator of $\delta_r(\varepsilon)$ is of the form $e + \varepsilon f$, while the denominator is of the form $g + \varepsilon h$. Now, with r given, the fact that ε is in $(2,\alpha_r)$ determines m, so that in this interval e, f, g, and h are constant.

$$\frac{d}{d\varepsilon}\delta_r(\varepsilon) = \frac{d}{d\varepsilon}\frac{e + f}{g + f} = \frac{f(g + h) - h(e + f)}{(g + h)} = \frac{fg - he}{(g + h)^2}.$$

So the sign of the derivative of δ_r is constant in $(2,\alpha_r)$. From the limits established above, we realize that δ_r is increasing throughout $(2,\alpha_r)$.

The same argument may be applied to the behavior of δ_r on (α_r,∞).

The cases $r = 1$, $r = 3$, $r \equiv 0 \pmod 4$ offer no challenge.

Example:

$$\delta_5(\varepsilon) = \begin{cases} 4, & \varepsilon = 1 \\ (3\varepsilon - 2)/(5 - 2\varepsilon), & 2 < \varepsilon < 2\frac{1}{2} \\ 1, & \varepsilon = 2\frac{1}{2} \\ (1 + 4\varepsilon)/(3 + \varepsilon), & \varepsilon > 2\frac{1}{2} \end{cases}$$

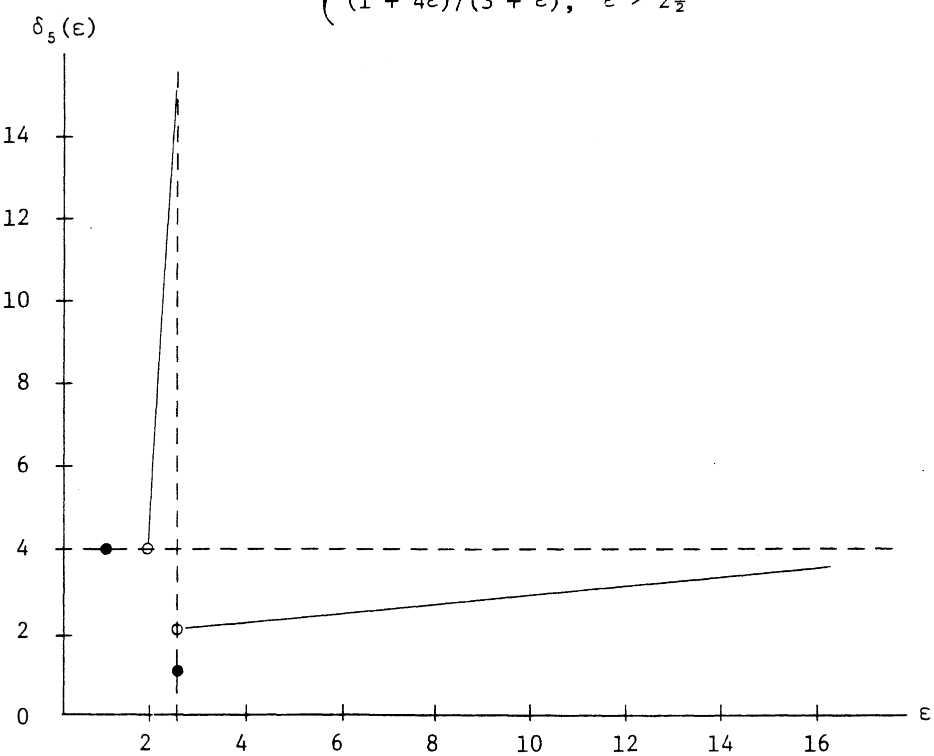

FIGURE 1

5. CHARACTERISTIC NUMBERS OF SECONDARY SEQUENCES

The concept of characteristic number of a Fibonacci sequence was introduced in [1] to structure the collection of Fibonacci sequences.

Definition: The *characteristic number* D_T of a Fibonacci sequence T is $D_T = |T_n^2 - T_{n-1}T_{n+1}|$.

Readers familiar with the elementary properties of Fibonacci sequences will recognize that the value of D_T is independent of the choice of n, so that D_T is well defined.

A table of characteristic numbers of primitive Fibonacci sequences for $D < 2000$ can be found in [4, pp. 42-44].

We summarize some useful properties of characteristic numbers in the following proposition.

Proposition 11: (i) $D_{kT} = k^2 D_T$; (ii) $D_V = 5D_T$; (iii) $D_{\bar{T}} = D_T$.

Proof: Left to the reader.

Proposition 12: (i) A natural number $n = a^2 b$, b square-free, is the characteristic number of a [primitive] Fibonacci sequence if and only if all prime factors of b are of the forms $10k \pm 1$ and 5 [and additionally, all prime factors of a are of the forms $10k \pm 1$]. (ii) Let D have n distinct-prime factors of the forms $10k \pm 1$. Then there are exactly 2^n primitive sequences with characteristic number D.

Proof: (i) Cf. Theorem 2 of [9, p. 78]. The same source gives an expression $r(D)$ for the number of inequivalent Fibonacci sequences having characteristic D. The only difference here is the observation that the only primes of the forms $5k \pm 1$ are indeed of the forms $10k \pm 1$.

Note that D_T may have square factors even for primitive T; for example, $D_{(3,13)} = 121 = 11^2$. (ii) See [3] and [9].

Theorem 6: Let $S \equiv {}^rT$. Then $D_S = D_T[L_r + 1 + (-1)^r]$.

Proof:
$$\begin{aligned}
D_{rT} &= |({}^rT_n)^2 - {}^rT_{n+1}{}^rT_{n-1} = ({}^rT_2)^2 - {}^rT_3{}^rT_1| \\
&= |(T_2 + T_{r+2})^2 - (T_3 + T_{r+3})(T_1 + T_{r+1})| \\
&= |T_2^2 + 2T_2T_{r+2} + T_{r+2}^2 - T_3T_1 - T_3T_{r+1} - T_1T_{r+3} - T_{r+3}T_{r+1}| \\
&= |(T_2^2 - T_3T_1) + (T_{r+2}^2 + T_{r+3}T_{r+1}) + 2T_2T_{r+2} - T_1T_{r+3} - T_3T_{r+1}| \\
&= |(T_2^2 - T_3T_1)[1 + (-1)^r] + 2T_2(F_rT_1 + F_{r+1}T_2) \\
&\qquad - T_1(F_{r+1}T_1 + F_{r+2}T_2) - (T_1 + T_2)(F_{r-1}T_1 + F_rT_2)| \\
&= |(T_2^2 - T_3T_1)[1 + (-1)^r] + T_2^2(2F_{r+1} - F_r) - T^2(F_{r+1} + F_{r-1}) \\
&\qquad - T_1T_2(F_{r+2} + F_{r+1} - 2F_r)| \\
&= |(T_2^2 - T_3T_1)[1 + (-1)^r] + T_2^2(F_{r+1} + F_{r-1}) - T_1^2(F_{r+1} + F_{r-1}) - T_1T_2(F_{r+1} + F_{r-1})| \\
&= |(T_2^2 - T_3T_1)[1 + (-1)^r] + L_i(T_2^2 - T_1^2 - T_1T_2)| \\
&= |(T_2^2 - T_3T_1)[1 + (-1)^r + L_i]| \\
&= D_T[1 + (-1)^r + L_i].
\end{aligned}$$

Corollary: Let $T \equiv {}^{1/r}S$. Then $D_T = D_S/[L_i + 1 + (-1)^r]$.

Corollary: D_TL_r square-free, r odd $\Rightarrow {}^rT$ primitive.

Proof: Immediate from Proposition 7(ii) and Theorem 6.

Corollary: Let $S \equiv {}^rT$. Then

$$D_S = \begin{cases} D_T \cdot (L_r + 2), & r \text{ even} = \begin{cases} D_TL_t^2, & r = 2t,\ t \text{ even} \\ D_T \cdot 5 \cdot F_t^2, & r = 2t,\ t \text{ odd} \end{cases} \\ D_T \cdot L_r, & r \text{ odd} \end{cases}$$

The question of which Fibonacci sequences occur as secondary sequences is completely settled by the work of Section 4, but only if we are willing to identify multiples of equivalent sequences; the answer then is that every sequence is, for every r, r-secondary. If, however, we decline to make the identification, our curiosity may be piqued by examples like the following.

Example: An examination of the table of characteristic numbers of primitive sequences provides the information:

Characteristic Number	Corresponding Sequences (in conjugate pairs)
11	(1,4)(2.5)
19	(1,5)(3,7)
209 = 11 · 19	{ (1,15)(13,27) / (5,18)(8,21) }

We note the following relations:

$^9(1,4) \equiv 2(8,21)$
$^9(2,5) \equiv 2(13,27)$ $\quad$ $^5(13,27) \equiv (1,5)$ $\quad$ $^9(5,18) \equiv (2,5)$
$^5(5,18) \equiv (3,7)$

$^5(1,5) \equiv (8,21)$
$^5(3,7) \equiv (1,15)$ $\quad$ $^9(1,15) \equiv (1,4)$

We may abstract this information into the table below, where a + represents that a secondary sequence of the sequence in the left column is equivalent to a multiple of the sequence in the top row; and a - represents the reverse.

	(1,15)	(13,27)	(5,18)	(8,21)
(1,4)	-			+
(2,5)		+	-	
(1,5)		-		+
(3,7)	+		-	

What is strange is that although one multiple each of (1,15) and (13,27) is equivalent to a secondary sequence, and (8,21) has this happen twice, it fails to happen at all for (5,18). At least, no multiple of (5,18) is secondary from an equivalent of what seem the most likely candidates: that is, the four primitive sequences with characteristic number dividing 209, the characteristic number of (5,18). It may come as a surprise that the characteristic number of a secondary sequence need not be a multiple of that of the sequence it is secondary from, and even that a sequence can be secondary from another of much larger characteristic number. The exact conditions are given in the theorem below.

Definition: Let a sequence T be a multiple of an equivalent of the primitive sequence U; we will refer to U as the *base* of T.

Proposition 13: Let $D_S = x^2y$, $x,y \in \mathbb{Q}^+$ (so S not necessarily integral), and let $\lambda S \equiv {}^rT$. Then D_T cannot be of the form u^2y, unless $r = 1$, $r = 3$, or $r \equiv 0 \pmod 4$.

Proof: Suppose $D_T = u^2y$. Then if r is odd, $\lambda^2D_S = \lambda^2x^2y = L_r \cdot u^2y = L_r \cdot D_T$, which implies L_r is a square. By [2], the only square Lucas numbers are $L_1 = 1$, $L_3 = 4$. If $r \equiv 2 \pmod 4$, then $\lambda^2D_S = \lambda^2x^2y = F^2_{r/2} \cdot 5 \cdot u^2y$, which is impossible.

Corollary: Let S be a primitive sequence with $D_S = m^2 > 1$. Then no multiple of S is secondary from an equivalent of F. That is, no secondary sequence of F has a base whose characteristic number is a perfect square greater than 1.

Proof: Secondary sequences of F of even order have either F or L as their base, and $D_F = 1$, $D_L = 5$. Suppose ${}^rF \equiv \lambda S$, r odd. Since $D_S = m^2 > 1$, but $D_F = 1$, then by the proposition we must have $r = 1$ or $r = 3$. But ${}^1F \equiv F$, which has $D = 1$, while ${}^3F \equiv 2F$, which is not primitive.

Example: $S = (7,17)$, $D_S = 11^2$. S is not secondary from any equivalent of F, nor from any sequence T with $D_T < 11^2$.

Theorem 7: Let r and S be given, $r \in \mathbb{N}$ and S a primitive sequence. Then the only solutions to $T \equiv \lambda S$ with T primitive are:

	λ	T
$r = 2t$, t odd:	F_t	$V(S)$
$r = 2t$, t even:	L_t	S
$r = 1$	1	S
$r = 3$	2	S
r odd, $r \geq 5$:	ij	$ij \cdot {}^{1/r}S$

where i and j are determined as follows:

Let $G = \text{GCD}(D_S, L_r)$, with $d = D_S/G$, $\ell = L_r/G$, and write ℓ as $\ell = i^2j$, j square-free.

Proof: It suffices to direct our attention to the last case listed, the others being straightforward consequences of earlier theorems.

If r is odd, $r \geq 5$, then λ must satisfy $L_r \cdot D_T = \lambda^2 D_S$ in such a fashion that D_T is integral. For such a λ, $T = \lambda^{1/r} S$ is guaranteed to be the unique solution of ${}^rT \equiv \lambda S$ by Proposition 7(iii). So, the only question is what values are admissible for λ.

Using the notation of the theorem, we have

$$D_T = \frac{\lambda^2 D_S}{L_r} = \frac{\lambda^2 dG}{\ell G} = \frac{\lambda^2 d}{\ell}.$$

Since GCD(d,ℓ), $\ell = i^2 j$ must divide λ^2. Any λ satisfying this requirement yields a solution; the smallest such λ is ij, and for some multiple of $\lambda = ij$, the sequence T is primitive. Larger values of λ lead to multiples of that sequence.

Definition: If a prime p divides some member of the Lucas sequence, then the first member L_n of L which p divides is known as the *entry point of p in L*, and p is called a *primitive prime divisor* of L_n. We say *p enters L* at index n.

Proposition 14: (i) If a prime p enters L at L_n, then $p|L_{n(2k-1)}$, $k \in \mathbb{N}$, and p divides no other members of L. (ii) a) The primes which enter $\{L_{2n}\}$ include all primes of the forms $20k + 3$, $20k + 7$, and some primes of the forms $20k + 1$, $20k + 9$; b) for $\{L_{2n+1}\}$, all primes of the forms $20k + 11$, $20k + 19$, and 2, and a different collection of primes of the forms $20k + 1$, $20k + 9$; c) for $\{F_{2n+1}\}$, all primes of the forms $20k + 13$, $20k + 17$, and 5, plus the remaining primes of the forms $20k + 1$, $20k + 9$; d) all primes enter $\{F_{2n}\}$.

Proof: Lucas was the first to prove (i) [10, p. 35]; he also proved most of (ii) [10, pp. 22-23], though Zeckendorf [13] was the first to prove it in the version given (it is usually called Zeckendorf's Theorem).

Corollary to the Theorem: Let S and T be primitives ${}^rT \equiv \lambda S$. If r is odd and not less than 5, and D_S has no prime factors which enter L at odd index, then D_T is a multiple of D_S.

Proof: Apart possibly from 2, the prime factors of L_r all enter L at odd index. Since $L_r \cdot D_T = \lambda^2 D_S$, and GCD$(D_S, L_r) = 1$ (S is primitive, so $2 \nmid D_S$), $D_S | D_T$.

Corollary: Given primitive sequences S and T, and given r, a necessary (but not sufficient) condition for ${}^rT \equiv \lambda S$ to hold is that sqf$(D_T D_S)$ = sqf(L_r), where for $n = e^2 f$, f square-free, sqf$(n) = f$.

Proof: If ${}^rT \equiv \lambda S$, sqf$(D_T D_S)$ = sqf$(\lambda^2 D_S D_T)$ = sqf$(L_r D_T)$ = sqf(L_r).

The sequence $S = (5,18)$ has $D_S = 209 = 11 \cdot 19$, and the sequence $T = (3,7)$ has $D_T = 19$, so that sqf$(D_S D_T) = 11 = L_5$. But we have seen that ${}^5(3,7) \equiv (1,15)$, which is not a multiple of (5,18).

The result of the second corollary tells us that characteristic number alone cannot give us a complete criterion for judging if one sequence is secondary from another. Of course, this was to be expected, since there are always at least two sequences with the same characteristic number (unless it is 1).

In the example in the proof of the corollary, everything would work out nicely if we were to identify conjugate sequences, for $\overline{S} = (8,21)$, $\overline{T} = (1,5)$, and ${}^5(1,5) = (8,21)$. This will not work in general, however. Consider any D_S, D_T, each with at least two prime factors apart from possibly 5. Then to each of D_S, D_T, there correspond at least two pairs of conjugate sequences, and it is easy to envision a "switch" that allows sqf$(D_S D_T)$ to be equal to sqf(L_r) without any of ${}^rT \equiv \lambda S$, ${}^r\overline{T} \equiv \lambda S$, ${}^r\overline{T} \equiv \lambda \overline{S}$, ${}^rT \equiv \lambda \overline{S}$ holding.

For concreteness, take $D_S = 589 = 19 \cdot 31$, $S = (7,29)$. $D_T = 209 = 11 \cdot 19$, $T = (5,18)$, $r = 15$, $L_r = 1364 = 4 \cdot 11 \cdot 31$. Then sqf$(D_T D_S) = 11 \cdot 31$ = sqf(L_r), but ${}^{15}T \equiv 22(3,26)$, ${}^{15}\overline{T} = {}^{15}(8,21) \equiv 2(84,325)$, while $\overline{S} = (15,37)$.

From among the four items r, λ, S, and T (S, T primitive), specification of any two either determines what the other two must be for there to be a solution to ${}^rT \equiv \lambda S$, or else determines that no solution exists.

Example: $L_{25} = 167761 = 11 \cdot 101 \cdot 151$.

Suppose $D_S = 101$, $D_T = 151$. Then $\lambda S \equiv {}^{25}T$ is impossible since $101 \cdot 151$ = sqf$(D_S D_T) \neq$ sqf $L_{25} = 11 \cdot 101 \cdot 151$.

However, since $L_5 = 11$, we are led to wonder if perhaps λS could be reached from T in two stages; for example, $151 \cdot S \stackrel{?}{=} {}^{1/5}({}^{25}T)$. This will be our next topic of investigation.

6. CHAIN-SECONDARY SEQUENCES

Definitions: ${}^-T = \overline{T}$

$\mathbb{E} = \{2n | n \in \mathbb{N}\} \cup \{1/2n | n \in \mathbb{N}\}$

$\mathbb{O} = \{2n - 1 | n \in \mathbb{N}\} \cup \{1/(2n - 1) | n \in \mathbb{N}\}$

$$B = E \cup 0 = \{n | n \in N\} \cup \{1/n | n \in N\}$$
$$X^- = X \cup \{\overline{X}\},\ X = E,\ 0,\ \text{or } B$$
$$^{r_k \cdots r_1}T = {}^{r_k}({}^{r_{k-1}} \ldots(\ldots {}^{r_2}({}^{r_1}T)\ \ldots)),\ r_i \in B^-.$$

Definition: A primitive sequence S is a *chain-secondary sequence* of a primitive sequence T if and only if there is a chain $\{{}_iT\}_{i=0}^k$ of (not necessarily integral) sequences such that

(i) S is the base of ${}_kT$, with ${}_kT \equiv \lambda_0 S$, $\lambda_0 \in Q^+$

(ii) $T \equiv {}_0T$

(iii) for each i between 1 and k inclusive, there are $\lambda_i \in B$ and $r_i \in B^-$ such that ${}^{r_i}_{i-1}T \equiv \lambda_i\ {}_iT$.

When such a chain exists, we say that S is *derivable* from T, writing $S \leftarrow T$.

Notice that allowing $\lambda_i \in Q^+$ would not achieve any greater generality, since we are free to have as many "links" in the chain with $r_i = 1$ as we like.

The definition in effect allows free substitution of a sequence for its conjugate in pursuing a derivation from T to S, without going so far as to identify the two conjugate sequences. We have already seen, following Theorem 6, an example of ${}^{-,r}T \not\equiv {}^{r,-}T$ when r is odd and $r \geq 5$; the introduction of conjugates in fact banishes us from the complete commutativity we would otherwise enjoy in conjugate-free chains:

Proposition 15: ${}^{s,r}T \equiv {}^{r,s}T$, for $r,s \in B$.

Proof: For $r,s \in N$:

$$\begin{aligned}({}^{s,r}T)_n &= {}^rT_{n+s} + {}^rT_n = (T_{n+s+r} + T_{n+s}) + (T_{n+r} + T_n)\\ &= (T_{n+r+s} + T_{n+r}) + (T_{n+s} + T_n)\\ &= {}^sT_{n+r} + {}^sT_n = ({}^{r,s}T)_n\end{aligned}$$
$${}^{s,1/r}T \equiv {}^{1/r,\,r,\,s,\,1/r}T \equiv {}^{1/r,\,s,\,r,\,1/r}T \equiv {}^{1/r,\,s}T$$

since we now know we are allowed to pass s all the way to the left.

The condition $S \leftarrow T$ is equivalent to the existence of k, and some $\lambda_i \in B$, $r_i \in B$, for $i = 1, \ldots, k$, $\lambda_0 \in Q^+$, such that

$$^{r_k \cdots r_1}T \equiv \left(\prod_{i=0}^{k} \lambda_i\right) S. \qquad (1)$$

Proposition 16: $S \leftarrow T$ if and only if there is a chain $\{{}_iT'\}_{i=0}^{k'}$ of (not necessarily integral) sequences such that

(i) S is the base of ${}_kT'$, with ${}_kT' \equiv \lambda_0' S$, $\lambda_0' \in Q^+$

(ii) $T \equiv {}_0T'$

(iii) for each i between 1 and k' inclusive, there are $\lambda_i' \in B$ and $r_i' \in Q^+$ such that ${}^{r_i'}_{i-1}T' \equiv \lambda_{i'}'\ {}_iT'$.

(iv) $r_1' = 1$ or $r_1' = 2$.

Proof: The operation ${}^r(\ )$ commutes with $(^-)$ for $r \in E$, by Proposition 5 and the definition of ${}^{1/r}(\ )$ for $r \in N$; and we have just seen in Proposition 15 that ${}^r(\ )$ commutes with ${}^s(\ )$, up to equivalence, for $r,s \in B$. The net effect of our remarks is that any "link" in the chain for which $r_i \in E$—call it an "even link"—may be repositioned elsewhere in the chain while preserving $S \leftarrow T$. In particular, we may permute the links of the chain so that all even links occur first, still preserving $S \leftarrow T$, provided we do not alter the order of succession of the remaining links. Even links are trivial, in that apart from altering ${}_{i-1}T$ by a factor F_t/λ_i or L_t/λ_i they do not affect it at all, except possibly to transform it to its Lucas dual. We conveniently absorb all of the multiplicative effect of the even links into λ_0'. We may then eliminate all of them except possibly for a single link with $r = 2$, since as an operation the Lucas dual has order 2.

Proposition 16: The relation $\leftarrow$ is an equivalence relation (and henceforth we will write it as $\leftrightarrow$).

Proof: Reflexivity and transitivity offer no difficulty. If $S \leftarrow T$, so that (1) holds, then ${}^{1/r_1,\ \ldots,\ 1/r_k}S = \left[\prod_{i=0}^{k}\left(\frac{1}{\lambda_i}\right)\right]T$ with the symbol $1/^-$ defined to be $^-$; and $T \leftarrow S$.

Definition: The equivalence classes into which $\leftrightarrow$ divides the set of all primitive sequences we will refer to as *families*. The *Brousseau number* of a family is the smallest of the characteristic numbers associated with members of the family; the corresponding sequence and its conjugate are the *founders* of the family. We will represent the set of Brousseau numbers by $\mathcal{B}$. The set $\mathcal{L}$ of *$\mathcal{L}$-factors* is the set

$$\mathcal{M} \cup \{5m \mid m \in \mathcal{M}\}$$

where $\mathcal{M}$ is the smallest subset of Q^+ containing all odd-index Lucas numbers which is closed under multiplication, division, and powers.

Examples: $L_{45} = 2537720636 = 4 \cdot 11 \cdot 19 \cdot 31 \cdot 97921$ gives rise to the following L-factors: $19 \cdot 97921$ (since $4 \cdot 11 \cdot 31 = L_{15}$), $31 \cdot 97921$ (since $4 \cdot 11 \cdot 19 = L_5L_9$), $31 \cdot 19 \cdot 97921$ (since $4 \cdot 11 = L_3L_5$), and L_{45} itself.

In light of Proposition 16, the condition that $S \leftarrow T$ is equivalent to (1) holding for some k and some $\lambda_0 \in Q^+$, $\lambda_i \in b$, $r_i \in 0$, $i = 1, \ldots, k$, with $r_1 = 1$ or 2. Converting to the corresponding necessary condition on characteristic numbers gives

$$D_T 5^\alpha \frac{\prod_{r_i \in N} L_{r_i}}{\prod_{1/r_i \in N} L_{1/r_i}} = \frac{q_1^2 \prod_{r_i \in N} \lambda_i^2}{q_2^2 \prod_{1/\lambda_i \in N} (1/\lambda_i)^2} D_S, \tag{2}$$

or

$$D_T \cdot 5^\alpha \cdot \prod_{r_i \in N} L_{r_i} \cdot q_2^2 \cdot \prod_{1/\lambda_i \in N} (1/\lambda_i)^2 = D_S \cdot \prod_{1/r_i \in N} L_{1/r_i} \cdot q_1^2 \cdot \prod_{\lambda_i \in N} \lambda_1^2$$

with $\alpha = 0$ or 1; and $\lambda = q_1/q_2$ in lowest terms, $q_i \in N$.

Proposition 17: Let S,T be primitive. Let p be a prime which is not an odd-index-entry Lucas prime. If $p^t | D_S$, then $p^t | D_T$, for $t \in N$.

Proof: The only other possibility is that p^t is "absorbed" by the denominator of the fraction on the R.H.S. of (2). Denote that denominator by B^2, and the corresponding numerator by A^2, and suppose that $p^t | B^2$, $p^t | A^2$. From (1) we know that $B^{r_k \cdots r_1}T \equiv A \cdot S$. Now, the fact that p does not divide any L_r, r odd, means that no term $(^{r_k \cdots r_1}T)$, written in lowest terms, can have p as a factor of its denominator, since $(^{r_k \cdots r_1}T)$ can incur only $\mathcal{L}$-factors there. Hence, $p|B$ implies $p|A$, because S is primitive; but this leads to the conclusion that $p^t | D_T$.

Consequently, a prime of the form $10k \pm 1$ which has odd-index entry in F or even-index entry in L is a Brousseau number, for some family of sequences. The product of powers of such primes is also a Brousseau number, and we will call such numbers *Brousseau numbers of the first kind*. Every sequence whose characteristic number is a Brousseau number of the first kind is the founder of a family.

The remaining Brousseau numbers are either products of powers of primes of odd-index Lucas entry (*the second kind*), or mixed products of Brousseau numbers of the first and second kinds (*the third kind*).

Example: $D_F = 1$, $D_{(1,7)} = 41$, $D_F \cdot D_{(1,7)} = 41$. But $41 | L_{10+20k}$ and no other Lucas numbers; hence $41 \notin \mathcal{L}$, so F and (1,7) must be in different equivalence classes.

Example: $L_{25} = 11 \cdot 101 \cdot 151 = L_5 \cdot 101 \cdot 151$. The primes 101 and 151 are both primitive prime divisors of L_{25}, and both have period 50. Each of them is a Brousseau number, but their product is an $\mathcal{L}$-factor.

Corollary: Two sequences with relatively prime Brousseau numbers belong to different families.

Theorem 8: If S and T are in the same family, then D_TD_S is an $\mathcal{L}$-factor times a rational square. If S and T are both primitive, then $\text{sqf}(D_TD_S) = \text{sqf}(\ell)$, ℓ an integral $\mathcal{L}$-factor.

Proof: Algebraic manipulation of (2) easily leads to the first conclusion, with, say, $D_TD_S = \frac{\ell_1}{\ell_2} \cdot \frac{s_1^2}{s_2^2}$, ℓ_1, ℓ_2 products of odd-index Lucas numbers, $s_1, s_2 \in N$, s_1/s_2 in lowest terms. If S and T are both primitive, $D_TD_S \in N$. Since $\text{GCD}(s_1, s_2) = 1$, we must have $\ell_2 | s_1^2$. Writing ℓ_2 as a^2b, b square-free, we obtain $s_1 = abc$ for some c, and

$$a^2s_2^2D_TD_S = \ell_1s_1^2a^2/\ell_2 = \ell_1a^2b^2c^2a^2/a^2b = \ell_1a^2bc = \ell_1\ell_2c^2$$

and $\text{sqf}(D_TD_S) = \text{sqf}(a^2s_2^2D_TD_S) = \text{sqf}(\ell_1\ell_2c^2) = \text{sqf}(\ell_1\ell_2)$, with $\ell_1\ell_2$ clearly an $\mathcal{L}$-factor.

We would like to find a criterion involving characteristic numbers which would enable us to determine if two sequences belong to the same family or not. We conclude with conjectures in this direction:

Conjecture 1: $D_S = D_T \Rightarrow S \leftrightarrow T$

Conjecture 2: $S \leftrightarrow T \Longleftrightarrow D_S D_T$ is an $\mathcal{L}$-factor times a rational square. It would also be desirable to have an algorithm to produce the derivation given the $\mathcal{L}$-factor.

Conjecture 3: p is a Brousseau number $\Rightarrow$ each of the powers of p corresponds to a distinct family of sequences.

REFERENCES

1. Brother U. Alfred [Brousseau]. "On the Ordering of Fibonacci Sequences." *The Fibonacci Quarterly* 1, No. 5 (1963):43-46.
2. Brother U. Alfred [Brousseau]. "On Square Lucas Numbers." *The Fibonacci Quarterly* 2 (1964):11-13.
3. Brother Alfred Brousseau. "A Note on the Number of Fibonacci Sequences." *The Fibonacci Quarterly* 10 (1972):657-658.
4. Brother Alfred Brousseau. *Fibonacci and Related Number Theoretic Tables.* The Fibonacci Association 1972.
5. Don Crowe. Personal communication, December, 1974.
6. Leonard Eugene Dickson. *History of the Theory of Numbers.* Vol. I: *Divisibility and Primality.* Chelsea, 1952.
7. Verner E. Hoggatt, Jr. *Fibonacci and Lucas Numbers.* Houghton-Mifflin, 1969.
8. A. F. Horadam. "A Generalized Fibonacci Sequence." *American Math. Monthly* 68 (1961): 455-459.
9. Eugene Levine. "Fibonacci Sequences with Identical Characteristic Values." *The Fibonacci Quarterly* 6, No. 5 (1968):75-80.
10. Édouard Lucas. *The Theory of Simply Periodic Numerical Functions.* The Fibonacci Association, 1969.
11. Christopher Moeller. "Secondary Fibonacci Sequences." Paracollege Senior Project, St. Olaf College, 1975.
12. A. Tagiuri. *Periodico di Matematiche* 16 (1901):1-12.
13. E. Zeckendorf. "Les facteurs premiers des nombres de Fibonacci et des nombres de Lucas." *Bulletin de la Société Royale des Sciences de Liège* 41 (1972):252-253.

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AN ESTIMATE FOR THE LENGTH OF A FINITE JACOBI ALGORITHM

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There are many papers concerning the length of the continued fraction expansion of a rational number (see, e.g., M. Mendès-France [2]). Following a method given by J. D. Dixon [1] in an elementary way, an estimate can be given for the length of the Jacobi algorithm of a rational point.

The Jacobi algorithm may be described in the following way: Let

$$B = \{x = (x_1, \ldots, x\ )\,|\,0 \le x_j < 1,\ 1 \le j \le n\}.$$

If $x = (0, \ldots, 0)$, then $Tx = x$. If $x_1 = \cdots = x_t = 0$, $x_{t+1} > 0$ for $0 \le t < n$, then,

$$T(0, \ldots, 0, x_{t+1}, \ldots, x_n) = (0, \ldots, 0, x_{t+2}/x_{t+1} - [x_{t+2}/x_{t+1}], \ldots, 1/x_{t+1} - [1/x_{t+1}]).$$

We define $x^{(g)} = T^g x$. We say that the algorithm of x has length $L(x) = G$ if

$$G = \min\{g \ge 0\ x^{(g)} = (0, \ldots, 0)\}.$$

Let $x^{(s)} = (0, \ldots, 0, x_{t+1}^{(s)}, \ldots, x_n^{(s)})$, then we define

$$k_0^{(s+1)} = \cdots = k_{t-1}^{(s+1)} = 0$$

$$k_t^{(s+1)} = 1 \text{ (if } t = 0, \text{ then } k_0^{(s+1)} = 1)$$

$$k_{t+1}^{(s+1)} = [x_{t+2}^{(s)}/x_{t+1}^{(s)}],\ \ldots,\ k_n^{(s+1)} = [1/x_{t+1}^{(s)}]$$

$$A_i^{(j)} = \delta_{ij} \text{ for } 0 \le i,\ j \le n$$

$$A_i^{(n+1)} = 0 \text{ for } 1 \le i \le n, \ A_0^{(n+1)} = 1$$

$$A_i^{(s+n+1)} = \sum_{j=0}^{n} A_i^{(s+j)} k_j^{(s)}, \quad 0 \le i \le n.$$

Then, an easy induction shows

$$x_i = \frac{A_i^{(s+n+1)} + \sum_{j=1}^{n} A_i^{(s+j)} x_j^{(s)}}{A_0^{(s+n+1)} + \sum_{j=1}^{n} A_0^{(s+j)} x_j^{(s)}}$$

for $1 \le i \le n$.

We want to prove the following

Theorem: Let $x = (a_1/b, \ldots, a_n/b) \in B$ be a rational point. Then

(1) Let $\theta > 1$ and $\theta^n + 1 = \theta^{n+1}$, then $L(x) \le (\log \theta)^{-1} \log b$.

(2) Let $0 < \sigma < 1$. Then there is an $\eta = \eta(\sigma) > 0$ with the following property: Denote by $N(z)$ the number of rational points x satisfying $b \le z$ such that $L(x) \le \eta \log b$, then $N(z) = O(z^{n+\sigma})$.

Remark: Since the order of magnitude of the number of rational points satisfying $b \le z$ is z^{n+1}, the result (2) states that in some sense almost all rational points satisfy $L(x) > \eta \log b$.

We first need a lemma, well known for the Jacobi algorithm without "Störungen" (that means $x_1^{(g)} \ne 0$ for all g; see O. Perron [3]).

Lemma: For $a \ge 0$,

$$(A_1^{(a+n)}, A_2^{(a+n)}, \ldots, A_n^{(a+n)}, A_0^{(a+n)}) = 1.$$

Proof: This is clear for $a = 0$. Therefore, we put $a = g + 1 \ge 1$. Suppose that $k_{t-1}^{(g)} = \cdots = k_0^{(g)} = 0$, $k_t^{(g)} = 1$ and $k_{s-1}^{(g-1)} = \cdots = k_0^{(g-1)} = 0$, $k_s^{(g-1)} = 1$, where $0 \le s \le t$. Then the following relations hold ($0 \le i \le n$):

$$A_i^{(g+n+1)} = A_i^{(g+n)} k_n^{(g)} + \cdots + A_i^{(g+t+1)} k_{t+1}^{(g)} + A_i^{(g+t)}$$

$$A_i^{(g+n)} = A_i^{(g-1+n)} k_n^{(g-1)} + \cdots + A_i^{(g+s)} k_{s+1}^{(g-1)} + A_i^{(g+s-1)}.$$

We introduce the matrices:

M_g with rows $(A_1^{(g+j)}, \ldots, A_n^{(g+j)}, A_0^{(g+j)})$, $s \le j \le n$;

M_{g+1} with rows $(A_1^{(g+1+h)}, \ldots, A_n^{(g+1+h)}, A_0^{(g+1+h)})$, $t \le h \le n$;

M_{g+1}^{*} with rows $(A_1^{(g+h)}, \ldots, A_n^{(g+h)}, A_0^{(g+h)})$, $t \le h \le n$.

Then M_g has rank $n + 1 - s$, and M_{g+1} and M_{g+1}^{*} both have rank $n + 1 - t$.

Let $d = (A_1^{(g+n+1)}, \ldots, A_n^{(g+n+1)}, A_0^{(g+n+1)})$ denote the greatest common divisor. Then d divides all $(n + 1 - t) \times (n + 1 - t)$ determinants of M_{g+1} and therefore of M_{g+1}^{*} as well.

Now the Laplacian expansion for determinants shows that d is a divisor of all $(n + 1 - s) \times (n + 1 - s)$ determinants of the matrix M_g. Repeating the argument, we finally see that d divides determinants of M_0, but $|\det M_0| = 1$.

Proof of the Theorem: If $L(x) = G$, then $a_i/b = A_i^{(G+n+1)}/A_0^{(G+n+1)}$ for $1 \le i \le n$. Therefore $b = d_G A_0^{(G+n+1)}$. From this, we first obtain

$$b \ge A_0^{(G+n+1)} \ge \theta^G$$

and

$$\log b \ge G \log \theta.$$

The number of rational points satisfying $b \le z$ is smaller than or equal to the number of allowed algorithms (see O. Perron [3] or F. Schweiger [4]) such that $d_G A_0^{(G+n+1)} \le z$.

Since $A_0^{(G+n+1)} \ge k_n^{(s)} \ldots k_n^{(G)}$ and given $k_n^{(g)}$ there are at most

$$(k_n^{(g)} + 1)^{n-1} \le 2^{n-1} (k_n^{(g)})^{n-1}$$

possible values for the digits $k_j^{(g)}$, $1 \le j \le n - 1$, we have the estimate (we write c_j instead of $k_n^{(j)}$):

$$N(z) \leq \sum_{G \leq \eta \log z} \left(\sum_{q_1 \ldots q_G d_G \leq z} (2^{n-1})^G (q_1 \ldots q_G)^{n-1} \left(\frac{z}{q_1 \ldots q_G d_G} \right)^s \right)$$

where $s > n$ will be chosen. This shows

$$N(z) = 0\left(z^s \sum_{G \leq \eta \log z} 2^{(n-1)G} \sum_{q_1 = 1}^{\infty} \cdots \sum_{q_G = 1}^{\infty} \sum_{q_G = 1}^{\infty} (q_1 \ldots q_G d_G)^{n-1-s} \right)$$

$$= 0\left(z^s \sum_{G \leq \eta \log z} (2^{n-1}\zeta(s+1-n))^{G+1} \right) = 0\left(z^s (2^{n-1}\zeta(s+1-n))^{\eta \log z} \right).$$

We put $s = n + \varepsilon$ and obtain $N(z) = 0(z^{\alpha})$ where

$$\alpha = n + \varepsilon + \eta(\log \zeta(1 + \varepsilon) + (n - 1)\log 2).$$

Choosing $\varepsilon > 0$ and $\eta = \eta(\varepsilon)$, we may obtain

$$\varepsilon + \eta[\log \zeta(1 + \varepsilon) + (n - 1)\log 2] \leq \sigma.$$

REFERENCES

1. J. D. Dixon. "A Simple Estimate for the Number of Steps in the Euclidean Algorithm." *American Math. Monthly* 78 (1971):374-376.
2. M. Mendès-France. "Sur les fractions continues limitées." *Acta Arith.* 23 (1973):207-215.
3. O. Perron. "Grundlagen für eine Theorie des Jacobischen Kettenbruchalgorithmus." *Math. Ann.* 64 (1907):1-76.
4. F. Schweiger. "The Metrical Theory of Jacobi-Perron Algorithm." *Lecture Notes in Mathematics* 334. Berlin-Heidelberg-New York: Springer, 1973.

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SOLUTION OF THE RECURRENT EQUATION $u_{n+1} = 2u_n - u_{n-1} + u_{n-3}$

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To find the general term of the sequence $\{u_n\}$, we introduce an auxiliary sequence $\{v_n\}$, intertwined with $\{u_n\}$ in the following way:

$$\begin{array}{ccccccccc} u_1 & & u_2 & \rightarrow u_3 & \cdots & u_{n-1} & & u_n & \rightarrow u_{n+1} \cdots \\ & \times & & \times & & & \times & & \\ v_1 & & v_2 & \rightarrow v_3 & \cdots & v_{n-1} & & v_n & \rightarrow v_{n+1} \cdots \end{array}$$

where

$$(1) \qquad \begin{cases} u_{n+1} = v_{n-1} + u_n, \\ v_{n+1} = u_{n-1} + v_n. \end{cases}$$

It is clear that both sequences are determined as soon as u_1, v_1 $(= u_3 - u_2)$, and u_2, v_2 $(= u_4 - u_3)$ are given. $\{u_n\}$ solves our problem since

$$u_{n+1} = v_{n-1} + u_n = u_{n-3} + v_{n-2} + u_n = u_{n-3} + (u_n - u_{n-1}) + u_n.$$

1. Adding the equations in (1) memberwise, we obtain:

$$u_{n+1} + v_{n+1} = (u_{n-1} + v_{n-1}) + (u_n + v_n),$$

which implies that $\{u_n + v_n\}$ *is a Fibonacci sequence* $\{F_n\}$ whose first two terms are

$$u_1 + v_1 \ (= u_1 - u_2 + u_3) \quad \text{and} \quad u_2 + v_2 \ (= u_2 - u_3 + u_4).$$

2. Our problem would be completely solved if we would have an expression for $u_n - v_n = \varepsilon_n$. Subtracting the equations in (1) memberwise, we obtain:

$$\begin{aligned} \varepsilon_{n+1} &= \varepsilon_n - \varepsilon_{n-1}, \\ &= (\varepsilon_{n-1} - \varepsilon_{n-2}) - \varepsilon_{n-1} \quad \text{(replacing } n \text{ by } n - 1 \text{ above)}, \\ &= -\varepsilon_{n-2}, \\ &= -(-\varepsilon_{n-5}) \quad \text{(replacing } n \text{ by } n - 3 \text{ above)}, \\ &= \varepsilon_{n-5}. \end{aligned}$$

Thus, $\{\varepsilon_n\}$ *is a periodic sequence*, with period 6 and

$$\varepsilon_1 = u_1 - v_1 = u_1 + u_2 - u_3,\quad \varepsilon_2 = u_2 - v_2 = u_2 + u_3 - u_4,\quad \varepsilon_3 = \varepsilon_2 - \varepsilon_1,$$
$$\varepsilon_4 = -\varepsilon_1,\qquad \varepsilon_5 = -\varepsilon_2,\qquad \varepsilon_6 = -\varepsilon_3.$$

3. Hence

$$u_n + v_n = F_n$$
$$u_n - v_n = \varepsilon_n = \varepsilon_{[n]} \quad \text{(where } [n] = n \text{ modulo 6)},$$

and

$$\boxed{u_n = \tfrac{1}{2}(F_n + \varepsilon_{[n]})} \qquad (n > 4).$$

Now F_n may be written in the form (using the Binet formula):

$$F_n = (u_1 - u_2 + u_3)N_{n-2} + (u_2 - u_3 + u_4)N_{n-1},$$

where N_n is the integer closest to

$$\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n$$

(see, for instance, N. N. Vorob'ev, *Fibonacci Numbers*, Blaisdell Publishing Company, 1961, page 22).

Remarks: 1. The method used makes obvious the following relations:

$$u_n + u_{n+3} = \tfrac{1}{2}(F_n + F_{n+3}) = F_{n+2},$$
$$u_{n+6} - u_n = \tfrac{1}{2}(F_{n+6} - F_n) = 2F_{n+3}, \ \ldots .$$

2. Any sequence $\{\varepsilon_n\}$ and any Fibonacci sequence are solutions of the given recurrent equation (directly or by our formula).

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PRIMENESS FOR THE GAUSSIAN INTEGERS

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Complex numbers of the form $a + bi$, where a and b are integers, are commonly called Gaussian Integers. It can be shown that the Gaussian Integers, denoted by G, along with addition and multiplication of complex numbers, form an integral domain. One might suspect that many properties about the integers, denoted by Z, carry over to G. This is indeed the case, and it is the purpose of this paper to examine the property of primeness in the Gaussian domain. The Fundamental Theorem of Arithmetic states that every integer is either a prime or can be uniquely factored into a product of primes, apart from the order in which the factors appear. This theorem also holds for G. It is also true that both G and Z are unique factorization domains. For Z, the units are 1 and -1, while the units for G are 1, -1, i, and $-i$. The job at hand, then, is to determine what elements of G are prime.

For each $\alpha \in G$, $\alpha \cdot \bar{\alpha}$, where $\bar{\alpha}$ is the conjugate of α, is called *the norm of* α and is denoted by $N(\alpha)$. Thus for $a,b \in Z$, $N(a + bi) = (a + bi)(a - bi) = a^2 + b^2$. It also follows that for $\alpha,\beta \in G$, $N(\alpha \cdot \beta) = N(\alpha) \cdot N(\beta)$.

Since G is a unique factorization domain, any $\alpha \in G$ can be factored into a product of primes. Therefore, suppose $\alpha = p_1 \cdot p_2 \cdot \ldots \cdot p_n$, where the p_i's ($i = 1, 2, \ldots, n$) are prime in G. We thus have $N(\alpha) = N(p_1) \cdot N(p_2) \cdot \ldots \cdot N(p_n)$. Hence, any factorization of $\alpha \in G$ leads to a corresponding factorization of $N(\alpha)$ in Z. As a result, α is prime in G if $N(\alpha)$ is prime in Z. As an illustration of these results, consider $\alpha = 3 + 7i$. Since $N(\alpha) = 9 + 49 = 58 = 2 \cdot 29$, $3 + 7i$ has at most two prime factors having norms 2 and 29. Those elements of G with norm 2 are $1 \pm i$. Selecting $1 + i$ and solving the equation $(3 + 7i) = (1 + i)(x + iy)$ for x and y, one discovers that $(3 + 7i) = (1 + i)(5 + 2i)$. If $1 - i$ were chosen, $3 + 7i = (1 - i)(-2 + 5i)$. This appears at first glance to be a different factorization, but observe that $(3 + 7i) = -i(1 - i)(5 + 2i)$ where $-i$ is a unit. Note also that $N(5 + 2i) = 29$. Hence, $(1 + i)(5 + 2i)$ is a prime factorization of $3 + 7i$.

We now have a procedure for determining whether a Gaussian integer of the form $a + bi$, $a,b \neq 0$, is prime in G. What remains is to find a method for determining whether or not an integral prime ($a + bi$, $b = 0$) is prime in G. Then the same method would apply for $a + bi$ when $a = 0$, since i is a unit.

If an integral prime p does not remain prime in G, then p can be written in the form $p = x^2 + y^2$ where $x,y \in Z$. This can be seen by letting $p = \alpha \cdot \beta$ where α,β are not units and $\alpha = a + bi$. Then $N(p) = N(\alpha) \cdot N(\beta)$ implies $p^2 = N(\alpha) \cdot N(\beta)$. As a result, $p = N(\alpha)$, since p is prime in Z. Hence, $p = a^2 + b^2$. As a consequence of this result, note, for example, 2, 5, 13, and 29 are not prime in G and $2 = 1^2 + 1^2$, $5 = 2^2 + 1^2$, $13 = 3^2 + 2^2$, and $29 = 5^2 + 2^2$. On the other hand, 3, for example, is prime in G and $3 \neq x^2 + y^2$ for any $x,y \in Z$.

A sufficient condition for $p \in Z$ to be prime in G is $p \equiv 3 \pmod 4$. To see why this is the case, let $p = 4n + 3$ for some $n \in Z$. Assume p is not prime in G. By the result just established above, $p = x^2 + y^2$. Thus $x^2 + y^2 = 4n + 3$ implies $x^2 + y^2 \equiv 3 \pmod 4$. Now if $x^2 + y^2 \equiv 3 \pmod 4$, x and y cannot both be even or odd. Therefore, without loss of generality, let $x = 2m + 1$ be odd and $y = 2r$ be even. Then $(2m + 1)^2 + (2r)^2 \equiv 3 \pmod 4$. But this implies $2(m^2 + m + r^2) \equiv 1 \pmod 2$, which is absurd. Hence, p is prime in G. As examples, note 3, 7, 11, and 19 are all congruent to 3 (mod 4) and 3, 7, 11, and 19 are primes in Z that are also prime in G.

It turns out that $p \equiv 3 \pmod 4$ is also a necessary condition for an integral prime to be prime in G. If p is an integral prime and either $p \equiv 1 \pmod 4$ or $p \equiv 2 \pmod 4$, then p is not prime in G. For if $p \equiv 2 \pmod 4$, then p is even and equals 2. But $2 = (1 + i)(1 - i)$ and hence is not prime in G. In order to establish the remaining case, the result "If $p \equiv 1$ (mod 4), then there exists an $x \in Z$ such that $x^2 \equiv -1 \pmod p$" will be used without proof (see Schockley, p. 139). Let p be an integral prime and $p \equiv 1 \pmod 4$. Therefore, there exists an $x \in Z$ such that $x^2 + 1 \equiv 0 \pmod p$. But this implies that $p|(x + i)(x - i)$. Moreover, if p is prime in G, then either $p|(x + i)$ or $p|(x - i)$. In either case, $p = \pm 1$, a contradiction. Hence p is not prime in G. As a consequence of this result, integral primes such as 5, 13, and 29 are not prime in G since 5, 13, and 29 are all congruent to 1 (mod 4).

If p is prime in Z and $p \equiv 1 \pmod 4$, then p is not prime in G and $p = x^2 + y^2$; this being a consequence of the above remarks. Now $x + iy$ is prime in G since $N(x + iy) = x^2 + y^2 = p$, which is prime in Z. Therefore, to determine a factorization of an integral prime p in G, one needs only obtain the perfect squares contained in p and test pairwise sums of squares. For example, consider 29, which is not prime in G. The perfect squares contained in 29 are 1, 4, 9, 16, and 25. Since $29 = 4 + 25$, $29 = (2 + 5i)(2 - 5i)$.

REFERENCE

James E. Shockley. *Introduction to Number Theory*. New York: Holt, Rinehart, and Winston, Inc., 1967.

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A NOTE ON ORDERING THE COMPLEX NUMBERS

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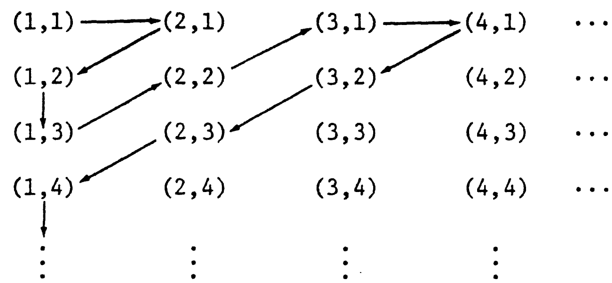
Many order relations can be defined on C. One of the most common orderings is the dictionary or lexicographical ordering. This order behaves in much the same way that the words are arranged in the dictionary. If the symbol "$\textcircled{<}$" is used to denote this order ("$\textcircled{<}$" is read "less than"), then $(a,b) \textcircled{<} (c,d)$ iff $a < c$, or $a = c$ and $b < d$. One can easily verify that $\textcircled{<}$ satisfies the definition of an order relation on C. Thus, $0 \textcircled{<} i$, $2 + 3i \textcircled{<} 3 + 16i$, $2 + 7i \textcircled{<} 2 + 10i$, $-3 - i \textcircled{<} 4$, etc.

Another ordering of C closely related to the dictionary ordering is the antilexicographical ordering. This ordering ($\boxed{<}$) is defined as: $(a,b) \boxed{<} (c,d)$ iff $b < d$ or $b = d$ and $a < c$. It is also an easy matter to verify that $\boxed{<}$ is an order relation on C.

As another illustration, one can show that Δ defined by $(a,b)\ \Delta\ (c,d)$ iff $\sqrt{a^2 + b^2} < \sqrt{c^2 + d^2}$, or $\sqrt{a^2 + b^2} = \sqrt{c^2 + d^2}$ and $\tan^{-1}(b/a) < \tan^{-1}(d/c)$ is an ordering of C. Thus $(1,2)\ \Delta\ (2,3)$ since $\sqrt{1^2 + 2^2} < \sqrt{2^2 + 3^2}$, and $(\sqrt{3},1)\ \Delta\ (\sqrt{2},\sqrt{2})$ since $\sqrt{(\sqrt{3})^2 + 1^2} = \sqrt{(\sqrt{2})^2 + (\sqrt{2})^2}$ and $\tan^{-1}(1/\sqrt{3}) = \pi/6$ tan $^{1}(\sqrt{2}/\sqrt{2}) = \pi/4$.

As a final illustration, any one-to-one correspondence between C and the members of an ordered set can be used to establish an order relation on C or any infinite subset of C, such as $G^+ = \{(a,b) \in Z \times Z | a,b > 0\}$. For example, consider the natural numbers with the usual ordering and the following list:



By using the above process, it is clear that there is a one-to-one correspondence between G^+ and the natural numbers. Thus, this correspondence induces the following order relation $\square$ on G^+: $(1,1) \square (2,1) \square (1,2) \square (1,3) \square (2,2) \square (3,1) \square \cdots$. Here $(1,3) \square (3,1)$ since $4 < 6$ (as natural numbers). Note that this ordering is not the dictionary ordering restricted to $G^+ \times G^+$ since $(2,2) \square (1,4)$ and $(1,4) \textcircled{<} (2,2)$.

It might also be observed that a field can be ordered as an ordered field if and only if no sum of squares of nonzero elements is zero (see Jacobson, p. 269). Since i and 1 are not zero and $i^2 + 1^2 = 0$, it follows that C with the usual operations can never be ordered as an ordered field.

Although C can be ordered, one should note that C with the order relation $\textcircled{<}$ does not satisfy the completeness property of the reals. The set

$$S = \{(3,1),\ (3.1,1),\ (3.14,1),\ (3,141,1),\ (3.1415,1),\ (3.14159,1),\ \ldots\},$$

for example, has $(\pi,1)$ as an upper bound. But $(\pi,.9)$ is also an upper bound and $(\pi,.9) \textcircled{<} (\pi,1)$. In fact, $(\pi,x) \textcircled{<} (\pi,1)$ if $x \textcircled{<} 1$. Since $\{x \in R | x < 1\}$ has no lower bound, S cannot have a least upper bound.

It can also be demonstrated that C with the order relation $\textcircled{<}$ does not possess the "Archimedean" property: If $(0,0) \textcircled{<} (a,b)$ and $(0,0) \textcircled{<} (c,d)$, then there exists a positive integer n such that $(c,d) \textcircled{<} n(a,b)$. For consider $(1,0)$ and $(0,1)$. Clearly $(0,1) \textcircled{<} (1,0)$, $(0,0) \textcircled{<} (1,0)$, and $(0,0) \textcircled{<} (0,1)$; but for no positive integer n can $(1,0) \textcircled{<} n(0,1)$.

It is interesting to note that C possesses a subset $G = \{a + bi | a,b \in Z\}$ that behaves in a similar fashion to the set $Z \times Z$ of pairs of integers; both structures are integral domains.

It is well known that Z with respect to $<$ (the usual order) is not dense, i.e., between any two integers there is not always another integer. This same result holds true for G. For example, consider (a,b) and $(a,b + 1)$. Since there is no integer between b and $b + 1$, G is not dense.

Between any two integers there is always a finite number of integers under the usual order. But this is not necessarily the case with the Gaussian integers, G. It is easily demonstrated that there are an infinite number of Gaussian integers (with respect to $\textcircled{<}$) between (a,b) and $(a + 1,b)$ where a,b are positive integers. Thus, one can easily deduce that G^+ under $\textcircled{<}$ is not well ordered, i.e., not every nonempty subset of G possesses a smallest element. On the other hand, by considering the ordering of G^+ induced by the above list which establishes a one-to-one correspondence between the natural numbers and G^+, one notes that G^+ is well ordered with respect to this order.

For the natural numbers, if $a < b$ then $a + 1 \leq b$. This property does not carry over to G^+. This can be seen by considering $(1,2) \textcircled{<} (1,3)$. $(1,2) + (1,0) = (2,2)$ and $(1,3) \textcircled{<} (2,2)$.

REFERENCE

N. Jacobson. *Lectures in Abstract Algebra.* Vol. II: *Theory of Fields.* Princeton: Van Nostrand, 1951.

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THEORY OF EXTRA NUMERICAL INFORMATION APPLIED TO THE FIBONACCI SUM

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In both logic and mathematics the comma is used to represent the *ordered* and *unordered* concepts of *and*. This equivocation in the use of the comma is bad notation which can lead to serious problems. It is also unwise to indicate ordering by changing brackets to parentheses. To avoid these problems, we will denote the *unordered and* by the common plus sign $+$, and the *ordered and* by the symbol $\dotplus$, to be called *proto-plus*. $+$ will be employed as ordinary addition when it is used with real and complex numbers. Obviously this creates a problem regarding the use of the *unordered and* in set theory. For example, instead of the set $\{2, 3, 4\}$, we would be obliged to write $\{2 + 3 + 4\}$, which would indicate adding 2, 3, and 4, yielding 9, which was not the original intention. This problem is resolved by building an enlightening new set theory out of the properties of its own elements. The first step in this direction is to introduce *ordered multiplication*, a noncommutative operation denoted by the symbol $\circ$. This operation will enable us to differentiate between concepts such as "two" and "a two" (one *of* two, or a pair, denoted by $1 \circ 2$). $2 \circ 3$ would then be understood as "two triples." The axioms for $+$ and $\dotplus$ will be given later. Next we introduce the concept of "*any* counting number," denoted by ω, where $\omega + \omega = \omega$. A set containing pencils (p) and erasers (e) would not be written as $\{p, e\}$, but as $\omega \circ p + \omega \circ e$. Naturally, $\omega \circ 2 + \omega \circ 3 + \omega \circ 4$ does not equal $\omega \circ 9$. Adding two operations of "choice" ($\subset$) and "anti-choice" ($\not\subset$) completes the list of operations necessary for the construction of this new set theory. One of the interesting consequences of this approach is that the operations of $+$ (ordinarily denoted by the comma between elements) and *union* (ordinarily denoted by $\cup$) are one and the same. Many other interesting insights arise from this approach.

The ordered collection "*a and then b and then c*" will be written as $a \dotplus b \dotplus c$, and we will introduce a sigma type notation, parallel to the common use of Σ, for iterated use of $\dotplus$, to be denoted by σ. $\displaystyle\mathop{\boldsymbol{\sigma}}_{i=1}^{n} f(n)$ will then denote $f(1) \dotplus f(2) \dotplus f(3) \dotplus \cdots \dotplus f(n)$, and will be called a *proto-sum*. If "a" and "b" are real numbers, "$a \dotplus b$" will be called a *proto-number* (as well as a *proto-sum*). Obviously

(1a) $$a \dotplus b \neq b \dotplus a$$

We define proto-minus $\dot{-}$ by

(1b) $$a \dotplus (-b) = a \,\dot{-}\, b$$

Note that $1 \,\dot{-}\, 1$ is *not* zero, but differs from it only by the *extra-numerical information* of *ordering*. We shall call such a term a *proto-null*, and, since it is not zero, we can use it as a divisor.

We will now present the axioms for $\dotplus$ and the *proto-numbers*. Given a collection of real numbers R, with elements "a" and "b", and a collection of proto-numbers P, with elements "p", "q", "r", and "s_i" (i any counting number), and three operations $+$, $\dotplus$, and $\cdot$ in R and P, then

(2a)	$(\forall a)(\forall b)$	$a \dotplus b \in P$
(2b)	$(\forall p)(\forall q)$	$p + q \in P$
(2c)	$(\forall p)(\forall q)$	$p \dotplus q \in P$
(2d)	$(\forall p)(\forall q)$	$p \cdot q \in P$
(2e)	$(\forall p)(\forall q)$	$p + q = q + p$
(2f)	$(\forall p)(\forall q \neq P)$	$p \dotplus q \neq q \dotplus p$
(2g)	$(\forall p)(\forall q)$	$p \cdot q = q \cdot p$
(2h)	$(\forall p)(\forall q)(\forall r)$	$p + (q + r) = (p + q) + r$
(2i)	$(\forall p)(\forall q)(\forall r)$	$p + (q \dotplus r) = (p + q) \dotplus r = p + q \dotplus r$*
(2j)	$(\forall p)(\forall q)(\forall r)$	$p \cdot (q \cdot r) = (p \cdot q) \cdot r$
(2k)	$(\forall p)(\exists 0)$	$p + 0 = 0 + p = p$
(2l)	$(\forall p)(\exists 0)$	$p \dotplus 0 = p$
(2m)	$(\forall p \neq 0)$	$0 \dotplus p \neq p$
(2n)	$(\forall p)(\exists -p)$	$-p + p = p + (-p) = 0$
(2o)	$(\forall p)(\forall q)(\forall r)$	$p \cdot (q + r) = p \cdot q + p \cdot r$
(2p)	$(\forall p)(\forall s_i)$	$p \cdot \displaystyle\mathop{\boldsymbol{\sigma}}_{i=1}^{\infty} (s_i) = \mathop{\boldsymbol{\sigma}}_{i=1}^{\infty} (p \cdot s_i)$

*$+$ has precedence of operation over $\dotplus$.

We would like to add the axiom

$$(2q) \qquad (\forall p \neq 0)(\exists p^{-1}) \qquad p^{-1} \cdot p = p \cdot p^{-1} = 1$$

but we need extra concepts to deal with the multiplicative inverses of $0 \mathbin{\vec{+}} 1$ and $1 \mathbin{\vec{-}} 1$.

The case of $0 \mathbin{\vec{+}} 1$ can be handled by introducing the ordered operation *retro-plus*, denoted by $\overleftarrow{+}$, which has an axiomatic system which is the exact mirror image of the axiomatic system developed for $\vec{+}$, with the condition that

$$(3a) \qquad 0 \mathbin{\vec{+}} (1 \mathbin{\overleftarrow{+}} 0) = 1$$

Then we can deal with *retro-numbers* which are ordered from right to left instead of from left to right. It follows that

$$(3b) \qquad 0 \mathbin{\overleftarrow{+}} 1 = \frac{1}{0 \mathbin{\vec{+}} 1}$$

and we find that $0 \mathbin{\overleftarrow{+}} 1$ is the multiplicative inverse of $0 \mathbin{\vec{+}} 1$.

Before discussing the multiplicative inverse of $1 \mathbin{\vec{-}} 1$, we must have more tools to work with. By the above axioms we can show that

$$(3c) \qquad \mathop{\boldsymbol{\sigma}}\limits_{i=1}^{n} (a_i + b_i) = \mathop{\boldsymbol{\sigma}}\limits_{i=1}^{n} a_i + \mathop{\boldsymbol{\sigma}}\limits_{i=1}^{n} b_i$$

$$(3d) \qquad \mathop{\boldsymbol{\sigma}}\limits_{i=1}^{n} (a_i \mathbin{\vec{+}} b_i) = \mathop{\boldsymbol{\sigma}}\limits_{i=1}^{n} a_i \mathbin{\vec{+}} \mathop{\boldsymbol{\sigma}}\limits_{i=1}^{n} b_i$$

$$(3e) \qquad \mathop{\boldsymbol{\sigma}}\limits_{i=1}^{m} \sum_{j=1}^{n} {}_i a_j = \sum_{j=1}^{n} \mathop{\boldsymbol{\sigma}}\limits_{i=1}^{m} {}_i a_j$$

$$(3f) \qquad \mathop{\boldsymbol{\sigma}}\limits_{j=1}^{n} \mathop{\boldsymbol{\sigma}}\limits_{i=1}^{m} {}_i a_j = \mathop{\boldsymbol{\sigma}}\limits_{i=1}^{m} \sum_{j=1}^{n} \left({}_{i-j+1}a_j \cdot {}^m\mathbf{z}_{i-j+1} \cdot {}^n\mathbf{z}_i \right), \text{ where } {}^m\mathbf{z}_i \begin{cases} = 1, & 0 < i \leq m^* \\ = 0, & 0 \geq i > n \end{cases}$$

$$(3g) \qquad \left(\mathop{\boldsymbol{\sigma}}\limits_{i=1}^{m} a_i \right) \cdot \left(\mathop{\boldsymbol{\sigma}}\limits_{j=1}^{n} b_i \right) = \mathop{\boldsymbol{\sigma}}\limits_{i=1}^{m} \sum_{j=1}^{n} \left(a_{i-j+1} \cdot b_j \cdot {}^m\mathbf{z}_{i-j+1} \cdot {}^n\mathbf{z}_j \right)$$

$$(3h) \qquad (a \mathbin{\vec{+}} b)^r = \mathop{\boldsymbol{\sigma}}\limits_{i=1}^{r+1} \left[\frac{r!}{(r - i + 1)!(i - 1)!} a^{r-i+1} b^{i-1} \right]$$

The binomial expansion is an operation between ordered sums and equation (3h) is its only legitimate expression. In this treatise we will only consider sums generated by the binomial expansion, giving us the basis for a theory of rational proto-numbers.

Another common operation between ordered sums is long division, and it must be considered a proto-algorithm. Henceforth we shall refer to it as proto-division. It turns out to be the operational inverse of the proto-multiplication given in equation (3g). As an example of equation (3g),

$$\begin{aligned} (3i) \quad & (1 \mathbin{\vec{+}} 2 \mathbin{\vec{+}} 3) \cdot (4 \mathbin{\vec{+}} 5 \mathbin{\vec{+}} 6) \\ & = 1 \cdot 4 \mathbin{\vec{+}} (1 \cdot 5 + 2 \cdot 4) \mathbin{\vec{+}} (1 \cdot 6 + 2 \cdot 5 + 3 \cdot 4) \mathbin{\vec{+}} (2 \cdot 6 + 3 \cdot 5) \mathbin{\vec{+}} 3 \cdot 6 \\ & = 4 \mathbin{\vec{+}} 13 \mathbin{\vec{+}} 28 \mathbin{\vec{+}} 27 \mathbin{\vec{+}} 18 \end{aligned}$$

The following demonstrates that proto-division is the inverse of this operation:

$$(3j) \qquad \begin{array}{r|l} & 4 \mathbin{\vec{+}} 5 \mathbin{\vec{+}} 6 \\ \hline 1 \mathbin{\vec{+}} 2 \mathbin{\vec{+}} 3 & 4 \mathbin{\vec{+}} 13 \mathbin{\vec{+}} 28 \mathbin{\vec{+}} 27 \mathbin{\vec{+}} 18 \\ & 4 \mathbin{\vec{+}} 8 \mathbin{\vec{+}} 12 \mathbin{\vec{+}} 0 \mathbin{\vec{+}} 0 \\ \hline & 5 \mathbin{\vec{+}} 16 \mathbin{\vec{+}} 27 \mathbin{\vec{+}} 18 \\ & 5 \mathbin{\vec{+}} 10 \mathbin{\vec{+}} 15 \mathbin{\vec{+}} 0 \\ \hline & 6 \mathbin{\vec{+}} 12 \mathbin{\vec{+}} 18 \\ & 6 \mathbin{\vec{+}} 12 \mathbin{\vec{+}} 18 \\ \hline & 0 \mathbin{\vec{+}} 0 \end{array}$$

Given the proto-numbers p, q ($\neq 0$), and r, we define

$$(4a) \qquad r = p \div q = \frac{p}{q}$$

$$(4b) \qquad \frac{p}{q} + \frac{r}{s} = \frac{p \cdot s + q \cdot r}{q \cdot s}$$

*The coefficients ${}^m\mathbf{z}_i$ can be more generally developed, but space does not permit further discussion.

(4c) $$\frac{p}{q} + \frac{r}{s} = \frac{p \cdot s + q \cdot r}{q \cdot s}$$

(4d) $$\frac{p}{q} \cdot \frac{r}{s} = \frac{p \cdot r}{q \cdot s}$$

(4e) $$\frac{p}{q} - \frac{r}{s} = \frac{p \cdot s - q \cdot r}{q \cdot s}$$

(4f) $$\frac{p}{q} \mathbin{\dot{-}} \frac{r}{s} = \frac{p \cdot s \mathbin{\dot{-}} q \cdot r}{q \cdot s}$$

(4g) $$\frac{p}{q} \div \frac{r}{s} = \frac{p \cdot s}{q \cdot r}$$

We will adhere to the additive index law for real powers of proto-numbers where, for $m,n \in R$,

(4h) $$p^n = \overbrace{p \cdot p \cdot p \cdot \cdots \cdot p}^{n}$$

(4i) $$p^m \cdot p^n = p^{m+n}$$

(4j) $$p^m \div p^n = \frac{p^m}{p^n} = p^{m-n}$$

(4k) $$p^0 = 1$$

Probably all ordered summation processes are related to this axiomatic system. For example, we cannot consider any specific infinite sum without ordering its terms. Thus, all infinite series as commonly used are actually infinite proto-sums. Consider the following example:

(5a) $$\frac{1}{1 - a} = \sum_{i=1}^{\infty} a^{i-1}$$

which should be written as

(5b) $$\frac{1}{1 \mathbin{\dot{-}} a} = \overset{\infty}{\underset{i=1}{\boldsymbol{\sigma}}}\, a^{i-1}$$

In equation (5a), $\frac{1}{1 - a}$ is the total of the infinite sum and is a real number if "a" is real. This sort of expression fails to differentiate between a sum and its numerical *total*. In the case of equation (5b), we must remember that $\frac{1}{1 \mathbin{\dot{-}} a}$ is *not* a real number, but a proto-number. To find a numerical *total* for the infinite proto-sum in equation (5b), we must devise a means of relating terms such as $\frac{1}{1 \mathbin{\dot{-}} a}$ and $\frac{1}{1 - a}$. This is not as simple as it appears, and requires a study of both the null and infinite proto-numbers.

To define infinite proto-numbers, we must consider the multiplicative inverse of the simplest proto-null $1 \mathbin{\dot{-}} 1$. Let us begin by denoting the multiplicative inverse of $1 \mathbin{\dot{-}} 1$ by $\underline{\text{И}}$,* which we will call proto-И. Then

(6a) $$\underline{\text{И}} = \frac{1}{1 \mathbin{\dot{-}} 1} = (1 \mathbin{\dot{-}} 1)^{-1}$$

(6b) $$\underline{\text{И}} \mathbin{\dot{-}} \underline{\text{И}} = \frac{1}{1 \mathbin{\dot{-}} 1} \mathbin{\dot{-}} \frac{1}{1 \mathbin{\dot{-}} 1} = \frac{1 \mathbin{\dot{-}} 1}{1 \mathbin{\dot{-}} 1} = 1$$

Similarly, we can show that

(6c) $$(\underline{\text{И}})^r \mathbin{\dot{-}} (\underline{\text{И}})^r = (\underline{\text{И}})^{r-1}$$

We can also prove that, as a consequence of the axiom in equation (2q),

(6d) $$\underline{\text{И}} = 1 + 1 + 1 + \cdots .$$

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*И is the letter for *yee* in the Russian alphabet.

We could have anticipated this result by dividing out $\frac{1}{1 - 1}$ by proto-division, and it can be shown that such a division in a proto-system leaves *no remainder*. Similarly, there is no remainder in a proto-system when we divide out $\frac{1}{1 - a}$ to get the result in equation (5b). Because of this, it is easily seen that the proto-binomial expansion in equation (3h) is true for *all* real values of a, b, and r, in a proto-system. Surprising as this result may be, we must bear in mind that an equation such as

(6e) $$\frac{1}{1 - 2} = \underset{i=1}{\overset{\infty}{\sigma}} 2^{i-1}$$

is not a contradiction since $\frac{1}{1 - 2} \neq -1$.

We are familiar with sequences having an open end; i.e., an infinite number of terms such as $f(1) + f(2) + \cdots + f(n) + \cdots$. Sequences having unique first and last terms, with an infinite number of terms in between, are less familiar, but have been employed, for example, by Cantor and others. We will say that sums and sequences of this sort have open middles.

Using an infinite proto-sum with an open middle, we define the *intrafinite integer* И by

(6f) $$\text{И} = \underset{i=1}{\overset{\text{И}}{\sigma}} (1) = \overbrace{1 + 1 + 1 + \cdots + 1}^{\text{И}}$$

Then we introduce a *principle* of *substitution* for И such that, if

(6g) $$g(n) = \underset{i=1}{\overset{n}{\sigma}} f(i)$$

then

(6h) $$g(\text{И}) = \underset{i=1}{\overset{\text{И}}{\sigma}} f(i)$$

For $f(i) = i$ this becomes

(6i) $$g(\text{И}) = \underset{i=1}{\overset{\text{И}}{\sigma}} i = 1 + 2 + 3 + 4 + \cdots + \text{И}$$

For every term in this proto-sum, up to and including И, we can associate its value with its rank. Obviously we are not dealing with the class of natural numbers, since the natural numbers are all finite in size, despite the fact that there are an infinite number of them. We will call our collection *the amorphous intrafinite numbers*.

Henceforth we will abbreviate $\underset{i=1}{\overset{\text{И}}{\sigma}}$ by $\underset{i}{\sigma}$ and will always use it in place of $\underset{i=1}{\overset{\infty}{\sigma}}$, which is actually meaningless due to the ambiguity of ∞. We could, in a sense, interpret И as being the number of all *counting numbers*.

Now we must construct a system of numeration of radix (base) И. Note that a system of numeration is also a form of proto-math. In a system of numeration of radix Γ, we proto-add "ones" Γ times, *and then* proto-add another Γ "ones," etc. The empty frame, with Γ positions to be filled by "ones" in such a system of numeration, will be called a *Collect** of radix Γ.

If $n < \Gamma$, then $\underset{i=1}{\overset{n}{\sigma}} (1)$ will be called a proto-digit, to be written as $\underset{\sim}{n}$; i.e.,

(6j) $$\underset{i=1}{\overset{n}{\sigma}} (1) = \underset{\sim}{n},\ n < \Gamma$$

By choosing a system of numeration of radix И for a general approach, all infinite sums will be considered as Collects of radix И, as well as all finite sums; i.e.,

(6k) $$\underset{i=1}{\overset{n}{\sigma}} f(i) = f(1) + f(2) + \cdots + f(n) + \overbrace{0 + 0 + \cdots + 0}^{\text{И} - n}$$

where the right-hand proto-sum will have И terms.

*Pronounced kólekt.

Notationwise, we will use the following convention:

(6l)
$$\underset{i}{\boldsymbol{\sigma}} f(i) = f(1) + f(2) + f(3) + \cdots + f(\text{И})$$
$$= f(1) + f(n) + f(3) + \cdots + f(n) + \cdots$$

employing either the open end or open middle notion as desired, considering them as equivalent.

By equations (3g) and (6d), we can show that

(6m)
$$\underline{\text{И}}^{r} = \frac{1}{(1 - 1)^{r}} = \underset{i}{\boldsymbol{\sigma}}\left[\frac{(r + i - 2)!}{(r - 1)!(i - 1)!}\right]$$

and

(6n)
$$\underline{\text{И}}^{-r} = (1 - 1)^{r} = \underset{i}{\boldsymbol{\sigma}}\left[\frac{(-1)^{i+1} r!}{(r - i + 1)!(i - 1)!}\right]$$

In order to obtain *totals* for infinite sums, we must relate proto-sums to their corresponding unordered sums. To do so, we must delete all information concerning ordering, introducing a certain degree of indeterminacy, so one can no longer differentiate between + and +. To accomplish this, note that

(6o)
$$\overbrace{0 + \cdots + 0}^{n} + a = a \cdot (0 + 1)^{n} = a(1 - \underline{\text{И}}^{-1})^{n}$$

Then, given $\overset{m}{\underset{i=1}{\boldsymbol{\sigma}}} a_i \underline{\text{И}}^{p}$, we see, by equation (6n) that

(6p)
$$\overset{m}{\underset{i=1}{\boldsymbol{\sigma}}} a_i \underline{\text{И}}^{p} = \sum_{i=1}^{m} a_i \underline{\text{И}}^{p} + B$$

where B is a linear, unordered sum of powers of $\underline{\text{И}}$, all less than p. If we drop all terms of lower potency than $\underline{\text{И}}^{p}$, there is no distinction between + and + (this reasoning also holds for $m = \text{И}$, provided the proto-total of $\underset{i}{\boldsymbol{\sigma}} a_i \underline{\text{И}}^{p}$ contains no term of potency greater than $\underline{\text{И}}^{p}$). Setting $\overset{m}{\underset{i=1}{\boldsymbol{\sigma}}} a_i \underline{\text{И}}^{p}$ equal to $\sum_{i=1}^{m} a_i \underline{\text{И}}^{p}$, we have introduced a certain indeterminacy concerning all additive terms of potency less than $\underline{\text{И}}^{p}$. Such a relation will be called a *reduced equation*, or an *isonomic relation*, which we will denote by a "p" under the equality sign, whence

(6q)
$$\overset{m}{\underset{i=1}{\boldsymbol{\sigma}}} a_i \underline{\text{И}}^{p} \underset{p}{=} \sum_{i=1}^{m} a_i \underline{\text{И}}^{p}$$

to be read, "$\overset{m}{\underset{i=1}{\boldsymbol{\sigma}}} a_i \underline{\text{И}}^{p}$ is *isonomic to* $\sum_{i=1}^{m} a_i \underline{\text{И}}^{p}$, in a *reduced equation* of order p."

Now, to relate И^{r} and $\underline{\text{И}}^{r}$, we must take into account the missing remainders which are peculiar to proto-math. The following example demonstrates the problem:

Applying the *principle of substitution* of И to

(6r)
$$\sum_{i=1}^{n} i = \frac{n(n + 1)}{2}$$

we have

(6s)
$$\sum_{i=1}^{\text{И}} i = \frac{\text{И}^2}{2} + \frac{\text{И}}{2}$$

This is the unordered sum that corresponds to the proto-sum $\underset{i}{\boldsymbol{\sigma}} i$ in equation (6i), which has as a proto-total $\underline{\text{И}}^2$. In a reduced equation of order 2, $\text{И}^2/2 + \text{И}/2$ becomes $\text{И}^2/2$ and we are forced to conclude that

(6t)
$$\frac{\text{И}^2}{2} \underset{2}{=} \underline{\text{И}}^2$$

More generally, we find that

(6u)
$$\text{И}^{r} \underset{r}{=} r!\,\underline{\text{И}}^{r}$$

a result that can be derived from consideration of equation (61) and its unordered counterpart. The same result can also be derived from consideration of the missing remainders in multiplication of powers of $\underline{\aleph}$, but present space does not permit this. We would expect to find a relation between $\aleph^{-r}$ and $\underline{\aleph}^{-r}$ corresponding to equation (6u), but this has not been accomplished yet. So we will take equation (6u) as our definition for $\aleph^{r}$, for $r \geq 0$.

Now we must investigate the type of proto-sum generated by dividing our $1/1 \dotplus a$ by proto-division, where $|a| > 1$. As in the case of equation (6e), we are faced, in reduced equations, with the enigma of absolutely divergent sums having finite totals. This result demonstrates that numbers of such magnitude as $2^{\aleph}$ cannot be dealt with realistically using summation, but necessitate the use of infinite products. That would be beyond the scope of this present work, but note in passing that it can be accomplished through the use of $\circ$, the operation of ordered multiplication, based upon the following additional axioms for the collection P of the proto-numbers:

(7a)	$(\forall p)(\forall q)$	$p \circ q \in P$
(7b)	$(\forall p)(\forall q \neq p)$	$p \circ q = q \circ p$
(7c)	$(\forall p)(\forall q)(\forall r)$	$p \cdot (q \circ r) = (p \cdot q) \circ r = p \cdot q \circ r$*
(7d)	$(\forall p)(\exists 0)$	$p \circ 0 = 0 \circ p = 0$
(7e)	$(\forall p)(\forall q)(\forall r)$	$(p + q) \circ r = p \circ r + q \circ r$
(7f)	$(\forall p)(\forall q)(\forall r)$	$(p \dotplus q) \circ r = p \circ r \dotplus q \circ r$
(7g)	$(\forall p)(\forall q)(\forall r)$	$p \circ (q + r) \neq p \circ q + p \circ r$
(7h)	$(\forall p)(\forall q)(\forall r)$	$p \circ (q \dotplus r) \neq p \circ q \dotplus p \circ r$
(7i)	$(\forall p)$	$p \circ 1 = p \neq 1 \circ p$
(7j)	$(\forall a \in R)(\forall b \in R)$	$a \circ b \in P$
(7k)		$a \circ b = a \cdot b^{0 \dotplus 1}$

These axioms enable us to include proto-numbers as exponents.

It is also easily seen that

(7l) $$a^{b} \circ a^{c} = a^{b \dotplus c}$$

(7m) $$1 \circ a = a^{0 \dotplus 1}$$

(7n) $$\overbrace{a \circ \cdots \circ a}^{n} = a^{\underline{n}}$$

(7o) $$a \circ (b \circ c) = (a \circ b) \circ (1 \circ c)$$

(7p) $$(a \circ b) \circ (c \circ d) = [a \circ (b \cdot c)] \circ (1 \circ d)$$

(7q) $$\overbrace{a^{\underline{m}} \circ \cdots \circ a^{\underline{m}}}^{n} = a^{\underline{m \cdot n}}$$

(7r) $$\lg_{e}(a \circ b) = \lg_{e} a \dotplus \lg_{e} b$$

In the case of proto-sums generated by proto-division such that

$$\frac{1}{1 \dotplus a} = \underset{i}{\sigma} a^{i-1}, \quad |a| > 1$$

the results obtained by employing reduced equations are all self-consistent within the system. Take the case of equation (6e): this reduces to

(8a) $$\sum_{i=1}^{n} 2^{i-1} \underset{0}{=} -1$$

(We treat this case as a reduced equation of order 0, since all such absolutely divergent sums act as though they were convergent; i.e., having zero-order totals.) This is why we say, as in equation (8a), that this properly divergent sum is *isonomic* to -1 (*not equal to* -1). This means that, in the proto-system, $\sum_{i=1}^{\aleph}(2^{i-1})$ has the same properties, or obeys the same laws, as -1.

As an example of this consistency within the proto-system, consider the following: Dividing a convergent sum by an absolutely divergent sum should give us a reduced total of zero, as in

*$\cdot$ has precedence of operation over $\circ$.

(8b) $$\frac{\underset{i}{\boldsymbol{\sigma}}(2^{-i})}{\underset{j}{\boldsymbol{\sigma}}(1)} = \frac{1}{2} \rightharpoonup \underset{k}{\boldsymbol{\sigma}}(2^{-k-1}) = \frac{1}{2} \rightharpoonup \frac{1}{2}\underset{k}{\boldsymbol{\sigma}}(2^{-k}) = \frac{1}{2} \rightharpoonup \frac{1}{2}\left(\frac{1}{2 \rightharpoonup 1}\right) = \frac{1 \rightharpoonup 1}{2 \rightharpoonup 1}$$

which obviously reduces to zero, as anticipated.

If, instead, we claim $\sum_{i=1}^{k} 2^{i-1}$ to have the properties of -1, then proto-dividing $\underset{i}{\boldsymbol{\sigma}}(2^{-i})$ by $\underset{j}{\boldsymbol{\sigma}}(2^{j-1})$ should produce a result that reduces to -1, since the first sum converges to 1. Indeed, by proto-division,

(8c) $$\frac{\underset{i}{\boldsymbol{\sigma}}(2^{-i})}{\underset{j}{\boldsymbol{\sigma}}(2^{i-1})} = \frac{1}{2} \rightharpoonup \underset{k}{\boldsymbol{\sigma}}\left(\frac{3}{2^{k+1}}\right) = \frac{1}{2} \rightharpoonup \frac{3}{2}\underset{k}{\boldsymbol{\sigma}}(2^{-k}) = \frac{1}{2} \rightharpoonup \frac{3}{2}\left(\frac{1}{2 \rightharpoonup 1}\right) = \frac{1 \rightharpoonup 2}{2 \rightharpoonup 1}$$

which reduces to -1, as anticipated.

All absolutely divergent sums of the type

(8d) $$\frac{1}{1 \rightharpoonup a} = \underset{i}{\boldsymbol{\sigma}} a^{i-1}, \quad |a| > 1$$

which behave as though they were convergent in reduced equations, will be called *co-vergent*.

The Fibonacci sum $\underset{i}{\boldsymbol{\sigma}} f(i)$ is generated by proto-division thusly:

(9a) $$F = \underset{i}{\boldsymbol{\sigma}} f(i) = \frac{1}{1 \rightharpoonup 1 \rightharpoonup 1}$$

Since

(9b) $$1 \rightharpoonup 1 \rightharpoonup 1 = \left(1 \rightharpoonup \frac{1+\sqrt{5}}{2}\right) \cdot \left(1 \rightharpoonup \frac{1-\sqrt{5}}{2}\right)$$

then

(9c) $$\underset{i}{\boldsymbol{\sigma}} f(i) = \frac{1}{\left(1 \rightharpoonup \frac{1+\sqrt{5}}{2}\right) \cdot \left(1 \rightharpoonup \frac{1-\sqrt{5}}{2}\right)}$$

whence the proto-Fibonacci sum is the product of two proto-sums:

(9d) $$c_1 = \underset{i}{\boldsymbol{\sigma}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{i-1}\right]$$

and

(9e) $$c_2 = \underset{i}{\boldsymbol{\sigma}}\left[\left(\frac{1-\sqrt{5}}{2}\right)^{i-1}\right]$$

c_1 is a co-vergent proto-sum, while c_2 is an alternating divergent proto-sum. In reduced equations of order zero, both sums are isonomic to finite numbers:

(9f) $$c_1 \overline{=}_{0} \frac{2}{1+\sqrt{5}}$$

and

(9g) $$c_2 \overline{=}_{0} \frac{2}{1-\sqrt{5}}$$

As one could anticipate, the product of these two "totals" is -1, the same as the reduced "total" of the Fibonacci proto-sum.

Let us form a new proto-sum by proto-adding every other term of the Fibonacci sum, beginning with the first term, to be denoted by F_1. Then let us form a second one by proto-adding every other term, beginning with the second term, to be denoted by F_2. Then

(9h) $$F_1 = \underset{i}{\boldsymbol{\sigma}} f(2i - 1)$$

and

(9i) $$F_2 = \underset{i}{\boldsymbol{\sigma}} f(2i)$$

It is easily seen, by proto-division, that

(9j) $$F_1 = \frac{1 \rightharpoonup 1}{1 \rightharpoonup 3 + 1}$$

and

(9k) $$F_2 = \frac{1}{1 - 3 + 1}$$

whence

(9l) $$F_2 = \underset{-}{\mathbf{N}} \cdot F_1$$

In reduced equations of zero order, F_1 will be isonomic to 0 and F_2 will be isonomic to -1.

For notational simplicity, let us introduce $\mathbf{z}_i$ defined by

(9m) $$\mathbf{z}_i = 1, \text{ for an odd integer}$$
$$= 0, \text{ for an even integer*}$$

Next, let us define F_1' and F_2' by

(9n) $$F_1' = \underset{i}{\boldsymbol{\sigma}}[\mathbf{z}_i f(i)]$$

and

(9o) $$F_2' = \underset{i}{\boldsymbol{\sigma}}[\mathbf{z}_i f(i + 1)]$$

Obviously, F_1' is the proto-sum F_1 with zeros inserted between all of its terms. The same is true for F_2' and F_2. It follows that

(9p) $$F_1' + F_2' = F$$

In zero-order reduced equations, the sum of the "totals" of F_1' and F_2' should be -1, the "total" of F. In substantiation of this, it is easily seen, by proto-division, that

(9q) $$F_1' = \frac{1 + 0 - 1}{1 + 0 - 3 + 0 + 1}$$

and

(9r) $$F_2' = \frac{1}{1 + 0 - 3 + 0 + 1}$$

Note that

(9s) $$1 + 0 - 1 = (1 + 1)(1 - 1)$$

and

(9t) $$1 + 0 - 3 + 0 + 1 = (1 - 1 - 1)(1 + 1 - 1)$$

The above equations substantiate equation (9p), since

(9u) $$F_1' + F_2' = \frac{1 + 0 - 1}{1 + 0 - 3 + 0 - 1} + \frac{1}{1 + 0 - 3 + 0 - 1} = \frac{(1 + 0 - 1) + 1}{1 + 0 - 3 + 0 + 1}$$
$$= \frac{1 + 1 - 1}{(1 - 1 - 1)(1 + 1 - 1)} = \frac{1}{1 - 1 - 1} = F \quad \text{[by (9a)]}.$$

If we define the alternating Fibonacci proto-sum $\overline{F}$ by

(9v) $$\overline{F} = \underset{i}{\boldsymbol{\sigma}}[(-1)^{i+1}f(i)] = \frac{1}{1 + 1 - 1}$$

it follows, by equations (9q) and (9r), that

(9w) $$F_1' = (1 + 0 - 1)F_2'$$

whence, by equations (9a), (9r), (9s), (9t), (9v), and (9w),

(9x) $$F_1' = (1 + 1)(1 - 1) \cdot F \cdot \overline{F}$$

Similarly,

(9y) $$F_2' = F \cdot \overline{F}.$$

Equation (9a) reduces to

(9z) $$F \underset{0}{=} -1$$

and equation (iv) reduces to

(10a) $$\overline{F} \underset{0}{=} 1.$$

Then, from equations (9x) and (9z) and (10a), we see that

(10b) $$F_1' \underset{0}{=} 0.$$

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*The concept of $\mathbf{z}$ can be generalized for some complex series.

Similarly,

(10c) $$F_2' \underset{0}{=} -1$$

whence

(10d) $$F_1' + F_2' \underset{0}{=} F_1' + F_2' \underset{0}{=} -1.$$

Comparing (9z) and (10d), we find another substantiation of (9p).

These operations on the Fibonacci proto-sum show how proto-math opens new vistas of research on infinite sums. It gives us the beginning of a nonconvergency approach to the summation of infinite series, and some of the classical methods of summing divergent series will be special cases of proto-math, one example of which is Cesaro's method. This is true since the sum of the partial sums of an infinite series is simply the operation of multiplying that proto-series by

(11a) $$\underline{\mathcal{N}} = \underset{i}{\sigma}(1).$$

In proto-math, the laws for regrouping terms in divergent series are easily found, and they vary according to the orders of the reduced equations. Given proto-sums, whose "totals" are linear sums of positive powers of $\underline{\mathcal{N}}$, we can find these totals by inspection of the nth terms. We can develop the differential calculus using 1 - 1 instead of infinitesimals, freeing us from the need for limiting processes. Indeed, it may even be possible to develop most of our present-day mathematics without recourse to limiting processes.

There seems to be a vague similarity between the approaches of proto-math and Non-Standard Analysis, but instead of using hyperreal numbers and infinitesimals (which lie on the real line, we use proto-numbers and proto-nulls (which do not lie on the real line). There is no Standard part for the infinite numbers in Non-Standard Analysis, but with our isonomic relations we seem to have achieved a generalization which enables us to enter the infinite range and deal with it in a realistic fashion.

There also seems to be a vague similarity to Cantor's work on the infinite, but there are many important differences. For example, in a proto-system of numeration of radix (base) $\mathcal{N}$, the number of digits is not the same as the number of rational numbers constructed from these digits, in distinct contrast to the Cantorial system, where the number of natural numbers cannot be distinguished from the number of rational numbers. Also, in the proto-system, rearranging the terms in a divergent sum gives us the same "total" as in the original sum. The need for *order types* and *ordinal numbers* does not arise.

Eventually, expressions such as

(11b) $$\mathrm{l}g_e(0) = \infty$$

and

(11c) $$\Gamma(0) = \infty$$

can surely be rendered obsolete, and expressions such as

(11d) $$e^{1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\cdots} \underset{1}{=} \mathcal{N}$$

can be given a rigorous foundation.

Since we have not basically employed set theory in the development of our infinite concepts, and this is a non-Boolean approach, we should expect major departure from the classical approach.

Granted that this work is still in an embryonic form, there is much yet to be done in firming up its foundations, but the promise in its unusual results and self-consistency make it worthy of further investigation.

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THE PENTANACCI NUMBERS

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The elegance of the Fibonacci sequence lies in the fact that its simple definition gives rise to a multitude of properties. Similar qualities can be found in a Pentanacci sequence defined as:

$$P_{(0)} = 0$$

$$P_{(n)} = \text{any five chosen integers, } n = 1, 2, 3, 4, 5$$

$$P_{(n)} = \sum_{m=n-5}^{n-1} P_{(m)}, \; n > 5$$

The generalized form of a Pentanacci sequence is, therefore:

$$p, q, r, s, t, (p + q + r + s + t), (p + 2q + 2r + 2s + 2t), (2p + 3q + 4r + 4s + 4t), \ldots$$

We will consider the specific Pentanacci sequence in which $p = q = r = s = t = 1$. This series begins 1, 1, 1, 1, 1, 5, 9, 17, 33, 65, 129, 253, ...

There is a simple recursive function for finding the sum of any n consecutive Pentanacci numbers. By definition,

$$\sum_{n=1}^{N} P_{(n)} > P_{(N+1)}, \; N > 5,$$

since

$$\begin{aligned} P_{(1)} + P_{(2)} + P_{(3)} + \cdots + P_{(N-4)} + P_{(N-3)} + P_{(N-2)} + P_{(N-1)} + P_{(N)} \\ > P_{(N-4)} + P_{(N-3)} + P_{(N-2)} + P_{(N-1)} + P_{(N)} \\ = P_{(N+1)} \end{aligned}$$

When we subtract $P_{(N-4)} + P_{(N-3)} + P_{(N-2)} + P_{(N-1)} + P_{(N)}$ from both sides, we arrive at $P_{(1)} + P_{(2)} + P_{(3)} + \cdots + P_{(N-5)} > 0$. This immediately leads to:

$$\sum_{n=1}^{N} P_{(n)} = P_{(N+1)} + \sum_{n=1}^{N-5} P_{(n)}, \; N > 5.$$

In general,

$$\begin{aligned} \sum_{k=M}^{N} P_{(k)} &= \sum_{k=1}^{N} P_{(k)} - \sum_{k=1}^{M-1} P_{(k)} \\ &= P_{(N+1)} + \sum_{k=1}^{N-5} P_{(k)} - P_{(M)} - \sum_{k=1}^{M-6} P_{(k)}. \end{aligned}$$

THE PENTANACCI RATIOS AND THEIR DEFINING FIFTH-POWER EQUATIONS

It is well known that the ratio of two consecutive Fibonacci numbers, $F_{(n+1)}/F_{(n)}$, approaches the limit $\frac{1+\sqrt{5}}{2} = 1.618034$ and its reciprocal approaches $\frac{1-\sqrt{5}}{2} = 0.618034$. These limits are the roots of $X^2 - X - 1 = 0$. The ratio of two consecutive Pentanacci numbers, $P_{(n+1)}/P_{(n)}$, approaches the limit 1.9659482 and its reciprocal approaches 0.5086604. These ratios are the only real roots of the fifth-power equation $X^5 - X^4 - X^3 - X^2 - X - 1 = 0$.

By definition, $P_{(n+1)} = P_{(n)} + P_{(n-1)} + P_{(n-2)} + P_{(n-3)} + P_{(n-4)}$. Dividing through by $P_{(n-1)}$, we define:

$$P_{(n)}/P_{(n-1)} = Z_1$$

$$P_{(n-1)}/P_{(n-3)} = Z_2 = Z_1^2 = P_{(n+1)}/P_{(n-1)}$$

$$P_{(n-1)}/P_{(n-4)} = Z_3 = Z_1^3$$

This gives us $Z_1^2 - Z_1 + 1 + 1/Z_1 + 1/Z_1^2 + 1/Z_1^3$, from which the quintic equation, $Z^5 - Z^4 - Z^3 - Z^2 - Z - 1 = 0$, is derived.

CONTINUED FRACTION EXPANSION OF PENTANACCI RATIOS

The ratios $P_{(n+1)}/P_{(n)}$ and $P_{(n)}/P_{(n+1)}$ can be expressed as finite continued fractions in order to demonstrate that they are rational numbers. In general, a continued fraction may be represented as:

$$[a_1, a_2, a_3, \ldots] \quad \text{or} \quad a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cfrac{1}{\ldots}}}}$$

The terms, a_i, are known as partial quotients. A finite continued fraction has a finite number of partial quotients and represents a rational number. Infinite continued fractions have an infinite number of partial quotients and represent irrational numbers.

It can be seen that:

(A) $$P_{(n+1)}/P_{(n)} = [1, a_2, a_3, a_4, \ldots, a_n]$$

(B) $$P_{(n)}/P_{(n+1)} = [0, 1, a_3, a_4, \ldots, a_{n+1}]$$

In equation (b), $a_{(k+1)}$ is the same as the a_k of equation (a) for all k, $1 \le k \le n$.

Consider a/b, where $a > b$ and both a and b are integers.

$$a/b = c + a/b - c,$$

$$a/b = c + (a - cb)/b$$

or

(C) $$a/b = c + \cfrac{1}{\cfrac{b}{a - cb}}$$

This can be expanded further.

Now consider b/a, where $a > b$ and both a and b are integers.

$$b/a = 0 + b/a,$$

so

(D) $$b/a = 0 + \cfrac{1}{\cfrac{a}{b}}$$

Applying equation (C) to equation (D) gives rise to

(E) $$b/a = 0 + \cfrac{1}{c + \cfrac{1}{\cfrac{b}{a - cb}}}$$

This also can be expanded by further manipulation of the $b/(a - cb)$ term.

THE GOLDEN RECTANGLE AND INTERMEDIATE SEQUENCES

Another property of the Fibonacci sequence is that two consecutive Fibonacci numbers represent the lengths of the sides of the Golden Rectangle. A Golden Rectangle is shown in Figure 1. Segments creating smaller Golden Rectangles and a square are included in the figure. The lengths of the sides of all the quadrangles are Fibonacci numbers.

A similar Pentanacci rectangle is shown in Figure 2. Note from Figure 2 that $a = P_{(n)} - P_{(n-1)}$, $b = P_{(n-1)} - P_{(n-2)}$, $c = P_{(n-2)} - P_{(n-3)}$, $d = P_{(n-3)} - P_{(n-4)}$ and $e = P_{(n-4)} - P_{(n-5)}$. In the Fibonacci sequence $F_{(n)} - F_{(n-1)} = F_{(n-2)}$. In the Pentanacci sequence, however, $P_{(n)} - P_{(n-1)} \neq P_{(n-2)}$. By subtracting two consecutive Pentanacci numbers, a new sequence called an Intermediate Sequence is formed.

The first few members of the Pentanacci sequence and of the first two intermediate sequences are:

1, 1, 1, 1, 1, 5, 9, 17, 33, 65, 129, 253, ...
0, 0, 0, 0, 4, 4, 8, 16, 32, 64, 124, ...
0, 0, 0, 4, 0, 4, 8, 16, 32, 60, ...

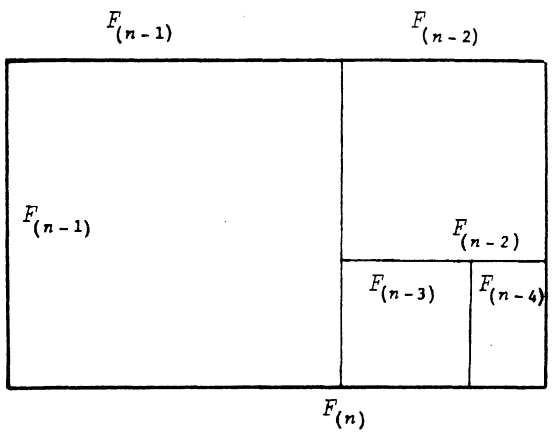

FIGURE 1

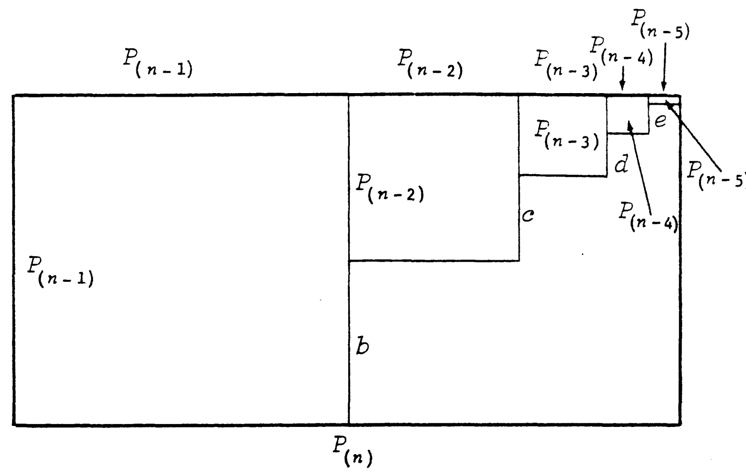

FIGURE 2

It can be shown that each intermediate sequence is a Pentanacci sequence. From the definition of the Pentanacci numbers:

$$P_{(k+1)} = P_{(k)} + P_{(k-1)} + P_{(k-2)} + P_{(k-3)} + P_{(k-4)}$$
$$P_{(k)} = P_{(k-1)} + P_{(k-2)} + P_{(k-3)} + P_{(k-4)} + P_{(k-5)}.$$

So, $P_{(k+1)} - P_{(k)} = P_{(k)} - P_{(k-5)}$ for $k \geq 5$, and for $k < 5$.

Given the following equations:

$$P_{(k+1)} - P_{(k)} = P_{(k)} - P_{(k-5)}$$
$$P_{(k)} - P_{(k-1)} = P_{(k-1)} - P_{(k-6)}$$
$$P_{(k-1)} - P_{(k-2)} = P_{(k-2)} - P_{(k-7)}$$
$$P_{(k-2)} - P_{(k-3)} = P_{(k-3)} - P_{(k-8)}$$
$$P_{(k-3)} - P_{(k-4)} = P_{(k-4)} - P_{(k-9)}$$

The sum of the right-hand side terms is $\sum_{n=k-4}^{k} P_{(n)} - \sum_{m=k-9}^{k-5} P_{(m)}$ which is equal to $P_{(k+1)} - P_{(k-4)}$, the sequence member following $P_{(k)} - P_{(k-5)}$ as defined by the definitions of both the Pentanacci sequence and an intermediate sequence.

The sum of the right-hand side terms, $P_{(k+1)} - P_{(k-4)}$, also equals $P_{(k+2)} - P_{(k+1)}$, the difference between the next two members of the Pentanacci sequence. Hence, we have shown, by applying the definitions of the Pentanacci and intermediate sequences that the latter is a subset of the former.

REFERENCES

Brother U. Alfred. *An Introduction to Fibonacci Discovery*. San Jose, California: The Fibonacci Association, 1965.

Mark Feinberg. "Fibonacci-Tribonacci." *The Fibonacci Quarterly* 1 (1963):71-74.

Verner E. Hoggatt, Jr. *Fibonacci and Lucas Numbers*. Boston: Houghton Mifflin Company, 1969.

C. D. Olds. *Continued Fractions*. New York: Random House, 1963.

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REMARKS ON THE DIOPHANTANIAN EQUATIONS $a^2 \pm ab + b^2 = c^2$

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The present note is concerned with properties of ordered triples (a,b,c) of nonnegative integers which satisfy one of the two equations given in the title. Such solutions of

(1) $$a^2 + ab + b^2 = c^2$$

will be referred to as obtuse Pythagorian triples; the corresponding solutions of

(2) $$a^2 - ab + b^2 = c^2$$

will be called acute Pythagorian triples. If a, b, and c are relatively prime, the triples will be termed "primitive."

These two Diophantanian equations arise in a variety of ways; as it will be shown, even the Fibonacci numbers can generate and be generated by solutions thereof. The following problems will further exemplify this diversity. The reader is encouraged to pursue them at least to the point of recognizing their relevance:

1. Find three Pythagorian triangles of the same area. This problem was resolved by Euler in about 1781 [3].
2. Find solutions for the Diophantanian equation $x^2 + y^2 + z^2 = 2A^2$. In doing so, A. Gerardin [4] resolved several other equations as well.
3. Find three squares as consecutive terms of an arithmetic progression with common difference k. This problem along with its ramifications was discussed by R. L. Goodstein [5].
4. Remove a square of side x from each corner of a rectangular cardboard so that the remaining portion can be folded into an open box of maximum volume. What dimensions for the rectangle yield integral x? The first part is an old calculus problem probably dating back to Lamb [8] or earlier.
5. Find fourth-degree polynomials with integral coefficients whose extrema and inflection points have integral coordinates and are easily found (i.e., the constant term of the first derivative is zero).
6. Find integral triangles (triangles, all of whose sides are of integral length) with a 60° or 120° angle. According to Dickson [3], this problem was first considered by A. Girard, whose solutions were rediscovered dozens of times over the past three hundred years.

In fact, except for rediscoveries of various formulas generating their solutions, the Diophantanian equations under consideration are almost totally neglected in the mathematical literature. We hope to fill this gap at least partially. As a basis for the results to follow, we restate here without proof a procedure that was originally given by Zuge [13] in a slightly different format:

Representation Theorem: Let m and n be relatively prime positive integers of different parity and assume that $3 \nmid m$. Let $(a,b,c) = (4mn, 2mn + |m^2 - 3n^2|, m^2 + 3n^2)$. Then all primitive acute Pythagorian triples are either of the form (a,b,c) or of the form $(|a - b|, \max\{a,b\}, c)$ and all primitive obtuse Pythagorian triples are of the form $(|a - b|, \min\{a,b\}, c)$.

During the course of this work, using this representation theorem, a computer program was prepared by Russell Still, an undergraduate student, generating all primitive acute and obtuse Pythagorian triples for which $m,n \le 50$. The author's gratitude is hereby expressed to Mr. Still for his valuable assistance. Copies of the printout are available from the author upon request.

The three types of triples given by the representation theorem may also be related by observing that if (a,b,c) is a solution of equation (1), then both $(a, a + b, c)$ and $(a + b, b, c)$ will satisfy equation (2). Consequently, in light of the geometrical interpretation afforded by Problem 6, they may be obtained from one another by the addition and/or subtraction of equilateral triangles. We shall further utilize this geometrical interpretation in regarding the triples as triangles and, in particular, in referring to c as the hypotenuse and to a and b as the legs of (a,b,c).

We first observe that since m and n are relatively prime, of different parity and $3 \nmid m$, the pair (m,n) must be congruent modulo 6 to one of the following pairs of numbers: (1,0), (1,2), (1,4), (2,1), (2,3), (2,5), (4,1), (4,3), (4,5), (5,0), (5,2), (5,4). Simple

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calculations show that in each case $m^2 + 3n^2 \equiv 1 \pmod 6$, that is, the hypotenuse of primitive obtuse and acute Pythagorian triples is always of the form $6k + 1$. This proves a conjecture by McArdle [9].

In fact, not only c, but every divisor of it must be of the same form. To prove this, let p be a prime divisor of $c = m^2 + 3n^2$. Observe first that $p \neq 2$ since m and n are of different parity, $p \neq 3$ since $3 \nmid m$, and $p \nmid m$ and $p \nmid n$ due to the relative primeness of m and n. Consequently, by raising both members of the congruence $m^2 \equiv -3n^2 \pmod p$ to the $(p-1)/2$th power and upon applying Fermat's theorem, one finds that $(-3)^{(p-1)/2} \equiv 1 \pmod p$.

Assume that $p = 6k + 5$. If k is even, say $k = 2s$, then $3^{6s+2} \equiv 1 \pmod{12s + 5}$ follows. If k is odd, say $k = 2s - 1$, one similarly obtains $3^{6s-1} \equiv -1 \pmod{12s - 1}$. Since both of these conclusions are contrary to known facts (see, for example, Theorem 20 on page 32 of [10]), the assumption that $p = 6k + 5$ is indeed untenable.

Conversely, if $c = 6k + 1$ is a prime, then it has a unique representation of the form $m^2 + 3n^2$ (see, for example, Theorem 5 on page 323 of [12]). Such m and n must clearly satisfy the restrictions stated in the Representation Theorem, hence each prime must appear as the hypotenuse of exactly one (two) primitive obtuse (acute) Pythagorian triple(s).

This last fact may be connected to a slight extension of Girard's results mentioned earlier, to conclude that each prime number of the form $6k + 1$ is uniquely expressible in both of the forms $x^2 \pm xy + y^2$, where x and y are positive integers. For example, one finds that the representations

$$7 = 1^2 + 1 \cdot 2 + 2^2 = 1^2 - 1 \cdot 3 + 3^2,$$
$$13 = 1^2 + 1 \cdot 3 + 3^2 = 1^2 - 1 \cdot 4 + 4^2, \text{ and}$$
$$19 = 2^2 + 2 \cdot 3 + 3^2 = 2^2 - 2 \cdot 5 + 5^2,$$

are unique.

If c has r distinct prime divisors, each of the form $6k + 1$, then repeated application of the well-known formula

$$(3) \qquad (m_1^2 + 3n_1^2)(m_2^2 + 3n_2^2) = (m_1m_2 \pm 3n_1n_2)^2 + 3(m_1n_2 \pm m_2n_1)^2$$

will yield exactly 2^{r-1} (2^r) primitive obtuse (acute) triples with hypotenuse c. Correspondingly, c will also have 2^{r-1} representations of each of the forms $x^2 \pm xy + y^2$. Equation (3) may also be regarded as a method of obtaining new triples out of old ones. Another such method is afforded by the matrix

$$M = \begin{pmatrix} -3 & 7 & 1 \\ 15 & 5 & 17 \\ 18 & 2 & 20 \end{pmatrix};$$

if (a,b,c) is an obtuse Pythagorian triple, then so is $(a,b,c)M$—viewed as a product of matrices.

Obtuse and acute Pythagorian triples may also be generated from Pythagorian triples by matrices. If we define

$$N = \begin{pmatrix} 2 & -2 & -1 \\ 1 & 1 & 0 \\ -1 & 1 & 2 \end{pmatrix}, \quad K = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}, \quad L = \begin{pmatrix} -1 & 1 & 0 \\ 2 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix},$$

then $(a,b,c)N$ is an obtuse and $(a,b,c)K$ and $(a,b,c)L$ are acute Pythagorian triples whenever $a^2 + b^2 = c^2$. Since N, K, and L are nonsingular, their inverses can also be utilized in transforming our results into the pythagorian setting.

The well-known [11] mechanical generation of sequences of Pythagorian triples from (21,220,221) and (41,840,841) by a systematic insertion of zeros may also be paralleled; each of the six sequences of triples given below are obtuse Pythagorian:

(120, 23, 133),
(10200, 203, 10303),
(1002000, 2003, 1003003), ...;

(129, 391, 469),
(1209, 39991, 40609),
(12009, 3999991, 4006009), ...;

(81, 1599, 1641),
(801, 159999, 160401),
(8001, 155999999, 16004001), ...;

(41, 399, 421),
(401, 39999, 40201),
(4001, 3999999, 4002001), ...;

(21 , 99 , 111), (201 , 9999 , 10101), (2001, 999999, 1001001), ...;   (80 , 19 , 91), (9800 , 199 , 9901), (998000, 1999, 999001), ...

Other interesting sequences of obtuse and acute Pythagorian triples were discussed in two earlier notes by the author [1, 2] in a more geometric setting. Still other modes of generating infinite sequences (a_k, b_k, c_k) of primitive obtuse Pythagorian triples with special properties are depicted in the tables below.

On the basis of Table 1, one may prove, for example, that there are infinitely many obtuse Pythagorian triples whose legs differ by unity. The proof of this fact has been posed by the author as a problem in *The Fibonacci Quarterly* in a slightly different setting. The corresponding problem concerning the existence of acute Pythagorean triples (a,b,c) with $a - b = 1$ has a totally different solution: there are no such triples. The proof of this fact is also left to the reader.

k	m	n	a	b	c
1	1	2	8	7	13
2	13	2	104	105	181
3	13	28	1456	1455	2521
4	181	28	20272	20273	35113
5	181	390	282360	282359	489061
6	2521	390	3932760	3932761	6811741
7	2521	5432	54776288	54776287	94875313
8	35113	5432	762935264	762935265	1321442641

TABLE 1

In Table 2, $a_k - b_k = 2$ for each k. Again, an infinite number of such triples can be recursively generated from the ones displayed. It may also be noticed that each m_{2k-1} (n_{2k}) of Table 2 is twice as large as the corresponding m_{2k-1} (n_{2k}) of Table 1, thus the two tables could be obtained from one another. The proof of the fact that in each case $m_{2k} = c_k$ reveals some analogy to the well-known Fibonacci identity $F_{2n+1} = F_n^2 + F_{n+1}^2$.

k	m	n	a	b	c
1	2	1	5	3	7
2	7	4	57	55	97
3	26	15	781	779	1351
4	97	56	10865	10863	18817
5	362	209	368517	368515	908287

TABLE 2

Continuing with the obtuse case, one may further observe that for each $k = 2, 3, 4, \ldots$, there exists a primitive obtuse Pythagorian triple (a,b,c) for which $c - b = k$; in fact, one such triple is given by

$$(2k - 1,\ 3k^2 - 4k + 1,\ 3k^2 - 3k + 1).$$

If, in addition, k is not a multiple of 3, then

$$(2k - 3,\ k^2 - 4k + 3,\ k^2 - 3k + 3)$$

is another such triple.

These triples may also serve as the basis for yet another observation: each odd number appears at least once as the shorter leg of a primitive obtuse Pythagorian triple. The two formulas above exhaust all such triples for powers of odd primes; with an increase in the number of divisors, one can observe a corresponding increase in the number of such triples.

One can also identify those primitive obtuse Pythagorian triples both of whose legs are odd. They are of the form

$$(2mn + m^2 - 3n^2,\ 2mn - m^2 + 3n^2,\ m^2 + 3n^2),$$

where $\frac{m}{3} < n < m$ and, as usual, n and m are relatively prime, of different parity, and $3 \nmid m$.

Conversely, if $n < \frac{m}{3}$ or $m < n$, then the primitive obtuse Pythagorian triples obtained via the Representation Theorem have an even leg. In fact, such a leg must be a multiple of 8 as it is readily shown via equation (1). For, suppose that a is even, say $a = 2x$. Then b and c must both be odd, say $b = 2y + 1$ and $c = 2z + 1$, and hence, from equation (1) we obtain that $2[x^2 + y(y + 1) - z(z + 1)] = x(2y + 1)$. This implies that x must be even. But then the left member of this equality is a multiple of 4, since $y(y + 1)$ and $z(z + 1)$ are clearly even. Therefore x is a multiple of 4 and, hence, a is a multiple of 8.

Incidentally, these observations provided a solution to a problem posed in the *American Mathematical Monthly* [7].

Furthermore, each multiple of 8 appears as the leg of a primitive obtuse Pythagorian triple. One such triple is given by the formula

$$(8k,\ 12k^2 - 4k - 1,\ 12k + 1)$$

where $k = 1, 2, 3, \ldots$. Again, not all such triples are given by this formula; for example, with the help of the printout one may verify that there are six different triples with a leg of 280.

If the triples are not required to be primitive, one may further observe that each of the following formulas yields obtuse Pythagorian triples for each $k = 1, 2, 3, \ldots$:

$$(8k + 2,\ 24k^2 + 8k,\ 24k^2 + 12k + 2),$$
$$(8k + 4,\ 12k^2 + 8k,\ 12k^2 + 12k + 4),$$
$$(8k + 6,\ 24k^2 + 32k + 10,\ 24k^2 + 36k + 14).$$

Since (6,10,14) is also such a triple, we may conclude that each positive integer except 1, 2, 4, and 8 can appear as the shorter leg of an obtuse Pythagorian triple (see [7]).

Concerning divisibility properties, we have the following two facts, which may be established by a case-by-case examination of all possible congruences:

(i) If (a,b,c) is an obtuse Pythagorial triple, then of the four numbers, a, b, $a + b$, and c, one is divisible by 3, one by 5, one by 7, and one by 8. Since (3,5,7) is one such triple, this result is the best possible.

(ii) If (a,b,c) is a primitive acute Pythagorian triple, and if $a + b$ is even, one has $a + b \equiv \pm 2 \pmod{12}$, while if $a + b$ is odd, the congruences $a + b \equiv \pm 1 \pmod{12}$ result.

In conclusion, paralleling results of Horadam [6], we associate the generalized Fibonacci sequences with the triples under consideration as follows. Let k be an arbitrary positive integer and assume that m and n satisfy the requirements set forth in the Representation Theorem. Define H_0 and H_1 by

$$H_0 = (-1)^{k+1}(F_k m - F_{k+1} n),\quad H_1 = (-1)^k (F_{k-1} m - F_k n),$$

and for $i \geq 2$ let $H_i = H_{i-1} + H_{i-2}$. Then it is easily shown that

$$H_k = n \quad \text{and} \quad H_{k+1} = m,$$

and thus H_k and H_{k+1} generate primitive obtuse and acute Pythagorian triples in the sense of the Representation Theorem. For example, the Fibonacci numbers may be associated with the triple (8,5,7) in the following manner:

$$(8,5,7) = (4F_2F_3,\ 2F_2F_3 + F_3^2 - 3F_2^2,\ F_3^2 + 3F_2^2).$$

REFERENCES

1. G. Berzsenyi. "Geometric Representation of the Sequence ⟨3, 7, 13, 21, 31, 43, 57, ...⟩," *J. Recreational Math.* 7 (1974):203-205.
2. G. Berzsenyi. "Integrally Inscribed Regular Triangles and Hexagons." *J. Recreational Math.* 8 (1975):12-15.
3. L. Dickson. *History of the Theory of Numbers*, Vol. II (see, in particular, pp. 173, 214-215, 405-406, and 434). New York: Chelsea Publishing Company, 1952.
4. A. Gerardin. "Trente sixieve Problems." *Sphinx-Oedipe* 6 (1911):22.
5. R. L. Goodstein. "Rational Triangles." *The Mathematical Gazette* 23 (1939):264-267.
6. A. F. Horadam. "Fibonacci Number Triples." *Amer. Math. Monthly* 68 (1961):751-753.
7. E. Kramer. Problem E2566. *Amer. Math. Monthly* 82 (1975):1010.
8. Sir H. Lamb. *An Elementary Course of Infinitesimal Calculus*. Cambridge, 1919.
9. E. McArdle. "The 'Cosine Rule.'" *J. Recreational Math.* 3, No. 1 (1970):122-123.
10. D. Shanks. *Solved and Unsolved Problems in Number Theory*. Vol. I. Washington, D.C.: Spartan Books, 1962.
11. W. Sierpinski. *Pythagorian Triangles*. New York: Yeshiva University, 1962.
12. W. Sierpinski. *Elementary Theory of Numbers*. Warsaw: Polish Academy of Science, 1964.
13. Zuge. "Allegemein-pythagoreische Zahlen." *Archiv der Mathematik und Physik* Series 2, 17 (1900):254-262.

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TRIANGULAR DISPLAYS OF INTEGERS

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The purpose of this article is to exhibit some properties of certain binomial coefficients that are generated in Theorem 1 below. We display the integers in a triangular form and show that their occurrence within that structure follows a regular pattern.

We make use of the kth difference $\Delta_h^k f(x)$ of a function, this difference being defined by

$$\Delta_h^k f(x) = f(x + kh) - \binom{k}{1} f[x + (k-1)h] + \cdots + (-1)^k f(x),$$

where h is a positive real number.

Although the following theorem is a special case of [1, Theorem 2], we present an independent proof that is more appropriate to the present context.

Theorem 1: Let $x_0, x_1, \ldots, x_k$ and $y_0, y_1, \ldots, y_{mk}$ be two sets of real numbers such that $x_0 < x_1 < \cdots < x_k$, $y_0 < y_1 < \cdots < y_{mk}$, $x_s = y_{ms}$, $s = 0, 1, \ldots, k$, and $y_i - y_{i-1} = h$, $i = 1, 2, \ldots, mk$. Then

$$\Delta_{mh}^k f(x_0) = \sum_{i=0}^{(m-1)k} \alpha_i \Delta_h^k f(y_i), \tag{1}$$

where the coefficients $\alpha_0, \alpha_1, \ldots, \alpha_{(m-1)k}$ are positive, symmetrical [that is, $\alpha_i = \alpha_{(m-1)k-i}$, $i = 0, 1, \ldots (m-1)k$], and have sum equal to m^k. More specifically,

$$\alpha_i = \begin{cases} \binom{i+k-1}{k-1}, & 0 \le i < m \\ \binom{i+k-1}{k-1} - \binom{k}{1}\binom{i-m+k-1}{k-1}, & m \le i < 2m \\ \vdots & \\ \binom{i+k-1}{k-1} - \binom{k}{1}\binom{i-m+k-1}{k-1} + \cdots + (-1)^q \binom{k}{q}\binom{i-mq+k-1}{k-1}, & qm \le i < (q+1)m, \end{cases}$$

where $(m-1)k = mq + r$, $0 \le r < m$.

Proof: In [2, Theorem 6, p. 150] it is proved that for any positive integer n,

$$\Delta_h^k f(x) = \sum_{i_1=0}^{n-1} \sum_{i_2=0}^{n-1} \cdots \sum_{i_k=0}^{n-1} \Delta_{\frac{h}{n}}^k f\left[x + (i_1 + \cdots + i_k)\frac{h}{n}\right],$$

from which we readily deduce that

$$\Delta_{mh}^k f(x_0) = \sum_{i_1=0}^{m-1} \sum_{i_2=0}^{m-1} \cdots \sum_{i_k=0}^{m-1} \Delta_h^k f[x + (i_1 + \cdots + i_k)h].$$

We now observe that α_p is equal to the number of ways in which p can be expressed as a sum $i_1 + \cdots + i_k$, where $0 \le i_t \le m-1$, $t = 1, 2, \ldots, k$. Consequently, α_p is equal to the coefficient of x^p in the expansion

$$\sum_{r=0}^{(m-1)k} \alpha_r x^r \equiv (1 + x + x^2 + \cdots + x^{m-1})^k = (1 - x^m)^k (1 - x)^{-k}. \tag{2}$$

It is now clear from (2) that the α_i are positive, symmetrical and have the form specified. That their sum is m^k follows by putting $x = 1$ in the left-hand side of (2).

When $k = 3$, for example, we display the coefficients α_i in the following triangular array:

	m																			
	1										1									
	2						1	3						3	1					
	3					1	3	6			7			6	3	1				
(3)	4				1	3	6	10	12				12	10	6	3	1			
	5			1	3	6	10	15	18		19		18	15	10	6	3	1		
	6		1	3	6	10	15	21	25	27		27	25	21	15	10	6	3	1	
	7	1	3	6	10	15	21	28	33	36	37	36	33	28	21	15	10	6	3	1
	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

We now make some observations in relation to the coefficients in (3). First, as predicted by Theorem 1, the sum of the integers in row r is equal to r^3. Second, each integer in the above table is either a multiple of 3 or leaves a remainder of +1 when divided by 3. Furthermore, for any particular row, the first entry, namely 1, and every third successive entry, are exactly those integers which leave remainder +1 when divided by 3. We summarize this discussion in the following theorem.

Theorem 2: Each integer in arrangement (3) is either a multiple of 3 or leaves a remainder of +1 on division by 3. If we label the integers in any one row as $\alpha_0, \alpha_1, \ldots$, then $\alpha_i \equiv 1 \pmod 3$ when $i \equiv 0 \pmod 3$, and $\alpha_i \equiv 0 \pmod 3$ when $i \not\equiv 0 \pmod 3$. Consequently, in row m, there are m coefficients which leave remainder +1 on division by 3, and $2(m - 1)$ which are a multiple of 3.

Proof: The form of the coefficients α_i is specified in Theorem 1. Since 3 is a prime number, the remainders after division by 3 are completely determined by the term

$$\binom{i+k-1}{k-1} = \binom{i+2}{2} = \frac{(i+1)(i+2)}{2}.$$

If $i \not\equiv 0 \pmod 3$, then i is of the form $3m - 1$ or $3m - 2$, where m is a positive integer. In either case, it is easy to see that $\frac{(i+1)(i+2)}{2}$ is divisible by 3. If, on the other hand, $i \equiv 0 \pmod 3$, then we can write $i = 3m$, and

$$\frac{(i+1)(i+2)}{2} = \frac{(3m+1)(3m+2)}{2}.$$

Consequently,

$$\frac{(3m+1)(3m+2)}{2} - 1 = \frac{9m(m+1)}{2},$$

and this is easily seen to be divisible by 3.

We can generalize the results of Theorem 2 as follows:

Theorem 3: Let k be a prime number. Then each coefficient α_i of Theorem 1 is either a multiple of k, or leaves a remainder of +1 on division by k. In any one row, $\alpha_i \equiv 1 \pmod k$ when $i \equiv 0 \pmod k$, and $\alpha_i \equiv 0 \pmod k$ when $i \not\equiv 0 \pmod k$. Consequently, in row m there are m coefficients which leave remainder +1 on division by k, and $(m - 1)(k - 1)$ which are a multiple of k.

The proof is similar to that of Theorem 2, and will not be included.

REFERENCES

1. A. M. Russell. "Functions of Bounded kth Variation." *Proc. London Math. Soc.* (3) 26 (1973):547-563.
2. D. V. Widder. *Laplace Transform.* Princeton: Princeton University Press, 1946.

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PYTHAGOREAN TRIANGLES AND MULTIPLE ANGLES

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In a paper dealing with Pythagorean triangles, Gruhn [1] asked how many pairs of primitive Pythagorean triangles exist in which the sine of one of the acute angles of the second triangle equals the sine of twice either of the acute angles of the first triangle. This question may be generalized to determining pairs of primitive Pythagorean triangles where an acute angle of the second is N times an acute angle of the first (here N can take on any positive integer value). In addition, it may be asked whether any relationship exists among the generators of such primitive Pythagorean triangles.

It is necessary to review first some known results from number theory and trigonometry. A Pythagorean triangle is a right triangle whose sides are positive integers. Such triangles will be designated by the triple (x,y,z) which satisfies the equation $x^2 + y^2 = z^2$. In the case where x and y are relatively prime, the triangle is said to be primitive. Formulas for the sides of primitive Pythagorean triangles in terms of generators m and n are (see [2]):

$$x = m^2 - n^2;\ y = 2mn;\ z = m^2 + n^2$$

where m and n are positive integers such that

$$m > n;\ (m,n) = 1;\ mn \text{ is even.}$$

For a given primitive triangle (x,y,z), the generators may be found from:

$$m = \sqrt{(z + x)/2};\ n = \sqrt{(z - x)/2}.$$

Some formulas for the expansion of sin NA and cos NA in terms of sin A and cos A are as follows (see [3, 4]):

(1) $$\sin NA = \sin A\left\{(2 \cos A)^{N-1} - \binom{N-2}{1}(2 \cos A)^{N-3} + \binom{N-3}{2}(2 \cos A)^{N-5} \ldots\right\}$$

(2) $$= \binom{N}{1}\sin A \cos^{N-1}A - \binom{N}{3}\sin^3 A \cos^{N-3}A + \binom{N}{5}\sin^5 A \cos^{N-5}A \ldots$$

(3) $$\cos NA = \binom{N}{0}\cos^N A - \binom{N}{2}\sin^2 A \cos^{N-2}A + \binom{N}{4}\sin^4 A \cos^{N-4}A \ldots .$$

The following conventions will be used throughout this paper:

1. θ: minimum of the acute angles of the original primitive Pythagorean triangle
2. N: a positive integer
3. $T_N = (x_N, y_N, z_N)$, y_N even: a primitive Pythagorean triangle where one of the acute angles is N times one of the acute angles of the original triangle
4. m_N, n_N: generators of T_N
5. $\sin\theta = \min(x_1/z_1, y_1/z_1)$

PRIMITIVENESS OF T_N

It is obvious that pairs of primitive Pythagorean triangles having an acute angle of the second N times an acute angle of the first may be obtained whenever $\theta < 90°/N$ or, equivalently, $\min(x_1/z_1, y_1/z_1) < \sin 90°/N$. In the following, therefore, when T_N is cited, it is assumed that this condition is satisfied.

Theorem 1: T_1 primitive implies T_N primitive. In order to prove this theorem, the following lemmas are needed.

Lemma 1: (i) If $x_1 < y_1$, then $\sin N\theta = \begin{cases} x_N/z_N, & N \text{ odd} \\ y_N/z_N, & N \text{ even;} \end{cases}$

(ii) If $x_1 > y_1$, then $\sin N\theta = y_N/z_N$.

Proof: Use is made of formula (1) for sin $N\theta$. For $x_1 < y_1$, $\sin\theta = x_1/z_1$. When N is even, every term in the bracket involves 2 cos θ. Thus, the sum and also sin $N\theta$ will be a fraction with an even numerator. The value of sin $N\theta$ can therefore be written as y_N/z_N. When N is odd, every term in the bracket except the last term will involve 2 cos θ. The last term has value one. Thus, the bracket will be a fraction with an odd numerator and sin $N\theta$ will be a fraction with an odd numerator, i.e., x_N/z_N. For $x_1 > y_1$, $\sin\theta = y_1/z_1$. Therefore, sin $N\theta$ will be a fraction with an even numerator, i.e., y_N/z_N.

Lemma 2: $(z_2, x_N) = (z_2, y_N) = 1$.

Proof: It is equivalent to show that

$$(z_2,\ z_N \sin N\theta) = (z_2,\ z_N \cos N\theta) = 1$$

or

$$(x_1^2 + y_1^2,\ z_N \sin N\theta) = (x_1^2 + y_1^2,\ z_N \cos N\theta) = 1.$$

Use is made of formulas (2) and (3) for sin $N\theta$ and cos $N\theta$. Initially, consider the case where $x_1 < y_1$, i.e., $\sin\theta = x_1/z_1$:

$$z_N \sin N\theta = \binom{N}{1}x_1 y_1^{N-1} - \binom{N}{3}x_1^3 y_1^{N-3} + \binom{N}{5}x_1^5 y_1^{N-5} \ldots$$

$$= (x_1^2 + y_1^2)Q(x_1, y_1) + x_1(2y_1)^{N-1}$$

$$= z_2 Q(x_1, y_1) + x_1(2y_1)^{N-1},$$

where Q is some polynomial function of x_1 and y_1. Any divisor of z_2 and z sin $N\theta$ must divide $x_1(2y_1)^{N-1}$. Now $(z_2, x_1) = (z_2, y_1) = 1$ since, otherwise, x_1 and y_1 would have a divisor greater than one contradicting the assumption that T_1 is primitive. Also $(z_2, 2) = 1$ since z_2 is odd. Thus $(z_2, x_1(2y_1)^{N-1}) = 1$ and this implies that $(z_2, z_N \sin N\theta) = 1$.

Similarly,

$$z_N \cos N\theta = \binom{N}{0} y_1^N - \binom{N}{2} x_1^2 y_1^{N-2} + \binom{N}{4} x_1^4 y_1^{N-4} \ldots$$
$$= (x_1^2 + y_1^2) R(x_1, y_1) + y_1 (2y_1)^{N-1},$$

where R is some polynomial function of x_1 and y_1. The same reasoning as before shows that $(z_2, z_N \cos N\theta) = 1$. The case where $x_1 > y_1$, i.e., $\sin\theta = y_1/z_1$, can be handled in the same manner.

The proof of Theorem 1 can be accomplished by mathematical induction. The theorem is trivially true for $N = 1$. Assume that it is true for $N = k$ and try to show its validity for $N = k + 1$. Use is made of the addition formulas:

(4) $$\begin{cases} \sin(k+1)\theta = \sin\theta\cos k\theta + \cos\theta\sin k\theta \\ \cos(k+1)\theta = \cos\theta\cos k\theta - \sin\theta\sin k\theta \end{cases}$$

There are three cases to consider: (i) $x_1 < y_1$, k odd; (ii) $x_1 < y_1$, k even; (iii) $x_1 > y_1$. In the first case, by use of Lemma 1, formulas (4) become

$$\frac{y_{k+1}}{z_{k+1}} = \frac{x_1 y_k}{z_1 z_k} + \frac{y_1 x_k}{z_1 z_k}$$

$$\frac{x_{k+1}}{z_{k+1}} = \frac{y_1 y_k}{z_1 z_k} - \frac{x_1 x_k}{z_1 z_k}$$

By taking $z_{k+1} = z_1 z_k$ and working only with the numerators, the equations become:

(5) $$\begin{cases} y_{k+1} = x_1 y_k + y_1 x_k \\ x_{k+1} = y_1 y_k - x_1 x_k \end{cases}$$

It must be shown that $(x_{k+1}, y_{k+1}) = 1$. Now, any divisor of x_k and y_k divides both x_{k+1} and y_{k+1}. Equations (5) can be rewritten as

$$z_2 x_k = y_1 y_{k+1} - x_1 x_{k+1}$$

$$z_2 y_k = x_1 y_{k+1} + y_1 x_{k+1}$$

Since, by Lemma 2, z_2 is relatively prime to both x_{k+1} and y_{k+1}, any common divisor of x_{k+1} and y_{k+1} must divide x_k and y_k. Therefore, $(x_{k+1}, y_{k+1}) = (x_k, y_k) = 1$. The reasoning in each of the other cases is identical, appropriate substitutions being made for the various trigonometric functions.

CALCULATION OF T_N

In order to compute T_N from a given triple T_1, it is first necessary to check that $\min(x_1/z_1, y_1/z_1) < \sin 90°/N$. If this condition is satisfied, then $z_N = z_1^N$. Formulas (2) and (3) can be used to calculate $z_N \sin N\theta$ and $z_N \cos N\theta$. For x_N take the odd number of this pair while for y_N take the even number. Table 1 lists formulas for $z_N \sin N\theta$, $z_N \cos N\theta$, z_N for $N = 1, \ldots, 7$ and $x_1 < y_1$. Formulas, identical to these, for sides of $T_2, \ldots, T_5$ were cited by Vieta in 1646 [5]. He called T_2—the triangle of the double angle, T_3—the triangle of the triple angle, etc.

Examples of calculated T_N values are given in Table 2. The T_2 examples serve further to refute Gruhn's original conjecture that (3,4,5) and (7,24,25) are the only pair of primitive Pythagorean triangles in which the sine of one of the acute angles of the second triangle equals the sine of twice either of the acute angles of the first triangle. It is to be noted that both Malament [6] and Beran [7] have separately corrected Gruhn's statement.

GENERATORS OF T_N

Table 2 also lists generator values for the triangles calculated. Recursive formulas for the generators are as follows:

Theorem 2: (i) N even:

$$m_N = \max\{m_1 n_{N-1} + m_{N-1} n_1,\ m_1 m_{N-1} - n_1 n_{N-1}\}$$

$$n_N = \min\{m_1 n_{N-1} + m_{N-1} n_1,\ m_1 m_{N-1} - n_1 n_{N-1}\}$$

(ii) N odd and greater than one:

$$m_N = m_1 m_{N-1} \pm n_1 n_{N-1}$$

$$n_N = |m_1 n_{N-1} \mp n_1 m_{N-1}|$$

Note: Use upper sign for $x_1 < y_1$, otherwise use lower sign.

TABLE 1. Typical Formulas for T_N, $x_1 < y_1$

	T_N		
N	$z_N \sin N\theta$	$z_N \cos N\theta$	z_N
1	x_1	y_1	$z_1 = (x_1^2 + y_1^2)^{1/2}$
2	$2x_1y_1$	$y_1^2 - x_1^2$	z_1^2
3	$3x_1y_1^2 - x_1^3$	$y_1^3 - 3x_1^2y_1$	z_1^3
4	$4x_1y_1^3 - 4x_1^3y_1$	$y_1^4 - 6x_1^2y_1^2 + x_1^4$	z_1^4
5	$5x_1y_1^4 - 10x_1^3y_1^2 + x_1^5$	$y_1^5 - 10x_1^2y_1^3 + 5x_1^4y_1$	z_1^5
6	$6x_1y_1^5 - 20x_1^3y_1^3 + 6x_1^5y_1$	$y_1^6 - 15x_1^2y_1^4 + 15x_1^4y_1^2 - x_1^6$	z_1^6
7	$7x_1y_1^6 - 35x_1^3y_1^4 + 21x_1^5y_1^2 - x_1^7$	$y_1^7 - 21x_1^2y_1^5 + 35x_1^4y_1^3 - 7x_1^6y_1$	z_1^7

TABLE 2. Some Examples of T_N

Example		T_1	T_2	T_3	T_4	T_5
A	x:	5	119	2035	-	-
	y:	12	120	828		
	z:	13	169	2197		
	m:	3	12	46		
	n:	2	5	9		
B	x:	7	527	11753	354144	9653287
	y:	24	336	10296	164833	1476984
	z:	25	625	15625	390625	9765625
	m:	4	24	117	527	3116
	n:	3	7	44	336	237
C	x:	35	1081	27755	462961	
	y:	12	840	42372	1816080	-
	z:	37	1396	50653	1874161	
	m:	6	35	198	1081	
	n:	1	12	107	840	
D	x:	3	7			
	y:	4	24	-	-	-
	z:	5	25			
	m:	2	4			
	n:	1	3			
E	x:	15	161	495		
	y:	8	240	4888	-	-
	z:	17	289	4913		
	m:	4	15	52		
	n:	1	8	47		

Proof of Theorem 2: Initially, consider the case where N is odd and $x_1 < y_1$. The remaining cases are proved in a similar manner. Using the addition formulas (4) for $\sin N\theta$ and $\cos N\theta$ and Lemma 1, the following values are obtained for the sides of T_N in terms of the generators of T_1 and T_{N-1}:

$$x_N = 4m_{N-1}n_{N-1}m_1n_1 + m_{N-1}^2m_1^2 - m_{N-1}^2n_1^2 - n_{N-1}^2m_1^2 + n_{N-1}^2n_1^2$$

$$y_N = 2[m_1n_1(m_{N-1}^2 - n_{N-1}^2) - m_{N-1}n_{N-1}(m_1^2 - n_1^2)]$$

$$z_N = m_{N-1}^2m_1^2 + m_{N-1}^2n_1^2 + n_{N-1}^2m_1^2 + n_{N-1}^2n_1^2$$

Consequently:

$$m_N = \sqrt{(z_N + x_N)/2} = m_1m_{N-1} + n_1n_{N-1}$$

$$n_N = \sqrt{(z_N - x_N)/2} = m_1n_{N-1} - n_1m_{N-1}$$

It is also to be noted that the sides of T_N serve as generators for T_{2N} where these exist. Thus, for instance, for $T_1 = (5,12,13)$, the sides 5 and 12 serve as generators for $T_2 = (119,120,169)$. Similarly, for $T_2 = (1081,840,1369)$, the sides serve as generators for $T_4 = (462961,1816080,1874161)$.

REFERENCES

1. E. W. Gruhn. "Parabolas and Pythagorean Triples." *Math. Teacher* (Dec. 1959):614-615.
2. Waclaw Sierpinski. *Pythagorean Triangles*. The Scripta Mathematica Studies, #9. New York: Graduate School of Science, Yeshiva University, 1962, pp. 6-7.
3. Murray R. Spiegel. *Mathematical Handbook*. Schaum's Outline Series. New York: McGraw-Hill Book Company, 1968 (Formula 1, p. 17).
4. E. W. Hobson. *A Treatise on Plane and Advanced Trigonometry*. 7th ed. New York: Dover Publications, Inc., 1928 (Formulas 2 and 3, p. 52).
5. F. Vietae. *Ad Logisticen speciosam, notae priores*. Prop. 46-51. In *Opera Mathematica*, 1646, pp. 34-37.
6. D. Malament. "Letters to Editor." *Math. Teacher* (May 1960):380.
7. R. Beran. "Letters to Editor." *Math. Teacher* (Oct. 1960):466.

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PROOF THAT THE AREA OF A PYTHAGOREAN TRIANGLE IS NEVER A SQUARE

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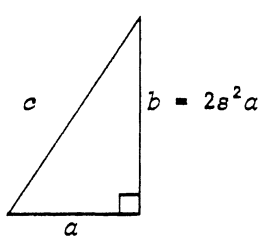
Prove that the area of an integral-sided (Pythagorean) triangle is never a square integer. In the diagrams provided below, the two triangles are equivalent. Thus, $a = a$, $b = n$, and $c = (n + k)$, where a, b, n, and k as well as s are integers. A = the area of the triangles.

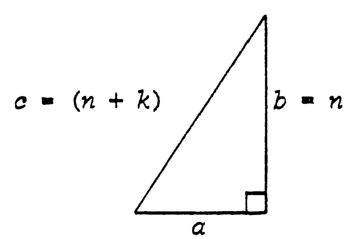


$A = \frac{1}{2}(2s^2a)a = s^2a^2$, which is a square

$a^2 + b^2 = c^2$

$a^2 + (2s^2a)^2 = c^2$

$a^2 + 4s^4a^2 = c^2$



$a^2 + b^2 = c^2$

$a^2 + n^2 = (n + k)^2$; $a^2 = 2kn + k^2$

$(2kn + k^2) + n^2 = (n + k)^2$

$a^2 + b^2 = c^2$ (Pythagorean Theorem)

$a^2 + n^2 = (n + k)^2$ (Pythagorean Theorem and equivalence of above diagrams)

$a^2 = 2kn + k^2$

$b = 2s^2a$ (from above diagrams)

$b^2 = n^2 = 4s^4(a^2)$ (since $b = n$ and $b = 2s^2a$)

$n^2 = 4s^4(2kn + k^2)$ (since $a^2 = 2kn + k^2$)

$n^2 = 8ks^4n + 4k^2s^4$

$n^2 - 8ks^4n - 4k^2s^4 = 0$

$$n = \frac{8ks^4 \pm \sqrt{64k^2s^8 - 4(-4k^2s^4)}}{2}$$

$$n = \frac{8ks^4 \pm \sqrt{64k^2s^8 + 16k^2s^4}}{2}$$

$$n = \frac{8ks^4 \pm \sqrt{16k^2s^4(4s^4 + 1)}}{2}$$

$$n = \frac{8ks^4 \pm 4ks^2\sqrt{4s^4 + 1}}{2}$$

$$n = 4ks^4 + 2ks^2\sqrt{4s^4 + 1}$$

From the above, we obtain $a^2 = 2kn + k^2$, $b^2 = n^2$, $c^2 = (n + k)^2$.

If n is irrational for all integral values of a, b, c, n, and k, then a^2, b^2, and c^2 cannot all be squares. If a^2, b^2, and c^2 are not squares, then a, b, and c are not integers, and the triangle is not an integral-sided, or Pythagorean, triangle. n can be an integer only if $\sqrt{4s^4 + 1}$ is an integer, and $\sqrt{4s^4 + 1}$ is an integer only if $s^4 = 0$—that is to say, if $s = 0$. From the diagrams, you can see that when $s = 0$, $b = 0$, and since the area of a triangle = $\frac{1}{2}ab$, this triangle has an area of 0.

Thus, dismissing the case when the area of the triangle is 0, the area of an integral-sided right triangle is never a square number.

This proof centers around the assumption that for integers a, n, and k, $a^2 + n^2 = (n + k)^2$. For example, when $a = 3$, $n = 4$, and $k = 1$, $3^2 + 4^2 = (4 + 1)^2$.

The following result—obtained by using a similar approach against Fermat's Last Theorem, where $x^n + y^n \neq z^n$ for integers when $n > 2$—is presented for the interest of the reader. For $n = 3$, $a^3 + n^3 = (n + k)^3$. Thus,

$$a^3 = 3kn^2 + 3k^2n + k^3$$

$$3kn^2 + 3k^2n + k^3 - a^3 = 0$$

$$n = \frac{-3k^2 \pm \sqrt{9k^4 - 4(3k)(k^3 - a^3)}}{6}$$

$$n = \frac{-3k^2 \pm \sqrt{12a^3k - 3k^4}}{6k}$$

I am not sure whether or not this result is of any use, or if it can be generalized for powers greater than the third power, but I intend to pursue this line of reasoning.

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RECONSIDERING A PROBLEM OF M. WARD

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ABSTRACT

In a recent issue of *The Fibonacci Quarterly*, Laxton proved a conjecture of Ward to the effect that integral linear recurrences which are not degenerate in a certain sense necessarily contain infinitely many distinct prime divisors. We point out that the result is an immediate corollary to an early theorem of Pólya published in 1921, and derive Ward's conjecture for a more general class of integral linear recurrences.

1. INTRODUCTION

Ward [6, 7] showed that nondegenerate integral linear recurrences of order 2 and 3 always contain infinitely many distinct prime divisors. Recently, Laxton [3] proved Ward's conjecture that a similar result must hold for recurrences of arbitrary higher order (again excluding some degenerated cases).

Let

$$w_{n+m} = a_{m-1}w_{n+m-1} + \cdots + a_1 w_{n+1} + a_0 w_n$$

(with $a_0 \neq 0$, $m > 0$, $n \geq 0$) be an mth order integral linear recurrence and let

$$P_w(x) = x^m - a_{m-1}x^{m-1} \cdots - a_0$$

be the associated characteristic (or spectral) polynomial.

Here is what was proved.

Theorem: Let $\{w_n\}$ be an integral linear recurrence of order $m > 1$. If all roots of $P_w(x)$ are distinct and if no ratio of distinct roots is a root of unity, then $\{w_n\}$ has infinitely many distinct prime divisors.

It turns out that the answer already did exist before the question. The very same result (and thereby the solution to Ward's conjecture) is an almost immediate corollary to a theorem of Pólya [5, Satz II'] dating back to 1921, which seems to have escaped attention.

We shall indicate how the theorem can be applied and use it to derive a stronger solution of Ward's problem.

2. POLYA'S THEOREM

We shall have to assume that the reader is familiar with some algebraic number theory (see Landau [2] or Pollard [4] for an excellent introduction).

First we observe

Lemma: Let K be an algebraic number field, D a nonzero algebraic integer in K, and $\{w_n\}$ a sequence of rational integers. $\{w_n\}$ has infinitely many prime divisors if and only if $\{Dw_n\}$ has infinitely many prime-ideal divisors.

We now combine Pólya's Satz III' [5, p. 15] and Satz II' [5, p. 17] to obtain

Theorem: Let $\alpha_1, \ldots, \alpha_r$ and all coefficients of the nontrivial polynomials $P_1(x), \ldots, P_r(x)$ be algebraic integers. Let $D \neq 0$ be an algebraic integer such that

$$F(x) = \frac{1}{D}(P_1(x)\alpha_1^x + \cdots + P_r(x)\alpha_r^x)$$

has rational integer values for $x = 0, 1, 2, \ldots$.

Assume that $r + \min \deg P(x) \leq 2$. If no ratio of distinct α's is a root of unity, then $F(x)$ has infinitely many prime-divisors.

Pólya showed the theorem for $D = 1$ (or any rational integer for that matter) but only slight modifications in the proof make it true for arbitrary algebraic integers.

For consider $G(x) = D \cdot F(x)$ and carry out the same proof. By the lemma, it follows that assuming that $F(x)$ only has finitely many prime-divisors (by way of contradiction, as Pólya does) is equivalent to assuming that $G(x)$ only has finitely many prime-ideal divisors. Where Pólya considers absolute values, one should use norms; where Pólya proceeds with analytic arguments related to the series $\Sigma F(n)z^n$, one can do exactly the same for G after factoring out D.

The theorem enables us to prove Ward's conjecture with the condition that all roots need to be distinct omitted!

Here is what we get.

Theorem: Let $\{w_n\}$ be an integral linear recurrence of order $m \geq 2$. If no ratio of distinct roots of $P_w(x)$ is a root of unity, then $\{w_n\}$ has infinitely many distinct prime divisors.

Here is how to prove it. Consider the recurrence equation for w_n. Following Gel'fond [1] (or other books on difference equations), the general solution can be expressed as

$$w_n = \frac{1}{D}(P_1(x)\alpha_1^x + \cdots + P_r(x)\alpha_r^x)$$

where $\alpha_1, \ldots, \alpha_r$ are the roots of $P_w(x)$, $P_i(x)$ a polynomial of degree equal to the multiplicity of w_i minus 1 and with algebraic integer coefficients, and D a nonzero determinant of algebraic integers (hence an algebraic integer as well). It easily follows that the conditions for Pólya's theorem are satisfied and $\{w_n\}$ must have infinitely many distinct prime divisors.

REFERENCES

1. A. O. Gel'fond. *Calculus of Finite Differences.* Delhi: Hindustan Publishing Corp., 1971.
2. E. Landau. *Vorlesungen über Zahlentheorie.* Band 3. Leipzig, 1927.
3. R. R. Laxton. "On a Problem of M. Ward." *The Fibonacci Quarterly* 12 (1974):41-44.
4. H. Pollard. *The Theorie of Algebraic Numbers.* Carus Math. Monographs 6. New York: AMS & Wiley, 1950.
5. G. Pólya. "Arithmetische Eigenschaften der Reihenentwicklungen rationaler Funktionen." *Journal für die reine und angewandte Mathematik* Band 151 (1921):1-31.
6. M. Ward. "Prime Divisors of Second-Order Recurring Sequences." *Duke Math. J.* 21 (1954):607-614.
7. M. Ward. "The Laws of Apparition and Repetition of Primes in a Cubic Recurrence." *Trans. AMS* 79 (1955):72-90.

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WHAT A DIFFERENCE A DIFFERENCE MAKES!

JERRY T. SULLIVAN

Two men are leaving the office when one remarks that both his wife and boy are celebrating their birthdays that night. The other wonders if it is his youngest son. "Yes," says the first, "but he's not so little anymore. His age, multiplied by my wife's age, is equal to the square of the difference of their ages plus one year." This problem, similar to an earlier one in *The Fibonacci Quarterly* [1], provides some surprising and amusing mathematical twists.

On the premise that many mothers are between 25 and 35 years of age, and also that a typical boy is about 10 years old, pairs of ages such as 10 and 30, 11 and 35, etc., can be tested. After a few trials, an answer is seen to be 13 and 34. Further thought shows that the problem can be handled algebraically. If the age of the wife is W and that of the boy is B, then

(1) $$WB = (W - B)^2 + 1.$$

The wife's age can be solved as a function of the boy's age:

(2) $$W = [3B \pm (5B^2 - 4)^{1/2}]/2.$$

Substituting $B = 13$ into equation (2) and using the positive square root gives the known answer $W = 34$. However, using the negative square root gives the answer $W = 5$. It is an unusual wife who is younger than her son, but the numbers 13 and 5 also satisfy equation (1). Using the number 5 in equation (2) and choosing the negative root gives the numbers 5 and 2 as another solution. Proceeding in this fashion results in the sequence

(3) $$1, 2, 5, 13, 34, 89, \ldots,$$

where each successive pair of numbers satisfies equation (1). The number 1 has the unusual property of giving the solutions 1 and 2 when substituted into equation (2). It does not give a solution lower than itself.

The above sequence is every other number of the usual Fibonacci sequence. Calling the initial age in the sequence A_0, the next A_1, etc., equation (1) may be rewritten as a difference equation,

(4) $$A_{N+1}A_N = (A_{N+1} - A_N)^2 + 1.$$

Equation (4) is a nonlinear difference euqtion, which fortunately can be simplified. First rewrite equation (4) as

$$A_{N+1}^2 - 3A_{N+1}A_N + A_N^2 = -1.$$

This must also hold for the next number pair so that

$$A_{N+2}^2 - 3A_{N+2}A_{N+1} + A_{N+1}^2 = A_{N+1}^2 - 3A_{N+1}A_N + A_N^2.$$

Cancelling like terms and rearranging gives

$$[A_{N+2} - A_N](A_{N+2} - 3A_{N+1} + A_N) = 0.$$

Setting the term in brackets equal to 0 would result in a repeating solution to equation (4), which would not generate the correct age sequence. The correct simplification of equation (4) is the linear difference equation

$$A_{N+2} - 3A_{N+1} + A_N = 0. \quad (5)$$

Equation (5) is solved by assuming that $A_N = R^N$. Substitution results in

$$R^N(R^2 - 3R + 1) = 0.$$

There are two roots which satisfy the quadratic equation,

$$R_+ = (3 + \sqrt{5})/2 \quad \text{and} \quad R_- = 1/R_+.$$

The solution to the difference equation (5) is

$$A_N = aR^N + bR^{-N},$$

and choosing the constants a and b so that $A_0 = 1$ and $A_1 = 2$ finally results in

$$A_N = (R/R + 1)R^N + (1/R + 1)R^{-N}. \quad (6)$$

The curious property that $A_0 = 1$ seemed to be a natural boundary for the problem, and is mirrored in the solution. Suppose there were lower solutions A_{-1}, A_{-2}, etc. Replacing N by $-N$ in equation (6) leads to

$$A_{-N} = \left(\frac{R}{R+1}\right)R^{-N} + \left(\frac{1}{R+1}\right)R^N = \left(\frac{R}{R+1}\right)R^{N-1} + \left(\frac{1}{R+1}\right)R^{-(N-1)} = A_{N-1},$$

so that all of the supposed lower solutions are actually equal to a higher one. Lastly, to actually compute A_N from equation (6) is not as formidable as it first appears. It is not necessary to compute large integer powers of $R = (3 + \sqrt{5})/2$, but merely to use the rules

$$R^2 = 3R - 1$$

$$R^3 = R(R^2) = 3R^2 - R = 8R - 3$$

$$R^4 = R(R^3) = 8R^2 - R = 21R - 8$$

$$\vdots$$

etc.

REFERENCE

1. J. A. H. Hunter. "Fibonacci to the Rescue." *The Fibonacci Quarterly* 8, No. 4 (1970): 406.

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CIRCULANTS AND HORADAM'S SEQUENCES

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In a certain problem in knot theory it became necessary to evaluate the following $n \times n$ determinant:

$$(1)\quad C_n(k,-(2k+1),k) = \begin{vmatrix} k & -(2k+1) & k & 0 & 0 & \cdots & & 0 \\ 0 & k & -(2k+1) & k & 0 & \cdots & & 0 \\ 0 & 0 & k & -(2k+1) & k & \cdots & & 0 \\ & \cdots & & \cdots & & & & \\ 0 & \cdots & & & k & & -(2k+1) & k \\ k & 0 & 0 & \cdots & & 0 & k & -(2k+1) \\ -(2k+1) & k & 0 & \cdots & & 0 & 0 & k \end{vmatrix}$$

where k is an integer. The purpose of this note is to express this determinant (and other determinants of the same form) in terms of Horadam's generalized sequences (see [5]).

$C_n(k,-(2k+1),k)$ belongs to the class of determinants known as "circulants." A determinant is a circulant if each row is a cyclic permutation of the preceding row. If the first row of an $n \times n$ circulant is $(a_0, a_1, \ldots, a_{n-1})$ then the second row will be $(a_{n-1}, a_0, a_1, \ldots, a_{n-2})$, the third $(a_{n-2}, a_{n-1}, a_0, \ldots, a_{n-3})$ and so on. If we let $C(a_0, a_1, \ldots, a_{n-1})$ denote the value of the $n \times n$ circulant with first row $(a_0, a_1, \ldots, a_{n-1})$ then the following pretty result holds (see Aitken [1, p. 123] or Muir [8, p. 445]):

Theorem 1: Let $\omega = \exp(2\pi i/n)$. Then

$$(2)\qquad C(a_0, a_1, \ldots, a_{n-1}) = \prod_{j=0}^{n-1} (a_0 + a_1\omega^j + a_2\omega^{2j} + \cdots + a_{n-1}\omega^{(n-1)j}).$$

For the particular case in which we are interested, all but 3 consecutive terms in each row of the determinant vanish. In agreement with (1), we will let $C_n(a_0, a_1, a_2)$ denote the value of the $n \times n$ circulant whose first row is $(a_0, a_1, a_2, 0, \ldots, 0)$. Equation (2) then reduces to

$$(3)\qquad C_n(a_0, a_1, a_2) = \prod_{j=0}^{n-1} (a_0 + a_1\omega^j + a_2\omega^{2j}).$$

Here a_0, a_1, and a_2 may be any real or complex numbers. We will assume throughout that $a_0 \neq 0$. It is also reasonable to assume that $n \geq 3$. It is clear that (up to sign) $C_n(a_0, a_1, a_2)$ is equal to $C(0, \ldots, 0, a_0, a_1, a_2, 0, \ldots, 0)$; i.e., it doesn't really matter where the 3 consecutive terms appear in the first row of the circulant.

As a consequence of Theorem 1 we get:

Corollary 2: Let x_1, x_2 be the roots of the quadratic equation

$$(4)\qquad a_0x^2 + a_1x + a_2 = 0 \quad (a_0 \neq 0).$$

Then

$$(5)\qquad C_n(a_0, a_1, a_2) = a_0^n(x_1^n - 1)(x_2^n - 1).$$

Proof: From (4) it follows that

$$(6)\qquad x_1 + x_2 = -a_1/a_0 \quad\text{and}\quad x_1x_2 = a_2/a_0.$$

Again let $\omega = \exp(2\pi i/n)$. Then

$$a_0^n(x_1^n - 1)(x_2^n - 1) = a_0^n\prod_{j=0}^{n-1}(x_1 - \omega^j)(x_2 - \omega^j) = a_0^n\prod_{j=0}^{n-1}(x_1x_2 - (x_1 + x_2)\omega^j + \omega^{2j})$$

$$= a_0^n\prod_{j=0}^{n-1}(a_2/a_0 + (a_1/a_0)\omega^j + \omega^{2j}) = \prod_{j=0}^{n-1}(a_2 + a_1\omega^j + a_0\omega^{2j}) = \prod_{j=0}^{n-1}(a_0 + a_1\omega^j + a_2\omega^{2j})$$

and the desired result then follows from (3).

Corollary 2 will suffice for our purposes. However, it should be noted that for an arbitrary circulant with first row $(a_0, a_1, \ldots, a_{n-1})$ an analogous result holds relating $C(a_0, \ldots, a_{n-1})$ and $\Pi(x_i^n - 1)$ where $x_1, x_2, \ldots, x_{n-1}$ are the roots of $a_0x^{n-1} + a_1x^{n-2} + \ldots + a_{n-1} = 0$ (compare Muir [8, p. 471]).

Following Horadam [5] for any integers p,q we define the sequences $u_n \equiv u_n(p,q)$ and $v_n \equiv v_n(p,q)$ (for $n \geq 0$) recursively by

(7) $$u_0 = 1, \quad u_1 = p, \quad u_n = pu_{n-1} - qu_{n-2} \quad (n \geq 2)$$

and

(8) $$v_0 = 2, \quad v_1 = p, \quad v_n = pv_{n-1} - qv_{n-2} \quad (n \geq 2)$$

In particular

(9) $$u_{n-1}(1,-1) = F_n \quad (\text{for } n \geq 1)$$

and

(10) $$v_n(1,-1) = L_n \quad (n \geq 2)$$

where $\{F_n\}$ is the ordinary Fibonacci sequence starting with $F_1 = F_2 = 1$ and

(11) $$L_n = F_{n+1} + F_{n-1} \quad (n \geq 2)$$

is the associated Lucas sequence.

The following can be verfied easily (see Horadam [5] and Bachmann [6, Chap. 2, pp. 73-78]):

Lemma 3: Let α,β be the roots of

(12) $$x^2 - px + q = 0$$

and let $d = +\sqrt{p^2 - 4q}$. Then for all $n \geq 0$

(13) $$\alpha^{n+1} - \beta^{n+1} = du_n(p,q)$$

and

(14) $$\alpha^n + \beta^n = v_n(p,q).$$

Equations (13) and (14) remain true even in the "degenerate" case $d = 0$ (i.e., $p^2 = 4q$ and $\alpha = \beta$), but then (13) is no longer useful for determining $u_n(p,q)$. Note further that although p,q are assumed to be rational integers, the recursion formulas (7) and (8) make equally good sense if we allow p and q to take real or complex values. Equations (13) and (14) (and most of the results stated below) remain valid in this more general setting. However, in this note we will restrict ourselves to integer Horadam sequences (and to circulants with integer entries).

Combining Corollary 2 and Lemma 3 gives:

Theorem 4: For any integers a, b, and c ($a \neq 0$),

(15) $$C_n(a,b,c) = a^n + c^n - v_n(-b,ac) \quad (n \geq 3).$$

Proof: From equation (5), we get

(16) $$C_n(a,b,c) = a^n(x_1^n - 1)(x_2^n - 1)$$

where x_1,x_2 are the roots of

(17) $$ax^2 + bx + c = 0.$$

Multiplying (17) by a and letting $z = ax$, we get

(18) $$z^2 + bz + ac = 0.$$

The roots of (18) are $z_1 = ax_1$ and $z_2 = ax_2$. Therefore,

(19) $$z_1 + z_2 = -b \quad \text{and} \quad z_1z_2 = ac.$$

If we let $p = -b$ and $q = ac$ in Lemma 3, then equation (14) becomes

(14′) $$z_1^n + z_2^n = v_n(-b,ac).$$

Now plug $x_i = z_i/a$ ($i = 1,2$) into (16) and use (14′) and (19) to get

$$\begin{aligned} C_n(a,b,c) &= a^n((z_1/a)^n - 1)((z_2/a)^n - 1) \\ &= a^n\left(\frac{(z_1z_2)^n}{a^{2n}} + 1 - \frac{(z_1^n + z_2^n)}{a^n}\right) \\ &= c^n + a^n - v_n(-b,ac), \text{ which is equation (15) above.} \end{aligned}$$

Thus we can use properties of $C_n(a,b,c)$ to give us information about $v_n(-b,ac)$ and vice versa. For example:

Corollary 5: For any integers r, a, b, and c ($a \neq 0$,

$$v_n(-rb,r^2ac) = r^n v_n(-b,ac). \tag{20}$$

Proof: Equation (15) implies that

$$C_n(ra,rb,rc) = r^n(a^n + c^n) - v_n(-rb,r^2ac). \tag{21}$$

But $C_n(ra,rb,rc)$ is an $n \times n$ determinant. Therefore,

$$C_n(ra,rb,rc) = r^n C_n(a,b,c) = r^n(a^n + c^n - v_n(-b,ac)) \tag{22}$$

and (20) follows.

Equation (20) can also be proved directly [i.e., without introducing $C_n(ra,rb,rc)$] by comparing $(\alpha^n + \beta^n)$ with $(\alpha_0^n + \beta_0^n)$ where α,β (resp. α_0,β_0) are the roots of

$$x^2 + rbx + r^2ac = 0 \quad (\text{resp. } x^2 + bx + ac = 0).$$

When $c = a$ [as is the case in (1) above], then we can express $C_n(c,b,c)$ in terms of Horadam sequences which are different from the sequence $\{v_n(-b,c^2)\}$ given by Theorem 4.

Theorem 6: Let b,c be integers $c \neq 0$. Let $r = -(b + 2c)$ and suppose $r \neq 0$. Let $u_n \equiv u_n(r,-rc)$ and $v_n \equiv v_n(r,-rc)$. Then for each $m \geq 2$,

$$C_{2m-1}(c,b,c) = -(v_{2m-1})^2/r^{2m-1} \tag{23}$$

and

$$C_{2m}(c,b,c) = -(b^2 - 4c^2)(u_{2m-1})^2/r^{2m}. \tag{24}$$

The proof of Theorem 6 depends on:

Lemma 7: Let $r = -(b + 2c)$. Then,

$$(v_{2m-1}(r,-rc))^2 = v_{2m-1}(-rb,(rc)^2) - 2(rc)^{2m-1} \tag{25}$$

and

$$(b^2 - 4c^2)(u_{2m-1}(r,-rc))^2 = v_{2m}(-rb,(rc)^2) - 2(rc)^{2m}. \tag{26}$$

Proof of Lemma 7: We will prove (26) by using (13) and (14). The proof of (25) is almost exactly the same and will be left as an exercise.

Let α,β be the roots of $x^2 - rx - rc = 0$. Then,

$$\alpha\beta = -rc. \tag{27}$$

Choose $\alpha = \dfrac{r + \sqrt{r^2 + 4rc}}{2}$ and $\beta = \dfrac{r - \sqrt{r^2 + 4rc}}{2}$. Note that $d^2 = r^2 + 4rc = b^2 - 4c^2$, since $r = -(b + 2c)$. Using this fact, it is easily verified that

$$\alpha^2 = r\alpha_0 \quad \text{and} \quad \beta^2 = r\beta_0, \tag{28}$$

where $\alpha_0 = \dfrac{-b + \sqrt{b^2 - 4c^2}}{2}$ and $\beta_0 = \dfrac{-b - \sqrt{b^2 - 4c^2}}{2}$ are the roots of $x^2 + bx + c^2 = 0$.

Now applying Lemma 3 (first with respect to α,β and then with respect to α_0,β_0), we get

$$\begin{aligned}
(b^2 - 4c^2)(u_{2m-1}(r,-rc))^2 &= (r^2 + 4rc)(u_{2m-1}(r,-rc))^2 \\
&= (du_{2m-1}(r,-rc))^2 = (\alpha^{2m} - \beta^{2m})^2 \\
&= (\alpha^2)^{2m} + (\beta^2)^{2m} - 2(\alpha\beta)^{2m} \\
&= r^{2m}(\alpha_0^{2m} + \beta_0^{2m}) - 2(-rc)^{2m} \quad \text{[using (27) and (28)]} \\
&= r^{2m}v_{2m}(-b,c^2) - 2(rc)^{2m} \\
&= v_{2m}(-rb,(rc)^2) - 2(rc)^{2m} \quad \text{[using (20)]}.
\end{aligned}$$

Proof of Theorem 6:

$$\begin{aligned}
r^{2m-1}C_{2m-1}(c,b,c) &= C_{2m-1}(rc,rb,rc) \\
&= 2(rc)^{2m-1} - v_{2m-1}(-rb,(rc)^2) \quad \text{[using (15)]} \\
&= -(v_{2m-1}(r,-rc))^2 \quad \text{[using (25)]}.
\end{aligned}$$

This proves (23). Equation (24) follows in the same way from (26).

When $|r| = s^2$, equations (23) and (24) can be rewritten in the following simpler form:

Corollary 8: If $(b + 2c) = \pm s^2$, then for all $m \geq 2$,

(29) $$C_{2m-1}(c,b,c) = \pm(v_{2m-1}(s,\pm c))^2$$

and

(30) $$C_{2m}(c,b,c) = \mp(b - 2c)(u_{2m-1}(s,\pm c))^2.$$

Proof: The proof of (30) depends on the fact that for any integers r, p, and q

(31) $$u_n(rp,r^2q) = r^n u_n(p,q).$$

This is analogous to (20) and is easily seen by comparing $(\alpha^{n+1} - \beta^{n+1})/d$ and $(\alpha_0^{n+1} - \beta_0^{n+1})/d_0$ where α,β (resp. α_0,β_0) are the roots and d (resp. $d_0 = d/r$) is the discriminant of $x^2 - rpx + r^2q = 0$ (resp. $x^2 - px + q = 0$).

Now if $r = -(b + 2c) = \mp s^2$, then it follows from (24) and (31) that

$$C_{2m}(c,b,c) = -(b^2 - 4c^2)(u_{2m-1}(\mp s^2,\pm s^2 c))^2/(\mp s^2)^{2m}$$

(32) $$= \frac{-(b^2 - 4c^2)}{s^2}(u_{2m-1}(\mp s,\pm c))^2 = \mp(b - 2c)(u_{2m-1}(\mp s,\pm c))^2,$$

since $-(b^2 - 4c^2) = r(b - 2c) = \mp s^2(b - 2c)$. It also follows from (31) that for any p,q

$$u_n(-p,q) = u_n((-1)p,(-1)^2q) = (-1)^n u_n(p,q).$$

Therefore,

$$(u_{2m-1}(-s,\pm c))^2 = (u_{2m-1}(+s,\pm c))^2$$

and it doesn't matter which sign we choose for s on the right side of (32). This proves (30). The proof of (29) is essentially the same.

Note that if we allow p and q to take on real or complex values in the recursion formulas (7) and (8) defining $u_n(p,q)$ and $v_n(p,q)$ then the above argument shows that (23) and (24) can always be simplified to

(29′) $$C_{2m-1}(c,b,c) = -(v_{2m-1}(\sqrt{r},-c))^2$$

(30′) $$C_{2m}(c,b,c) = (b - 2c)(u_{2m-1}(\sqrt{r},-c))^2$$

where $r = -(b + 2c)$.

If in Corollary 8 we let $b + 2c = p^2$ and $c = q$, then (29) and (30) can be rewritten as

(33) $$C_{2m-1}(q,p^2 - 2q,q) = (v_{2m-1}(p,q))^2$$

and

(34) $$C_{2m}(q,p^2 - 2q,q) = -(p^2 - 4q)(u_{2m-1}(p,q))^2.$$

The cases $b + 2c = \pm 1$ are of particular interest.

If $b + 2c = +1$ and we let $c = k + 1$, then (29) and (30) become

(35) $$C_{2m-1}(k+1,-(2k+1),k+1) = (v_{2m-1}(1,k+1))^2$$

and

(36) $$C_{2m}(k+1,-(2k+1),k+1) = (4k+3)(u_{2m-1}(1,k+1))^2.$$

If $b + 2c = -1$ and $c = k$, then we get

(37) $$C_{2m-1}(k,-(2k+1),k) = -(v_{2m-1}(1,-k))^2$$

and

(38) $$C_{2m}(k,-(2k+1),k) = -(4k+1)(u_{2m-1}(1,-k))^2.$$

For $k = 1$, equations (37) and (38) reduce to

(37′) $$C_{2m-1}(1,-3,1) = -L_{2m-1}^2$$

and

(38′) $$C_{2m}(1,-3,1) = -5F_{2m}^2.$$

(Compare Fielder [2, p. 356].) The determinant dealt with in Fielder's paper is an example of a "continuant"—another important class of determinants (see Muir [8, Chap. XIII]).

The circulants $C_n(k,-(2k+1),k)$ and $C_n(k+1,-(2k+1),k+1)$ arose in the following topological problem: To each pair of odd integers a,b satisfying $a \geq 3$, $|b| < a$, $(a,b) = 1$, there can be associated a "knot with two bridges" (see Schubert [10]). Let $M(n,a,b)$ denote the n sheeted branched cyclic covering the two-bridge knot associated with the pair $\{a, b\}$. Then it can be shown (Minkus [7]) that the one-dimensional integral homology group of

$M(n,\ 4k + 1,\ 4k - 1)$ is an abelian group on n generators $A_1, A_2, \ldots, A_n$ subject to the n defining relations $kA_i - (2k + 1)A_{i+1} + kA_{i+2} = 0$ $(i = 1, 2, \ldots, n)$, subscripts reduced mod n when necessary. Similarly, the homology group of $M(n,\ 4k + 3,\ 4k + 1)$ has defining relations $(k + 1)A_i - (2k + 1)A_{i+1} + (k + 1)A_{i+2} = 0$ $(i = 1, 2, \ldots, n)$. Thus, $C_n(k, -(2k + 1), k)$ and $C_n(k + 1, -(2k + 1), k + 1)$ are the determinants of the "relation matrices" of these groups. When these circulants are nonzero, they are (in absolute value) equal to the orders of these groups (compare Fox [3, p. 149]). Note that $C_n(k + 1, -(2k + 1), k + 1)$ and $-C_n(k, -(2k + 1), k)$ are perfect squares for odd values of n, in agreement with the theorem of Plans [9]. In the case $k = 1$ [equations (37′) and (38′) above], the two-bridge knot of type $\{5, 3\}$ is just the figure-eight knot. The homology groups of the branched cyclic coverings of this knot have been determined by Fox and agree with (37′) and (38′) (see [4, p. 193]).

REFERENCES

1. A. C. Aitkin. *Determinants and Matrices*. New York: Interscience Publishers, 1951.
2. D. C. Fielder. "Fibonacci Numbers in Tree Counts for Sector and Related Graphs." *The Fibonacci Quarterly* 12, No. 4 (1974):355-359.
3. R. H. Fox. "A Quick Trip through Knot Theory." In *Topology of 3-Manifolds and Related Topics* (Proc. Univ. of Georgia Inst., 1961), ed. M. K. Fort. Englewood Cliffs, N.J.: Prentice-Hall, Inc., 1962, pp. 120-167.
4. R. H. Fox. "The Homology Characters of the Cyclic Coverings of Knots of Genus One." *Annals of Mathematics* 71, No. 1 (1960):187-196.
5. H. F. Horadam. "Basic Properties of a Certain Generalized Sequence of Numbers." *The Fibonacci Quarterly* 3, No. 3 (1965):161-176.
6. P. Bachmann. *Niedere Zahlentheorie*. Vol. 2. Leipzig: Teubner, 1910.
7. J. B. Minkus. "On the Branched Cyclic Coverings of Knots with Two Bridges." Unpublished material.
8. T. Muir. *A Treatise on the Theory of Determinants*. New York: Dover, 1960.
9. C. McA. Gordon. "A Short Proof of a Theorem of Plans on the Homology of the Branched Cyclic Coverings of a Knot." *Bull. Amer. Math. Soc.* 77, No. 1 (1971).
10. H. Schubert. "Knoten mit zwei Brucken." *Mathematische Zeitschrift* 65 (1956):133-170.

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AN EXPANSION OF GOLUBEV'S 11 × 11 MAGIC SQUARE OF PRIMES TO ITS MAXIMUM, 21 × 21

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Edgar Karst, in the December 1972 issue of *The Fibonacci Quarterly* presented Golubev's magic square of order 11 consisting of prime numbers of the form $30x + 17$ and asked whether someone is able to attach a frame of order 13. The characteristics in Golubev's square are additionally "magic" in several ways which are repeated from the article cited. The stated requirements imposed were that:

1. All n rows, n columns, and 2 major diagonals have the same sum equal to n x the central number (n x 63317 in Golubev's square).
2. All included numbers be prime numbers equal to 17 plus an integral multiple of 30, with the multiple not divisible integrally by 17.
3. The sums of each pair of opposite (top and bottom or left and right) borders, excepting corner numbers, equal 2 x (the order less 2) x the central number [here 2 x 7 x 63317 or 2 x $(n - 2)$ x 63317].
4. The sums of opposite outer elements in any row or column equal 2 x the central number, for any order.
5. The opposite corner primes in the squares of each order have the sum 2 x the central prime (2 x 63317).

The addition of frames of the order 13 through 21 was as far as I could go with positive primes of form $30x + 17$ centered about 63317, following the rules imposed above. There were about 46 unused primes left over in the series. This is of course not enough for another (23rd-order) frame, but the availability of more primes in the progression suggests the possibility of rearrangements of complementary pairs and that an additional degree of magicality might be accomplished in the 21 x 21 square.

The 21 x 21 square is shown in Figures 1a, 1b, and 1c, which are to be considered as the left, middle, and right thirds of the square, respectively.

87587	38867	91757	34457	95087	30137	98897
87887	64037	62417	18257	48947	84377	41687
37397	79337	79817	45587	85577	40787	91097
92297	37217	83177	78797	47777	85247	41117
34127	91367	43427	79427	63647	62627	65147
99527	33767	88667	46877	64667	73547	52757
23567	97607	37547	84737	49367	80177	59447
103217	27767	94397	41777	80747	80897	73127
22637	103307	31907	89867	17957	81077	53117
106907	23057	100787	36527	92987	81647	52727
19577	121157	24677	94907	33587	44927	74507
109517	18047	106487	31607	104327	44417	51257
16787	112577	19997	99707	22037	43787	101537
112247	13877	111977	25367	112877	84437	46187
2957	116867	2897	106277	13217	27917	57947
114827	9467	117497	20327	120977	53657	73877
11087	120647	8837	111347	46727	64007	61487
119027	5717	122867	2207	78857	41387	85517
6047	124847	1427	81047	41057	85847	35537
121487	2357	64217	108377	77687	42257	84947
947	87767	34877	92177	31547	96497	27737

FIGURE 1a. Left-Hand Third of Square

27617	102677	23627	49937	19697	110567	14537
90527	35747	96137	30467	102317	24137	107687
35507	96587	29837	43037	24197	107837	18287
32327	34667	98327	27827	104987	21467	111497
57077	67427	56807	70157	49157	75227	49877
52457	74567	51287	75767	49787	49727	24527
54767	71987	54167	72647	53597	50147	84407
67217	60527	60257	58427	59387	70937	66467
75437	64877	60497	54347	71147	65717	51197
55967	60017	64577	61637	63737	66617	70667
69737	72707	62477	63317	64157	53927	56897
57737	58067	62897	64997	62057	68567	68897
56957	60917	66137	72287	55487	61757	69677
60167	66107	66377	68207	67247	55697	59417
71867	54647	72467	53987	73037	76487	42227
74177	52067	75347	50867	76847	76907	102107
69557	59207	69827	56477	77477	51407	76757
94307	91967	28307	98807	21647	105167	15137
91127	30047	96797	83597	102437	18797	108347
36107	90887	30497	96167	24317	102497	18947
99017	23957	103007	76697	106937	16067	112097

FIGURE 1b. Middle Third of Square

116027	10457	119057	7187	122117	3677	125687
18917	113147	13457	118757	7727	124277	38747
114197	12227	119657	6947	125207	47297	89237
14867	118037	8387	124427	43457	89417	34337
77867	48197	79907	47207	83207	35267	92507
119087	72977	61967	79757	37967	92867	27107
68687	46457	77267	41897	89087	29027	103067
53507	45737	45887	84857	32237	98867	23417
73517	45557	108677	36767	94727	23327	103997
73907	44987	33647	90107	25847	103577	19727
52127	81707	93047	31727	101957	5477	107057
75377	82217	22307	95027	20147	108587	17117
25097	82847	104597	26927	106637	14057	109847
80447	42197	13757	101267	14657	112757	14387
67187	98717	113417	20357	123737	9767	123677
7547	53087	5657	106307	9137	117167	11807
48767	78437	62987	15287	117797	5987	115547
111767	8597	118247	47837	3767	120917	7607
12437	114407	6977	119687	46817	1787	120587
107717	13487	113177	7877	118907	62597	5147
10607	116177	7577	119447	4517	122957	39047

FIGURE 1c. Right-Hand Third of Square

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SOME EXTENSIONS OF PROPERTIES OF THE SEQUENCE OF FIBONACCI POLYNOMIALS

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Sequences of functions, $\langle g_n \rangle$, that satisfy the recursion formula

(1) $$g_{n+2}(x) = axg_{n+1}(x) + bg_n(x)$$

where a and b are constants, inherit many of the properties of the sequence of Fibonacci polynomials [1]. This paper is intended to present some of these extensions.

1. BASIC DEFINITIONS AND PROPERTIES

Suppose that a and b are numbers. Let R denote the set of real numbers and C denote the set of complex numbers.

Definition 1: If $V \subseteq R$, $\mathcal{S}_{(a,b)}(V) = \{\langle g_n \rangle |$. For each natural number $p, g_p : V \to C$ and $g_{p+2}(x) = axg_{p+1}(x) + bg_p(x)$ for each $x \in V\}$.

If $V_1 \subseteq V_2$, it is easy to verify that if $\langle g_n \rangle \in \mathcal{S}_{(a,b)}(V_2)$, the corresponding sequence of restrictions is an element of $\mathcal{S}_{(a,b)}(V_1)$.

Theorem 1: If $\langle g_n \rangle$ and $\langle h_n \rangle$ are members of $\mathcal{S}_{(a,b)}(V)$ and $s:V \to C$ and $t:V \to C$, then $\langle sg_n + th_n \rangle \in \mathcal{S}_{(a,b)}(V)$. The proof for Theorem 1 is a straightforward computation.

Theorem 2: If $\{\langle g_n \rangle, \langle h_n \rangle\} \subseteq \mathcal{S}_{(a,b)}(V)$, then $\langle g_n \rangle = \langle h_n \rangle$ if and only if $g_1 = h_1$ and $g_2 = h_2$.

The proof of one of the implications of Theorem 2 is an application of the definition of equality of sequences. The other implication is an easy induction proof.

The elements of $\mathcal{S}_{(a,b)}(V)$ share a common summation formula.

Theorem 3: Suppose that for each natural number p, $g_p : V \to C$. $\langle g_n \rangle \in \mathcal{S}_{(a,b)}(V)$ if and only if for each natural number p,

$$(ax + b - 1)\sum_{j=1}^{p} g_j(x) = g_{p+1}(x) + bg_p(x) + (ax - 1)g_1(x) - g_2(x).$$

Proof: If $\langle g_n \rangle \in \mathcal{S}_{(a,b)}(V)$, the summation formula can be proved by a simple inductive argument. If $\langle g_n \rangle$ is a sequence of complex-valued functions on V with the given summation formula, then for each natural number p, the identity

$$(ax + b - 1)g_{p+1}(x) = (ax + b - 1)\left[\sum_{j=1}^{p+1} g_j(x) - \sum_{j=1}^{p} g_j(x)\right]$$

can be transformed into the equation $axg_{p+1}(x) = g_{p+2}(x) - bg_p(x)$ and thus
$$\langle g_n \rangle \in \mathcal{S}_{(a,b)}(V).$$

One element of $\mathcal{S}_{(a,b)}(R)$ seems to correspond to the sequence of Fibonacci polynomials.

Definition 2: Let $W_{(a,b)} = \langle w_n \rangle$ be the element of $\mathcal{S}_{(a,b)}(R)$ defined by $w_1(x) = 1$ and $w_2(x) = ax$.

$W_{(a,b)}$ is well defined as a consequence of Theorem 2. $W_{(1,1)}$, for example, is the sequence of Fibonacci polynomials. If $a \neq 0$ and $b \neq 0$, M. N. S. Swamy's formula [2] for the Fibonacci polynomials can be modified to give the following formula:

$$w_p(x) = \sum_{j=0}^{[(p-1)/2]} \binom{p-1-j}{j} (ax)^{p-1-2j} b^j.$$

The importance of $W_{(a,b)}$ is illustrated by the following theorem, which can easily be proved by induction.

Theorem 4: Suppose $V \subseteq R$ and that $\langle g_n \rangle$ is a sequence of complex-valued functions on V. $\langle g_n \rangle \in \mathcal{S}_{(a,b)}(V)$ if and only if $g_{p+2} = bg_1 w_p + g_2 w_{p+1}$ for each natural number p.

2. THE BINET FORMS FOR $W_{(a,b)}$

Definition 3: Let $A(x) = \dfrac{ax + \sqrt{a^2x^2 + 4b}}{2}$ and $B(x) = \dfrac{ax - \sqrt{a^2x^2 + 4b}}{2}$.

Theorem 5: $\langle 1, A, A^2, A^3, \ldots \rangle$ and $\langle 1, B, B^2, B^3, \ldots \rangle$ are elements of $\mathcal{S}_{(a,b)}(R)$.

Proof: $A^2(x) = axA(x) + b$ and $B^2(x) = axB(x) + b$. Using these two facts,
$$A^{p+2}(x) = A^2(x)A^p(x) = axA(x)A^p(x) + bA^p(x) = axA^{p+1}(x) + bA^p(x)$$
and
$$B^{p+2}(x) = B^2(x)B^p(x) = axB(x)B^p(x) + bB^p(x) = axB^{p+1}(x) + bB^p(x).$$

Theorem 6: For each natural number p, $(A - B)w_p = A^p - B^p$.

Proof: For each natural number p, let $h_p = (A - B)w_p$ and $g_p = A^p - B^p$. As a consequence of Theorem 1 and Theorem 5,
$$\{\langle h_n \rangle \cdot \langle g_n \rangle\} \subseteq \mathcal{S}_{(a,b)}^{(R)}.$$
By direct computation, $h_1 = g_1$ and $h_2 = g_2$. By Theorem 2, $\langle g_n \rangle = \langle h_n \rangle$ and the result follows by equating corresponding terms.

3. MATRIX GENERATORS

Let $Q = \begin{pmatrix} ax & 1 \\ b & 0 \end{pmatrix}$

Theorem 7: If $\langle g_n \rangle \in \mathcal{S}_{(a,b)}^{(V)}$, then for each natural number p,
$$\begin{pmatrix} g_{p+2} & g_{p+1} \\ g_{p+1} & g_p \end{pmatrix} = \begin{pmatrix} g_3 & g_2 \\ g_2 & g_1 \end{pmatrix} Q^{p-1}$$

Theorem 7 can be proved with a simple induction argument. Using Theorem 7, many identities analogous to familiar identities for the sequence of Fibonacci polynomials can be shown by standard methods. For example, the following statement is a result of computing the determinants of the matrices in Theorem 7.

Corollary: If $\langle g_n \rangle \in \mathcal{S}_{(a,b)}^{(V)}$ and p is a natural number,
$$g_{p+2}g_p - g_{p+1}^2 = (-b)^{p-1}(g_3 g_1 - g_2^2).$$

For the sequence $W_{(a,b)}$, the identity in the corollary above reduces to
$$w_{p+2}w_p - w_{p+1}^2 = -(-b)^p.$$

If Theorem 7 is specialized to $W_{(a,b)}$ and the result simplified, the following corollary results.

Corollary: For each natural number p,
$$\begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} w_{p+2} & w_{p+1} \\ w_{p+1} & w_p \end{pmatrix} = Q^{p+1}.$$

Theorem 8: If p and q are natural numbers, $w_{p+q+1} = w_{p+1} \cdot w_{q+1} + bw_p \cdot w_q$.

Proof: When Q^{p+q} is computed directly using the corollary above, w_{p+q+1} is the first row, first column entry. When Q^p and Q^q are computed using the corollary above, and the results multiplied, the first row, first column entry of $Q^p \cdot Q^q$ is $w_{p+1} \cdot w_{q+1} + bw_p \cdot w_q$.

Corollary: If m, n, and j are natural numbers and $n > j$, then

$$w_{m+n+1} = w_{m+j+1}w_{n-j+1} + bw_{m+j}w_{n-j}.$$

This corollary can be proved by simply letting $p = m + j$ and $q = n - j$ in Theorem 8.
Theorem 8 may be used to prove another generalization of itself.

Corollary: If $\{u, v, p\}$ is a set of natural numbers,

$$w_{u+p}w_{v+p} - (-b)^p w_u w_v = w_p w_{u+v+p}.$$

Proof: The proof is by induction on p. If $p = 1$, the corollary reduces to Theorem 8. Suppose that k is a natural number such that $w_{u+k}w_{v+k} - (-b)^k w_u w_v = w_k w_{u+v+k}$.

$$\begin{aligned} w_{u+k+1}w_{v+k+1} - (-b)^{k+1}w_u w_v &= (w_{u+v+2k+1} - bw_{u+k}w_{v+k}) - (-b)^{k+1}w_u w_v \\ &= w_{u+v+2k+1} - b(w_{u+k}w_{v+k} - (-b)^k w_u w_v) \\ &= w_{u+v+2k+1} - bw_k w_{u+v+k} \\ &= w_{u+v+k+1}w_{k+1} + bw_k w_{u+v+k} - bw_k w_{u+v+k} \\ &= w_{k+1}w_{u+v+k+1}. \end{aligned}$$

This corollary can be rearranged to give the following identity, analogous to one previously published for the sequence of Fibonacci numbers [3].

$$w_{u+p}w_{v+p} - w_p w_{p+u+v} = (-b)^p w_u w_v.$$

4. DIVISIBILITY PROPERTIES OF $W_{(a,b)}$

If $b = 0$, $W_{(a,b)} = \langle (ax)^{n-1} \rangle$. If $a = 0$, $W_{(a,b)} = \langle 1, 0, b, 0, b^2, \ldots \rangle$. Divisibility properties for each of these types of sequences are easily studied as separate cases. As a result, throughout the remainder of Section 4, a and b will be assumed to be nonzero numbers.

Theorem 9: If p and q are natural numbers, $w_p | w_{pq}$.

This theorem can be proved by induction, using Theorem 8 and writing

$$w_{p(k+1)} = w_{(kp-1)} + p + 1$$

in the induction step. The converse of Theorem 9 relies on Theorem 9 and a sequence of lemmas.

Lemma 1: If p is a natural number and $p > 1$ and U is a polynomial that divides both w_p and w_{p+1}, then $U | w_{p-1}$.

Proof: Suppose S and T are polynomials and $w_p = U \cdot S$ and $w_{p+1} = U \cdot T$:

$$w_{p-1}(x) = (1/b)(U(x))(T(x) - axS(x)).$$

Lemma 2: If U is a polynomial and there exists a natural number p such that $U | w_p$ and $U | w_{p+1}$, then U has degree 0.

Proof: If $p = 1$, $U | w_1$ and the conclusion follows from the fact that $w_1 = 1$. If $p > 1$, Lemma 1 may be applied repeatedly to show that $U | w_1$.

Lemma 3: If $\{n, p, q, r\}$ is a set of natural numbers and $p > 1$ and $q = np + r$ and $w_p | w_q$, then $w_p | w_r$.

Proof: Since $p > 1$, $np - 1 > 0$, $q = (np - 1) + r + 1$, and so by Theorem 8,

$$w_q = w_{np} \cdot w_{r+1} + bw_{np-1} \cdot w_r.$$

By hypothesis, $w_p | w_q$, and by Theorem 9, $w_p | w_{np}$ and hence $w_p | w_{np} w_{r+1}$. Thus, $w_p | bw_{np-1} w_r$. The greatest common divisor of w_{np} and w_{np-1} is a constant (Lemma 2), and so the greatest common divisor of w_p and w_{np-1} is a constant. Therefore, $w_p | w_r$.

Theorem 10: If p and q are natural numbers and $w_p | w_q$, then $p | q$.

Proof: If $p = 1$, the conclusion is obvious. Suppose $p > 1$. $q > p$, so there exists a pair of nonnegative integers, n and r, such that $q = np + r$ and $0 \leq r < p$. $r = 0$ since, if $r > 0$, Lemma 3 establishes that $w_p | w_r$, which is a contradiction, since $r < p$.

REFERENCES

1. Marjorie Bicknell. "A Primer for the Fibonacci Numbers. Part VII: An Introduction to Fibonacci Polynomials and Their Divisibility Properties." *The Fibonacci Quarterly* 8, No. 4 (1970):407-420.
2. M. N. S. Swamy. Problem B-74. *The Fibonacci Quarterly* 3, No. 3 (1965):236.
3. Problem E 1396. *American Math. Monthly* 67 (1960):81-82, 694.

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A DIVISIBILITY PROPERTY OF BINOMIAL COEFFICIENTS

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Let p be a prime number. Let the integers $a_{n\ell}$ be defined by the identity

$$\binom{py}{n} = \sum_{\ell} a_{n\ell}\binom{y}{\ell}.$$

The purpose of this note is to prove that the exponent to which p divides $a_{n\ell}$ is at least $\ell - (n - \ell)/(p - 1)$.

Let Y be a set with y elements. Let $Y_1, \ldots, Y_p$ be disjoint sets, each equipped with a fixed bijection to Y. We wish to count the subsets N of $Y_1 \cup \cdots \cup Y_p$ having exactly n elements. For such a setset N, denote by N_i the image of $N \cap Y_i$ in Y.

If j is an m-tuple $(i_1, \ldots, i_m)$ with $1 \le i_1 < i_2 < \cdots < i_m \le p$, write $i \in \operatorname{supp} j$ if $i = i_k$ for some k.

Let $S_j^m = \{x \in \cup N_i \,|\, x \in N_i \text{ if and only if } i \in \operatorname{supp} j\}$. The sets S_j^m are pair-wise disjoint, and $N_i = \cup\{S_j^m \,|\, i \in \operatorname{supp} j\}$. Moreover, it is easily seen that any change in the ordered p-tuple $(N_1, \ldots, N_p)$ of subsets must change some S_j^m. So producing the sets $N_1, \ldots, N_p$ is the same as producing the sets S_j^m.

Let $L = \cup N_i$, and let ℓ be its cardinality. Let $S^m = \bigcup_j S_j^m$; then S^m consists of the points of L that correspond to exactly m points of N. If t_m is the cardinality of S^m, therefore, one has $n = \ell + \sum_{m=2}^{p} (m - 1)t_m$, and $n/p \le \ell \le n$.

We construct as follows. First select a subset L of Y with cardinality ℓ between n/p and n. Then select a subset S^p of L with cardinality t_p at most $(p - 1)^{-1}(n - \ell)$. Then select a subset S^{p-1} of $L - S^p$ with cardinality t_{p-1} at most $(p - 2)^{-1}(n - \ell - (p - 1)t_p)$. Continue in this way until S^3 has been selected as a subset of $L - S^p - \cdots - S^4$ with cardinality t_3 at most $2^{-1}(n - \ell - (p - 1)t_p - \cdots - 3t_4)$. Now select a subset S^2 of $L - S^p - \cdots - S^3$ with cardinality t_2 equal to

$$n - \ell - \sum_{m=3}^{p} (m - 1)t_m.$$

Define $S^1 = L - S^p - \cdots - S^2$ with cardinality t_1. Finally, select a partition of each S^m into $\binom{p}{m}$ subsets S_j^m.

The above procedure yields the following expression for $\binom{py}{n}$:

$$\sum_{\ell}\binom{y}{\ell}\sum_{t_p}\binom{\ell}{t_p}\sum_{t_{p-1}}\binom{\ell - t}{t_{p-1}} \cdots \binom{\ell - t_p - \cdots - t_3}{t_2}\binom{p}{1}^{t_1} \cdots \binom{p}{p-1}^{t_{p-1}},$$

in which the numbers ℓ and t_m are constrained by the equalities and inequalities of the preceding paragraph. In this expression, each term in the coefficient of $\binom{y}{\ell}$ includes a power of p at least $t_1 + \cdots + t_{p-1} = \ell - t_p \ge \ell - (p - 1)^{-1}(n - \ell)$.

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FIBONACCI FEVER

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This is an account of a strange case of infibonacciation suffered recently by the author, the only remedy for which was found to be a dose of HP-35 followed by SR-50 taken at intervals of 1.618 hours. It all began in Egypt, of course, as so many things do, and specifically with the construction of the Great Pyramid of Khufu or Cheops (no mean task—geometrically, as it turns out). Much has been written on the contributions supposedly made to its design by knowledge Egyptian mathematicians may have had of Pi or Phi, generally considered to have been pretty fibal. Imagine my surprise when, under the influence of the contagion afflicting me both phi-sickally and mentally, I looked up the values of the trigonometric functions in the neighborhood of the well-known Great Pyramid angle of approximately 51°50' (and its complement) and found what I have not seen in print anywhere, namely

$$\begin{array}{ll} \sin A = \sqrt{b} & \sin B = b \\ \cos A = b & \cos B = \sqrt{b} \\ \tan A = \sqrt{a} & \tan B = \sqrt{b} \\ \cot A = \sqrt{b} & \cot B = \sqrt{a} \\ \sec A = a & \sec B = \sqrt{a} \\ \csc A = \sqrt{a} & \csc B = a \end{array}$$

where $a = 1.618033989\ldots$ and $b = a - 1$. Interpolation in the tables or use of one of the new pocket calculators quickly yields exact values for the angles:

$$A = 51°49'38''.253 \qquad B = 38°10'21''.747$$

This observation, that the values of the trigonometric functions at which their plotted curves intersect are all, except for the familiar values 0, ±1, $\pm\sqrt{2}$, and $\pm\sqrt{2}/2$, of magnitude a, b, or their square roots, should be sufficient to launch the new science of Fibonometry or Phigonometry, according to taste. Our basic right phiangle is then the one with unit hypotenuse and base b, which has the property that its altitude is the mean proportional between its base and its hypotenuse. This altitude, $\sqrt{b}$, is the approximation to $\pi/4$ that has led to the association of the Great Pyramid with an attempt to represent π.

$$\sqrt{b} = 0.78615\ 13778$$

$$\pi/4 = 0.78539\ 81634$$

This approximation is good to 0.1%. Some other phigonometric approximations that have been noted by pyramidographers qualify as genuine Fibonacci curiosities. They are:

$$A \approx 1/7 \text{ circle} \quad (\text{error } 0.8\%)$$

$$A \approx 9/10 \text{ radian} \quad (\text{error } 0.5\%)$$

$$B \approx 2/3 \text{ radian} \quad (\text{error } 0.06\%)$$

Further numerical approximations that have been noted are

$$6a^2/5 = 3.1416408 \approx \pi \quad (\text{error } 0.0015\%)$$

and

$$\arctan \sqrt{2}/2 \approx b \text{ radian}$$

$$0.61547971 \approx 0.61803399 \quad (\text{error } 0.4\%)$$

this latter deliriously close, but to what, is uncertain.

It is, and very likely will remain, an open question as to which of these approximations the Egyptians may have had in mind, if any, but it is nevertheless extremely curious that most of them fall within the probable limits imposed on the precision of construction of the Pyramid by the technology and surveying techniques available at the time.

The numerological ramifications of this question are quite prodigious, and demand the introduction at this point of some measurements of the actual Pyramid. Values published by Petrie and by Bruchet have been chosen here as representative of determinations made in both English and metric units, respectively, and the rounded values in cubits, given in the last column are based on generally accepted conversion factors.

	Feet	Meters	Cubits
Half-base	377.86	115.24	220
Height	481.33	146.60	280
Apothem of face	611.93	186.47	356

One cubit = 7 palms = 28 fingers, and the inclination of a pyramid's side is expressed (e.g., in the Rhind papyrus) as so many palms horizontal recession of the face for one cubit vertical rise. This quantity, the *seked*, is thus 7 cot A, where A is the inclination angle measured at the foot of the apothem. The angle A has actually been measured from one or two intact casing stones from the buried portion of the Pyramid and compares well with the estimates made from overall measurements. From the values given in the last column of the above table, we determine the *seked* of Cheops to be precisely 5.5 (or 5 palms, 2 fingers). This is not only a nice simple number but in relation to the cubit of 7 palms suggests, as do the ratios of the sides of the Pyramid triangle, the value of 22/28 as an approximation to either $\pi/4$ or $\sqrt{b}$, or both, as you wish. The error in either case is less than 0.06%.

$$A = \text{arc cot } 22/28 = .90482\ 70894 \text{ radian} = 51°50'34''$$

$$\sin A = 0.78631\ 83388$$

Measured angle of casing-stones = 51°51'

It has also been suggested that the Pyramid was designed to have a rise of 9 units in 10 taken along the edge of a face rather than at its center. This is easily checked for the triangle that forms a vertical section through a diagonal of the (almost perfectly) square base. Calling this corner angle of inclination C, we find

$$\tan C = 280/311.127 = 0.89995\ 40851$$

which verifies this hypothesis as well, to within 0.01%! The angle C turns out to be 41°59'09" or only 0.03% from the neat angle of 42° (which may recommend itself to hexagesimalists because it is 7/60 of a circle).

In case these excursions into the real world prove too enervating, let us indulge in a lettle ideal-pyramid designing, starting from our basic right phiangle whose sides are in the ratio $1{:}\sqrt{b} = 1.27202$ very nearly. Rounding this to 1.272, we might let our base be 1000 units and our height 1272, giving us a face apothem for the pyramid, or hypotenuse of our triangle, of 1618, a familiar number indeed. These numbers are all divisible by 2, so we get 500, 636, 809 (the last is prime). If we choose to be a little sloppy (will the Greeks detect it?), we can settle for 50, 64, 81, which has the beauty that the full base is then 100 units, and our numbers are simply 8^2, 9^2, 10^2. However, we would then have to settle for a pyramid angle of almost exactly 52° (only good for 13-fetishists or card players) with its rather poor 3.15 for π and 0.621 for b. There are those who will claim this design is justified for its 81/64 approximation to $\sqrt{a}$, which squares to 1.6018 for a itself. But then, some prefer bent pyramids to straight.

Now there is one place where all may find good values for the extrema in our triangular section, the base and hypotenuse, because their ratio if 1.618, that of the Fibonacci sequence. The sequence itself yields the pairs we need, and they get progressively better as we go to higher members, only requiring that we select for near-integer values of the mean proportional. A little play with early members is rewarding: we immediately find the ancient 3, 5 pair with its perfectly Pythagorean companion 4. Pyramid angle 53°08' and very primitive 3.2 for π. We might dream of 8, 5 with its convenient 1.6 ratio, but we left 3.2 behind in the last triangle, so we can't work a deal for $\pi = 2a$. It is at this point that it just dawns on us for no apparent reason that we can get a fair estimate of the middle value (the height of our pyramid) from the expression

$$\frac{\frac{1}{2}F_{i+2} + 2F_{i-1}}{2}$$

which shows us that F_{i+2} must be divisible by 4 to give an integer middle term. The Pythagoreans insist we write this as

$$\frac{5F_{i+1} - 3F_i}{4}$$

for obvious reasons, and as it is the same thing, we don't object. Now we cannot only construct right phiangles, but Phiophantine ones as well. (Except for the 3, 4, 5 case, we must not call them Diophantine, as the closer approximation to Phi precludes Di—still they will serve the useful purpose of providing a suitable tomb should a Pharaoh Die.) Since every F_n for n = 6, 12, 18, 24, ... is divisible by 4 we have an unlimited supply of models.

And what do we find almost as soon as we begin the painstaking task of examining this infinity of models? The first one after the 3, 4, 5 model is the actual Great Pyramid of Khufu! 55, 89 with the interpolated 70, upon multiplication by 4, yield the values 220, 280, 356, the very dimensions in Egyptian royal cubits that most people have found acceptable. But already the astute observer will have found a hint of another pair that looks interesting (do we all have our Fibonacci sequences out?), namely 377, 610, simply because these numbers are so close to the Pyramid measurements in English feet! True, this pair doesn't yield an integer value for the middle term (here 479.75) but the Fibonacci expression does give a value about halfway between the Pythagorean 479.55 and the round 480. Good British foot-rules must have been scarce in ancient Egypt, and the architect had the bad luck to choose one that was too long by 0.27%, though whether this was an effect of the higher mean temperature at Giza or due to more esoteric considerations, such as the ratio of the sidereal to the solar day (1.00274), the inhabitants of sunny Egypt preferring the longer solar foot and thus assigning fewer units to a given length than Britons, whose work beneath the moon and in the cooler northern dawns at Stonehenge and Avebury might naturally have led them to employ the sidereal foot, remains to be determined by future investigators. In any case, the reduction required is so slight that it can scarcely conceal the fact that Khufu was built on the English system.

But wait! Had you already noticed that doubling our measurements in meters, or expressing them in semimeters (perhaps in deference to Semiramis, always so phinegy with details): 230.48, 293.20, 372.94 begins to look alarmingly as if the French too had landed on the banks of the Nile and had the situation well in hand—compare 233, 377 and the Fibonterpolated value 296.5? The expansion of the French rule appears to have been greater, amounting to 1.1%, though it might be argued it was no less just.

It may be that further study will show vaguer correspondences with rough-hewn Norse wooden rulers or sly yardsticks of China, but at least we have pointed the way. On the vexed question of what the Egyptians hoped to achieve by their design, my own opinion is that their architects made a wise decision to split the difference between a very accurate representation of π and a very exact approximation to the Golden Ratio by choosing the very neat 55, 70, 89 triad with its traditional 22/7 compromise, showing that after all they knew perfectly well you can't square the circle but you can come as close as a scarab-beetle's left front leg to doing it, and in the process keep thousands of generations of people, amateurs and savants alike, guessing and struggling with the data to resolve the issue. No edifice of lesser mass and durability than Cheops could have been relied upon to do the job of preserving the sharp edge of the blade of discrimination between subtle geometric hypotheses for thousands of years.

In a lighter vein, we noticed one day as the fever was wearing off and we were relaxing to the sound of the oud, that much of the world's music can be represented, with regard to pitches of degrees of the scale, by simple powers of ratios between 1 and 2 (the unison and octave), with the perfect fifth (3/2) doing yeoman's work ever since the days of Pythagoras, who probably learned about it in Egypt, according to legend. Musics of China, India, Persia, Arabia, Byzantium, and Greece can be represented by using sufficiently high powers of 1.5 alone (try it some time, merely taking care to reduce values that exceed 2 by the appropriate division by a power of 2 so that the set of tones remains within the octave—negative powers should be included in a symmetrical manner). Those who appreciate the value of common cents in musicology will want to see results expressed in this medium of exchange currently being favored at 1200 to the octave according to the formula

$$\text{Cents} = (1200/\log_{10}2.0)\log_{10}R$$

where R is the frequency ratio of two pitches of interest, say any note and the fundamental or tonic. If R is some power of a constant ratio between 1 and 2, say

$$R = r^j/2^k \qquad j = 0, \pm 1, \pm 2, \ldots, \pm n$$

and k is chosen such that $1 < R < 2$,

$$\text{Cents} = 3986.314(j \log_{10}r - k \log_{10}2).$$

The point for Fibonaccians is, of course, what happens if we choose $r = 1.618...$? The result is curious. After reordering successive powers into a monotonic sequence, we have, in cents:

30.2	69.1	<u>99.3</u>	129.4	168.4	<u>198.5</u>	
237.5	276.6	<u>297.8</u>	336.7	366.9	<u>397.1</u>	
436.0	466.2	<u>496.4</u>	535.3	565.5	<u>595.6</u>	(604.4)

and so on for the upper half of the octave. These values are within a few cents of forming a

36-tone tempered scale, so that every third member of the sequence is very nearly one of the twelve tones of our present musical scale. For perfect correspondence, such that every third tone is 100, 200, 300, etc. cents, the value of r should be 1.618261.

The usual method of constructing tempered scales is to use a ratio r which is the nth root of 2 to obtain a scale of n equidistant tones. $\sqrt[36]{2} = 1.019440644$. The ratio 1.618261 is a power of this, in fact the 25th power. It is interesting to note that 1.618... itself is not a frequency ratio that corresponds to a tone of our 12-tone scale, for it gives 833 cents, far enough from 800 to sound sharp and give discords. Other attempts to relate the Golden Ratio to musical pitch have overlooked this hard musical fact. The present discussion may serve to reinstate the Divine Proportion into the Divine Harmony.

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EXPONENTIAL GENERATION OF BASIC LINEAR IDENTITIES*

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Generalizing results of Fibonacci and Lucas numbers has been an occupation of a large number of mathematicians down through the years. Frequently, one approach taken is to first prove a result involving the Fibonacci sequence $\{F_n\}_{n=0}^{\infty}$ and the Lucas sequence $\{L_n\}_{n=0}^{\infty}$ and then extend it to a result or results of special cases of the sequences $\{F_{nk+r}\}_{n=0}^{\infty}$ and $\{L_{nk+r}\}_{n=0}^{\infty}$, where k and r are fixed integers. In this paper attention is focused on deriving identities related to these latter sequences. Such results, called linear because of the subscripts, are surveyed in [1]. The exponential generating functions for these latter sequences are now shown to be most productive in deriving basic linear identities that the author believes to be new. In addition, alternate derivations of several known results will be given to show the great usefulness of these generating functions in attacking a variety of Fibonacci and Lucas problems.

Recalling the Maclaurin series expansion for e^x:

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

and hence

$$e^{Ax} = 1 + \frac{Ax}{1!} + \frac{(Ax)^2}{2!} + \cdots = \sum_{n=0}^{\infty} A^n\frac{x^n}{n!}, \tag{1}$$

for any constant A, we note that the exponential generating functions for the first mentioned sequences are

$$\sum_{n=0}^{\infty} F_n \frac{x^n}{n!} = \frac{e^{\alpha x} - e^{\beta x}}{\alpha - \beta}$$

and

$$\sum_{n=0}^{\infty} L_n \frac{x^n}{n!} = e^{\alpha x} + e^{\beta x}$$

where $\alpha = \dfrac{1+\sqrt{5}}{2}$ and $\beta = \dfrac{1-\sqrt{5}}{2}$.

The exponential generating functions of the sequences of interest in this paper are found by use of (1) to be

$$\sum_{n=0}^{\infty} F_{nk+r}\frac{x^n}{n!} = \frac{\alpha^r e^{\alpha^k x} - \beta^r e^{\beta^k x}}{\alpha - \beta} \tag{2}$$

$$\sum_{n=0}^{\infty} L_{nk+r}\frac{x^n}{n!} = \alpha^r e^{\alpha^k x} + \beta^r e^{\beta^k x} \tag{3}$$

$$\sum_{n=0}^{\infty} (-1)^n F_{nk+r}\frac{x^n}{n!} = \frac{\alpha^r e^{-\alpha^k x} - \beta^r e^{-\beta^k x}}{\alpha - \beta} \tag{4}$$

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(5) $$\sum_{n=0}^{\infty}(-1)^n L_{nk+r}\frac{x^n}{n!} = \alpha^r e^{-\alpha^k x} + \beta^r e^{-\beta^k x}$$

(6) $$\sum_{n=0}^{\infty}\alpha^n F_{nk+r}\frac{x^n}{n!} = \frac{\alpha^r e^{\alpha^{k+1}x} - \beta^r e^{\alpha\beta^k x}}{\alpha - \beta}$$

(7) $$\sum_{n=0}^{\infty}\beta^n F_{nk+r}\frac{x^n}{n!} = \frac{\alpha^r e^{\alpha^k \beta x} - \beta^r e^{\beta^{k+1}x}}{\alpha - \beta}$$

(8) $$\sum_{n=0}^{\infty}\alpha^n L_{nk+r}\frac{x^n}{n!} = \alpha^r e^{\alpha^{k+1}x} + \beta^r e^{\alpha\beta^k x}$$

(9) $$\sum_{n=0}^{\infty}\beta^n L_{nk+r}\frac{x^n}{n!} = \alpha^r e^{\alpha^k \beta x} + \beta^r e^{\beta^{k+1}x}.$$

Exponential generating functions are given a considerable workout in [2] in deriving many Fibonacci and Lucas identities.

By convoluting any pair of the above series and then equating like coefficients, a linear identity is found. To begin we convolute series (2) with itself.

$$\sum_{n=0}^{\infty}F_{nk+r}\frac{x^n}{n!}\cdot\sum_{n=0}^{\infty}F_{nk+r}\frac{x^n}{n!} = \sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{n}{j}F_{jk+r}F_{(n-j)k+r}\frac{x^n}{n!}$$

and

$$\left(\frac{\alpha^r e^{\alpha^k x} - \beta^r e^{\beta^k x}}{\alpha - \beta}\right)^2 = \frac{1}{5}\left[\left(\alpha^{2r}e^{2\alpha^k x} + \beta^{2r}e^{2\beta^k x}\right) - 2(\alpha\beta)^r e^{(\alpha^k+\beta^k)x}\right]$$

$$= \sum_{n=0}^{\infty}\frac{1}{5}\left[2^n L_{nk+2r} + 2(-1)^{r+1}L_k^n\right]\frac{x^n}{n!}.$$

Hence

(10) $$\sum_{j=0}^{n}\binom{n}{j}F_{jk+r}F_{(n-j)k+r} = \frac{1}{5}\left[2^n L_{nk+2r} + 2(-1)^{r+1}L_k^n\right].$$

The convolutions of series (3) with itself and then series (2) with (3) yield the following results:

(11) $$\sum_{j=0}^{n}\binom{n}{j}L_{jk+r}L_{(n-j)k+r} = 2^n L_{nk+2r} + 2(-1)^r L_k^n$$

(12) $$\sum_{j=0}^{n}\binom{n}{j}F_{jk+r}L_{(n-j)k+r} = 2^n F_{nk+2r}.$$

Several additional summations which reduce to simple expressions are found following the same procedure. Convolutions of (4) with (2), (4) with (3), (6) with (7), and (8) with (9), respectively, yield a representative class of the identities easily derived from the given generating functions.

$$\sum_{n=0}^{\infty}(-1)^n F_{nk+r}\frac{x^n}{n!}\cdot\sum_{n=0}^{\infty}F_{nk+r}\frac{x^n}{n!} = \sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{n}{j}(-1)^j F_{jk+r}F_{(n-j)k+r}\frac{x^n}{n!}$$

and

$$\left(\frac{\alpha^r e^{-\alpha^k x} - \beta^r e^{-\beta^k x}}{\alpha - \beta}\right)\left(\frac{\alpha^r e^{\alpha^k x} - \beta^r e^{\beta^k x}}{\alpha - \beta}\right) = \frac{1}{5}\left[\left(\alpha^{2r} + \beta^{2r}\right) - (\alpha\beta)^r\left(e^{(-\alpha^k+\beta^k)x} + e^{(\alpha^k-\beta^k)x}\right)\right]$$

$$= \frac{1}{5}\left[L_{2r} + (-1)^{r+1}\left(e^{-\sqrt{5}F_k x} + e^{\sqrt{5}F_k x}\right)\right]$$

$$= \frac{1}{5}\left\{L_{2r} + (-1)^{r+1}\sum_{n=0}^{\infty}5^{n/2}F_k^n[(-1)^n + 1]\frac{x^n}{n!}\right\}.$$

By equating like coefficients, we have

(13) $$\sum_{j=0}^{2n}\binom{2n}{j}(-1)^j F_{jk+r}F_{(2n-j)k+r} = 2(-1)^{r+1}5^{n-1}F_k^{2n},\ \text{for } n > 0,$$

and

$$(14)\qquad \sum_{j=0}^{2n+1}\binom{2n+1}{j}(-1)^j F_{jk+r}F_{(2n-j+1)k+r} = 0, \text{ for } n \geq 0.$$

Now considering series (4) with (3), the identities

$$(15)\qquad \sum_{j=0}^{2n}\binom{2n}{j}(-1)^j F_{jk+r}L_{(2n-j)k+r} = 0, \text{ for } n > 0,$$

and

$$(16)\qquad \sum_{j=0}^{2n+1}\binom{2n+1}{j}(-1)^j F_{jk+r}L_{(2n-j+1)k+r} = 2(-1)^{r+1}5^n F_k^{2n+1}, \text{ for } n \geq 0,$$

are deduced.

Similarly, we find

$$(17)\qquad \sum_{j=0}^{n}\binom{n}{j}\alpha^j\beta^{n-j}F_{jk+r}F_{(n-j)k+r} = \frac{1}{\sqrt{5}}\left\{F_{nk+2r} + \frac{(-1)^r}{\sqrt{5}}\left[L_{k+1}^n + (-L_{k-1})^n\right]\right\}$$

and

$$(18)\qquad \sum_{j=0}^{n}\binom{n}{j}\alpha^j\beta^{n-j}L_{jk+r}L_{(n-j)k+r} = L_{nk+2r} + (-1)^r\left[L_{k+1}^n + (-L_{k-1})^n\right].$$

A direction of generalization of the given results as well as derivation of new results is to find additional generating functions. Then aided by several lemmas that simplify the exponents of e resulting from convolutions, many linear identities are found.

To generalize the given generating functions we begin with series (2). Replacing α^k by $\alpha^k F_m$ and β^k by $\beta^k F_m$ where m is a fixed nonzero integer, leads to

$$\frac{\alpha^r e^{\alpha^k F_m x} - \beta^r e^{\beta^k F_m x}}{\alpha - \beta} = \frac{\alpha^r \sum_{n=0}^{\infty}(\alpha^k F_m)^n \frac{x^n}{n!} - \beta^r \sum_{n=0}^{\infty}(\beta^k F_m)^n \frac{x^n}{n!}}{\alpha - \beta} = \sum_{n=0}^{\infty} F_m^n \frac{(\alpha^{nk+r} - \beta^{nk+r})}{\alpha - \beta}\frac{x^n}{n!},$$

and hence

$$(19)\qquad \sum_{n=0}^{\infty} F_m^n F_{nk+r}\frac{x^n}{n!} = \frac{\alpha^r e^{\alpha^k F_m x} - \beta^r e^{\beta^k F_m x}}{\alpha - \beta}.$$

Each additional generating function given is similarly derived. (Note: Letting $m = 1$, we have $F_m^n = 1$ and then are back to the original generating function.) Only three additional generalized generating functions are listed.

$$(20)\qquad \sum_{n=0}^{\infty} L_m^n F_{nk+r}\frac{x^n}{n!} = \frac{\alpha^r e^{\alpha^k L_m x} - \beta^r e^{\beta^k L_m x}}{\alpha - \beta}$$

$$(21)\qquad \sum_{n=0}^{\infty} L_m^n L_{nk+r}\frac{x^n}{n!} = \alpha^r e^{\alpha^k L_m x} + \beta^r e^{\beta^k L_m x}$$

$$(22)\qquad \sum_{n=0}^{\infty} F_m^n L_{nk+r}\frac{x^n}{n!} = \alpha^r e^{\alpha^k F_m x} + \beta^r e^{\beta^k F_m x}.$$

The Binet definition of the numbers involved proves several useful lemmas.

Lemma 1: $\alpha^k = \alpha F_k + F_{k-1}$, $\beta^k = \beta F_k + F_{k-1}$, $\alpha^k = \frac{1}{\sqrt{5}}(\alpha L_k + L_{k-1})$, and $\beta^k = -\frac{1}{\sqrt{5}}(\beta L_k + L_{k-1})$, for any integer k.

Lemma 2: $\alpha^k F_m = F_{m+k} - \beta^m F_k$, $\beta^k F_m = F_{m+k} - \alpha^m F_k$, $\alpha^k L_m = L_{m+k} + \beta^m\sqrt{5}F_k$, and $\beta^k L_m = L_{m+k} - \alpha^m\sqrt{5}F_k$, for any integers k and m.

Substitution of these results into the given generating functions yields identities of interest in themselves. For example, consider series (2) and (19). From Lemma 1, it follows that

$$\sum_{n=0}^{\infty} F_{nk+r}\frac{x^n}{n!} = \frac{\alpha^r e^{(\alpha F_k + F_{k-1})x} - \beta^r e^{(\beta F_k + F_{k-1})x}}{\alpha - \beta} = \frac{\alpha^r \sum_{n=0}^{\infty}(\alpha F_k + F_{k-1})^n \frac{x^n}{n!} - \beta^r \sum_{n=0}^{\infty}(\beta F_k + F_{k-1})^n \frac{x^n}{n!}}{\alpha - \beta}$$

$$= \sum_{n=0}^{\infty} \sum_{j=0}^{n} \binom{n}{j} F_k^j F_{j+r} F_{k-1}^{n-j} \frac{x^n}{n!},$$

which yields

$$F_{nk+r} = \sum_{j=0}^{n} \binom{n}{j} F_k^j F_{j+r} F_{k-1}^{n-j}. \tag{23}$$

This identity has been derived by distinct approaches in [3] and [4].

$$\sum_{n=0}^{\infty} F_m^n F_{nk+r}\frac{x^n}{n!} = \frac{\alpha^r e^{\alpha^k F_m x} - \beta^r e^{\beta^k F_m x}}{\alpha - \beta} = \frac{\alpha^r e^{(F_{m+k} - \beta^m F_k)x} - \beta^r e^{(F_{m+k} - \alpha^m F_k)x}}{\alpha - \beta}$$

$$= \frac{\alpha^r \sum_{n=0}^{\infty}(F_{m+k} - \beta^m F_k)^n \frac{x^n}{n!} - \beta^r \sum_{n=0}^{\infty}(F_{m+k} - \alpha^m F_k)^n \frac{x^n}{n!}}{\alpha - \beta}$$

$$= \sum_{n=0}^{\infty} \sum_{j=0}^{n} \binom{n}{j} (-1)^{n-j+r+1} F_{m+k}^j F_k^{n-j} F_{m(n-j)-r}$$

and so

$$F_m^n F_{nk+r} = \sum_{j=0}^{\infty} \binom{n}{j} (-1)^{n-j+r+1} F_{m+k}^j F_k^{n-j} F_{m(n-j)-r}. \tag{24}$$

The corresponding Lucas number results are

$$L_{2nk+r} = \frac{1}{5^n} \sum_{j=0}^{n} \binom{2n}{j} L_k^j L_{k-1}^{2n-j} L_{j+r}, \tag{25}$$

$$L_{(2n+1)k+r} = \frac{1}{5^n} \sum_{j=0}^{2n+1} \binom{2n+1}{j} L_k^j L_{k-1}^{2n-j+1} F_{j+r}, \text{ and} \tag{26}$$

$$L_m^n L_{nk+r} = \sum_{j=0}^{n} \binom{n}{j} (-1)^r 5^{(n-j)/2} L_{m+k}^j F_j^{n-j} \left[\beta^{m(n-j)-r} + (-1)^{n-j} \alpha^{m(n-j)-r}\right]. \tag{27}$$

An alternate approach to identities of similar form is given in [2].

Several basic identities given early in the paper are now generalized by use of generating functions (19) to (22). It is of much interest to compare the original results with their generalized form. We now consider the convolution of series (19) with (20).

$$\sum_{n=0}^{\infty} \sum_{j=0}^{n} \binom{n}{j} F_m^j L_m^{n-j} F_{jk+r} F_{(n-j)k+r} \frac{x^n}{n!} = \left(\frac{\alpha^r e^{\alpha^k F_m x} - \beta^r e^{\beta^k F_m x}}{\alpha - \beta}\right)\left(\frac{\alpha^r e^{\alpha^k L_m x} - \beta^r e^{\beta^k L_m x}}{\alpha - \beta}\right)$$

$$= \sum_{n=0}^{\infty} \frac{1}{5}\left[2^n F_{m+1}^n L_{nk+2r} + \sum_{j=0}^{n} \binom{n}{j} (-1)^{jk+r+1} F_m^j L_m^{n-j} L_{(n-2j)k}\right] \frac{x^n}{n!}.$$

Hence,

$$\sum_{j=0}^{n} \binom{n}{j} F_m^j L_m^{n-j} F_{jk+r} F_{(n-j)k+r} = \frac{1}{5}\left[2^n F_{m+1}^n L_{nk+r} + \sum_{j=0}^{n} \binom{n}{j} (-1)^{jk+r+1} F_m^j L_m^{n-j} L_{(n-2j)k}\right] \tag{28}$$

and so

$$\sum_{j=0}^{n} \binom{n}{j} F_m^j L_m^{n-j} \left[F_{jk+r} F_{(n-j)k+r} + \frac{(-1)^{jk+r}}{5} L_{(n-2j)k}\right] = \frac{2^n}{5} F_{m+1}^n L_{nk+2r}. \tag{29}$$

Results of similar form may be derived by utilization of the other generating functions. For example, from series (19) and (21), we obtain

$$\sum_{j=0}^{n} \binom{n}{j} F_m^j L_m^{n-j} F_{jk+r} L_{(n-j)k+r} = 2^n F_{m+1}^n F_{nk+2r} + \sum_{j=0}^{n} \binom{n}{j} (-1)^{jk+r+1} F_m^j L_m^{n-j} F_{(n-2j)k} \tag{30}$$

and

$$(31)\qquad \sum_{j=0}^{n}\binom{n}{j}F_m^j L_m^{n-j}\left[F_{jk+r}L_{(n-j)k+r} + (-1)^{jk+r}F_{(n-2j)k}\right] = 2^n F_{m+1}^n F_{nk+2r}.$$

REFERENCES

1. R. T. Hansen. "General Identities for Linear Fibonacci and Lucas Summations." *The Fibonacci Quarterly* 16 (1978):121-128.
2. H. T. Leonard, Jr. Fibonacci and Lucas Identities and Generating Functions." Master's thesis, San Jose State College, 1969.
3. J. H. Halton. "On a General Fibonacci Identity." *The Fibonacci Quarterly* 3 (1965): 31-43.
4. V. E. Hoggatt, Jr. "A New Angle on Pascal's Triangle." *The Fibonacci Quarterly* 6 (1968):221-234.
5. David Zeitlin. "General Identities for Recurrent Sequences of Order Two." *The Fibonacci Quarterly* 9 (1971):357-388.
6. V. E. Hoggatt, Jr., J. W. Phillips, and H. T. Leonard, Jr. "Twenty-four Master Identities." *The Fibonacci Quarterly* 9 (1971):1-17.
7. L. Carlitz and H. H. Ferns. "Some Fibonacci and Lucas Identities." *The Fibonacci Quarterly* 8 (1970):61-73.
8. V. E. Hoggatt, Jr. "Some Special Fibonacci and Lucas Generating Functions." *The Fibonacci Quarterly* 9 (1971):121-133.
9. Marjorie Bicknell and C. A. Church, Jr. "Exponential Generating Functions for Fibonacci Identities." *The Fibonacci Quarterly* 11 (1973):275-281.

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IDENTITIES OF A GENERALIZED FIBONACCI SEQUENCE

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The purpose of this note is to give identities of third power and above of the generalized Fibonacci sequence with nth term H_n satisfying the recurrence relation $H_n = pF_n + qF_{n-1}$ and $H_0 = q$ where F_n denotes the nth classical Fibonacci number.

We refer to the following identities of A. F. Horadam [1]:

$$(1)\qquad H_nH_{n+2} - H_{n+1}^2 = (-1)^n e$$

$$(2)\qquad H_{m+h}H_{m+k} - H_mH_{m+h+k} = (-1)^m eF_hF_k$$

$$(3)\qquad H_m = F_{k+1}H_{m-k} + F_kH_{m-k-1}$$

and also use

$$(4)\qquad H_{k+1}H_{k+2}H_{k+4}H_{k+3} = H_{k+5}^4 - e^2$$

where $e = p^2 - pq - q^2$.

<u>*Identity 1*</u>: $H_n^4 - 2H_{n+1}^3H_n - H_{n+1}^2H_n^2 + 2H_n^3H_{n+1} + H_{n+1}^4 = e^2.$

<u>*Identity 2*</u>: $H_{n+4}^4 - 4H_{n+3}^4 - 19H_{n+2}^4 - 4H_{n+1}^4 + H_n^4 = -6e^2.$

<u>*Identity 3*</u>: $H_{n+5}^4 = 5H_{n+4}^4 + 15H_{n+3}^4 - 15H_{n+2}^4 - 5H_{n+1}^4 + H_n^4.$

<u>*Identity 4*</u>: $25\sum_{k=0}^{n} H_k^4 = H_{n+3}^4 - 3H_{n+2}^4 - 22H_{n+1}^4 - H_n^4 + 6e^2(n-1) + A$

where $A = 15p^4 - 32p^3q - 12p^2q^2 + 16pq^3 + 34q^4$.

<u>*Identity 5*</u>: A. $18\sum_{k=1}^{n}(-1)^k H_k^4 = (-1)^n(H_{n+4}^4 - 6H_{n+3}^4 - 9H_{n+2}^4 + 24H_{n+1}^4 - H_n^4);$

B. $9\sum_{k=1}^{n}(-1)^k H_k^4 = (-1)^n(-H_{n+3}^4 + 5H_{n+2}^4 + 14H_{n+1}^4 - H_n^4 - 3e^2).$

<u>*Identity 6*</u>: $25\sum H_{k+1}H_{k+2}H_{k+4}H_{k+5} = 26H_{n+3}^4 + 22H_{n+2}^4 + 3H_{n+1}^4 - H_n^4 - C,$

where $C = 19e^2n + (66p^4 + 70p^3q + 131p^2q^2 + 146pq^3 + 47q^4)$.

Identity 7: $9\sum_{0}^{2n-1}(-1)^k H_{k+1}H_{k+2}H_{k+4}H_{k+5} = H^4_{2n+5} - 5H^4_{2n+4} - 14H^4_{2n+3} + H^4_{2n+2} + 3e^2 + D,$

where $D = q(4p^3 + 6p^2q + 4pq^2 + q^3)$.

The proof of Identities 1-7 follow along the same lines as in [1], hence the details are omitted here.

Some more identities that are easily verifiable by induction follow:

(a) $2\sum_{0}^{n}(-1)^r H_{m+3r} = (-1)^n H_{m+3n+1} + H_{m-2}$ $\quad m = 2, 3, \ldots;$

(b) $3\sum_{0}^{n}(-1)^r H_{m+4r} = (-1)^n H_{m+4n+2} + H_{m-2}$ $\quad m = 2, 3, \ldots;$

(c) $11\sum_{0}^{n}(-1)^r H_{m+5r} = (-1)^n(5H_{m+5n+1} + 2H_{m+5n}) + 4H_m - 5H_{m-1}$ $\quad m = 1, 2, \ldots;$

(d) $4\sum_{0}^{n}H_k H_{2k+1} + 2H_0^2 = H_{2n+3}H_n + H_{2n}H_{n+3};$

(e) $3\sum_{0}^{n}(-1)^r H^2_{m+2r} = (-1)^n H_{m+2n}H_{m+2n+2} + H_m H_{m-2}$ $\quad m = 2, 3, \ldots;$

(f) $7\sum_{0}^{n}(-1)^r H^2_{m+4r} = (-1)^n H_{m+4n}H_{m+4n+4} + H_m H_{m-4}$ $\quad m = 4, 5, \ldots;$

(g) $2\sum_{1}^{n}H_{k+2}H^2_{k+1} = H_{n+3}H_{n+2}H_{n+1} - H_0H_1H_2;$

(h) $2\sum_{1}^{n}(-1)^r H_r H^2_{r+1} = (-1)^n H_n H_{n+1}H_{n+2} - H_0H_1H_2;$

(i) $2\sum_{1}^{n}(-1)^r H^3_{r+1} = (-1)\ (H^2_{n+1}H_{n+4} - H_nH_{n+2}H_{n+3}) - E,$

where $E = p^3 - 3pq^2 - q^3$.

REFERENCES

1. A. F. Horadam. "A Generalized Fibonacci Sequence." *American Math. Monthly* 68, No. 5, pp. 455-459.
2. David Zeitlin. "On Identities for Fibonacci Numbers." *American Math. Monthly* 70, No. 9, pp. 987-991.

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DIVISIBILITY PROPERTIES OF A GENERALIZED FIBONACCI SEQUENCE

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This note gives some divisitility properties of the generalized Fibonacci numbers viz $H_0 = q$, $H_1 = p$, $H_{n+1} = bH_n + cH_{n-1}$ $(n \geq 1)$, denoted henceforth by (b, c, p, q) GF sequence. The results have similarity to those of Dov Jarden [1].

For the Horadam generalized Fibonacci sequence: $H_0 = q$, $H_1 = p$, $H_{n+1} = H_n + H_{n-1}$ $(n \geq 1)$, we have

Theorem 1: $H_{n+k} + (-1)\ H_{n-k}$ is divisible by H for all $n \geq k$.

Proof: The proof easily follows from the identity

(1) $$H_{n+k} + (-1)^k H_{n-k} = L_k H_n.$$

Corollary a: $H^2_{n+k} + (-1)^{2k+1}H^2_{n-k}$ is divisible by H_n; and

Corollary b: $H^3_{n+k} + (-1)^{3k+2}H^3_{n-k}$ is divisible by H_n.

Divisibility properties of (b, c, p, q) GF sequence.

Theorem 2: If $(m,n) = 1$ and $q = 0$, H_mH_n/H_{mn}.

Proof: $H_n = (gr^n - hs^n)/(r - s)$ and $H_{mn} = (gr^{mn} - hs^{mn})/(r - s)$, where r and s are the roots of $x^2 - bx - c = 0$ and $g = p - sq$ and $h = p - rq$.

It is easily seen that H_m or H_n divides H_{mn} if $g = h$. Since $r = s$ leads to the degenerate case, we must have $q = 0$. Also, it is necessary that $(m,n) = 1$.

Theorem 3: If $p^2 - bpq - cq^2 = 0$, then $H_m H_n / H_{mn}$.

Proof: By the identity

$$H_n^2 - H_{n+1}H_{n-1} = (-c)^{n-1}e, \tag{2}$$

where $e = p^2 - bpq - cq^2$, the desired result follows.

Theorem 4: For $p = cq(1 - b)/(b^2 + c + 1 - b)$, if $c^2 = (-1 - b)(1 + 2c)$, then $H_m H_n / H_{mn}$.

It is known from [2] that $H_n = pU_n + cqU_{n-1}$, where the nth member of the U sequence is defined by $U_0 = 0$, $U_1 = 1$, and $U_{n+2} = bU_{n+1} + cU_n$ $(n > 0)$.

On suitably combining this relation with

$$2(pU_n + cqU_{n-1}) = (pU_{n+1} + cqU_n) + (pU_{n-1} + cqU_{n-2}), \tag{3}$$

it is easy to see that (b, c, p, q) GF sequence results in an A.P. Therefore, if $H_m H_n$ were to divide H_{mn}, we would get

$$c^2 = (1 - b)(1 + 2c).$$

Further equating the initial term of the A.P. with the common difference, we get either $c = 0$ or $p(b^2 + c + 1 - b) = cq(1 - n)$.

The case $c = 0$ is already discussed in Theorem 3; hence, the other condition gives the desired result of divisibility.

REFERENCES

1. Dov Jarden. "Recurring Sequences." *Riveon Lematematika Jerusalem* (Israel, 1966).
2. C. C. Yalavigi. "A Further Generalization of the Fibonacci Sequence." Private communication.

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PYTHAGOREAN PENTIDS

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1. INTRODUCTION

Let $T_n = n(n + 1)/2$ denote the nth triangular number. Then we have

$$(T_{2r})^2 + (T_{2r} + 1)^2 + (T_{2r} + 2)^2 + \cdots + (T_{2r} + r) = (T_{2r} + r + 1)^2 + (T_{2r} + r + 2)^2 + \cdots + (T_{2r} + 2r)^2 \tag{1.1}$$

and

$$(T_{2r} + 9k)^2 + (T_{2r} + 1 + 12k)^2 + \cdots + (T_{2r} + r + 12k)^2 = (T_{2r} + r + 1 + 12k)^2 + (T_{2r} + r + 2 + 12k)^2 + \cdots + (T_{2r} + 2r + 15k)^2, \tag{1.2}$$

$$r = 1, 2, 3, \ldots; \; k = 1, 2, 3, \ldots .$$

This gives a generalized identity of squares of numbers with $r + 1$ terms on the left-hand side and r terms on the right-hand side. But the triangular numbers are a particular case of the generalized Tribonacci sequence having a recurrence relation

$$X_{n+3} = 3X_{n+2} - 3X_{n+1} + X_n, \; n \geq 0, \text{ with } X_0 = 0, X_1 = 1, \text{ and } X_2 = 3. \tag{1.3}$$

Therefore, the properties of the generalized Tribonacci sequence are also properties of the triangular numbers.

The case $r = 1$ in equation (1.1) gives the well-known Pythagorean triad (3, 4, 5). For $r = 2$, we have the Pythagorean pentid (10, 11, 12, 13, 14). Pythagorean triads have been studied by various authors, particularly by Teigen and Hadwin [6] and by Shannon and Horadam [5]. The object of this note is to extend the results of the above-mentioned authors to the Pythagorean pentids. Similar extensions are also possible for the general Pythagorean n-tids of (1.1).

2. GENERALIZED FIBONACCI PENTIDS

The Horadam [2] generalized Fibonacci sequence satisfies the recurrence relation

$$H_{n+2} = H_{n+1} + H_n \quad (n \geq 1).$$

For this sequence, we have the identity

$$(2.1) \qquad H_{n+3}^2 + (H_{n+2} + H_n)^2 = (H_{n+2} - H_n)^2 + (2H_{n+2})^2 + H_n^2.$$

This can be easily checked by the substitutions

$$H_n = p, \; H_{n+1} = p + q.$$

Then corresponding to a result of Shannon [4], we have the identity

$$(2.2) \qquad U_{n+4}^2 + (U_{n+3} + U_n)^2 = (U_{n+3} - U_n)^2 + (2U_{n+3})^2 + U_n^2,$$

where U_n is the nth term of the Tribonacci [1] sequence whose recurrence relation is $U_{n+3} = U_{n+2} + U_{n+1} + U_n$, with U_1, U_2, and U_3 as the initial terms.

Proof: On using the recurrence relation, we obtain

$$(A) \qquad U_{n+4} - U_n = 2(U_{n+2} + U_{n+1}) \; \ldots$$

and

$$(B) \qquad U_{n+4} + U_n = 2U_{n+3} \; \ldots .$$

On multiplying (A) and (B), we have

$$U_{n+4}^3 - U_n^2 = 4U_{n+3}[U_{n+2} + U_{n+1}]$$

or

$$U_{n+4}^2 = U_n^2 + 2U_{n+3}[U_{n+4} - U_n]$$
$$= U_n^2 + [2U_{n+3}]^2 + [U_{n+3} - U_n]^2 - [U_{n+3} + U_n]^2,$$

from which the desired result follows. Comparison of (2.1) with (2.2) suggests a similar identity for the general recurring sequence V_n of order r with

$$V_{n+r} = \sum_{i=0}^{r-1} V_{n+i}, \; n \geq 1,$$

with the initial values $V_1, V_2, \ldots, V_r$.

Identity

$$(2.3) \qquad V_{n+r+1}^2 + [V_{n+r} + V_n]^2 = [V_{n+r} - V_n]^2 + [2V_{n+r}]^2 + V_n^2,$$

in which $r = 2$, gives (2.1), and $r = 3$ gives (2.2).

For the generalized Fibonacci sequence $W_n(a, b, p, q)$ of Horadam [3], we have

$$(2.4) \qquad \{QW_{n+3}\}^2 + \{2PW_{n+2} + W_n\}^2 = \{2PW_{n+2} - W_n\}^2 + \{4PW_{n+2}\}^2 + W_n^2$$

where $Q = p/q^2$ and $P = (p^2 - q)/2q^2$. This follows easily from a lemma of Shannon [5]:

$$(2.5) \qquad (p^2 - q)W_{n+2} - pW_{n+3} = q^2 W_n.$$

But (2.4) is in a form which can be generalized for higher-order recurrence relations. Therefore, we have the following:

Theorem 1: All Pythagorean pentids are recurrence pentids.

3. PYTHAGOREAN n-TIDS

In this section, the method of Teigen and Hadwin [6] is extended to Pythagorean n-tids. Teigen and Hadwin proved that the Pythagorean triad (a, b, c) can be represented by

(3.1) $a = x + z$, $b = y + z$, $c = x + y + z$, where x, y, z are positive and $2xy = z^2$, z even.

For the Pythagorean pentid (a, b, c, d, e), we have

(3.2) $a = x + y + z$, $b = y + z + t$, $c = z + t + u$, $d = x + y + z + t$ and $e = y + z + t + u$, where x, y, z, t, u are positive, and

(3.3) $z^2 = 2(xy + yt + yu)$, z even.

Similarly, for the Pythagorean septid (a, b, c, d, e, f, g), we have

(3.4) $a = x + y + z + t$, $b = y + z + t + u$, $c = z + t + u + v$, $d = t + u + y + w$, $e = x + y + z + t + u$, $f = y + z + t + u + v$, and $g = z + t + u + v + w$

where all the right-hand side parameters are positive, and

(3.5) $t^2 = 2(xu + yv + zw + zu + zv)$, t even.

Similar extensions follow for the n-tids.

An alternate method of generating infinite numbers of Pythagorean n-tids from a given n-tid is discussed in [7].

REFERENCES

1. M. Feinberg. "Fibonacci-Tribonacci." *The Fibonacci Quarterly* 1, No. 3 (1963):71-74.
2. A. F. Horadam. "A Generalized Fibonacci Sequence." *American Math. Monthly* 68 (1961): 455-456.
3. A. F. Horadam. "Special Properties of the Sequence $W_n(a, b, p, q)$." *The Fibonacci Quarterly* 5 (1967):424-434.
4. A. G. Shannon and A. F. Horadam. "A Generalized Pythagorean Theorem." *The Fibonacci Quarterly* 9 (1971):307-312.
5. A. G. Shannon and A. F. Horadam. "Generalized Fibonacci Number Triples." *American Math. Monthly* 80 (1973):178-190.
6. M. G. Teigen and D. W. Hadwin. "On Generating Pythagorean Triples." *American Math. Monthly* 78 (1971):378-379.
7. H. V. Krishna. "On Generating Pythagorean Pentids." *Proceedings* of the 42nd I.M.S. Conference, 1976.

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A TRIANGLE FOR THE BELL NUMBERS

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The Bell, or exponential, numbers B_n are defined by

(1) $$B_n = \frac{1}{e}\sum_{k=0}^{\infty}\frac{k^n}{k!} = \frac{1}{e}\left(\frac{0^n}{0!} + \frac{1^n}{1!} + \frac{2^n}{2!} + \cdots\right)$$

The first twelve Bell numbers are given in the following table:

TABLE 1. Bell Numbers

n	B_n
0	1
1	1
2	2
3	5
4	15
5	52
6	203
7	877
8	4140
9	21147
10	115975
11	678570

The Bell numbers also appear in the Maclaurin expansion of e^{e^x}:

(2) $$e^{e^x} = e\sum_{k=0}^{\infty}\frac{B_k x^k}{k!} = e\left(1 + \frac{x}{1!} + \frac{2x^2}{2!} + \frac{5x^3}{3!} + \frac{15x^4}{4!} + \cdots\right)$$

The Bell numbers can be generated recursively by an interesting method described in [2]. If we take the array described in this article and "flip" it about and then reorient it, the following triangle appears. This triangle is similar in form to Pascal's triangle. We shall call it the "Bell Triangle," and denote each element by $B'(n,r)$. This notation is similar to $C(n,r)$ for Pascal's triangle. There are three rules of formation for this triangle.

(3) $$B'(0,0) = 1$$

(4) $$B'(n,0) = B'(n-1,n-1) \qquad (n \geq 1)$$

(5) $$B'(n,r) = B'(n,r-1) + B'(n-1,r-1) \qquad (1 \leq r \leq n)$$

$B'(3,2) + B'(4,2) = B'(4,3)$

Row								
0	1							
1	1	2						
2	2	3	5					
3	5	7	10	15				
4	15	20	27	37	52			
5	52	67	87	114	151	203		
6	203	255	322	409	523	674	877	
7	877	1080	1335	1657	2066	2589	3263	4140

The Bell numbers form the left and right sides of the triangle. In fact,

(6) $$B'(n,n) = B_{n+1}$$

(7) $$B'(n,0) = B_n$$

Equations (6) and (7) follow from the two equivalent identites for Bell numbers:

(8) $$B_n = \binom{n}{0}B_{n+1} - \binom{n}{1}B_n + \binom{n}{2}B_{n-1} - \cdots \pm \binom{n}{n}B_1$$

(9) $$B_n = nB_{n-1} - \binom{n-1}{2}B_{n-2} + \binom{n-1}{3}B_{n-3} - \cdots \pm \binom{n-1}{n-1}B_1$$

The Bell triangle has many interesting properties. Here we present several new identities:

(10) $$\sum_{k=a}^{b} B'(n,k) = B'(n+1,b+1) - B'(n+1,a).$$

For $a = 0$ and $b = n$, this reduces to

(11) $$\sum_{k=0}^{n} B'(n,k) = B'(n+1,n+1) - B'(n+1,0) = B'(n+1,n) = B_{n+2} - B_{n+1},$$

(12) $$\sum_{k=x}^{n} B'(k+a,k) = B'(n+a,n+1) - B'(x+a-1,x),$$

(13) $$\sum_{k=x}^{n} (-1)^{n-k}\binom{n-x}{k-x}B'(k+a,k) = B'(n+a,x).$$

For $a = 0$ and $x = 0$, equation (13) reduces to (8).

(14) $$\sum_{k=x}^{n} \binom{n-x}{k-x}B'(k,a) = B'(n,a+n-x).$$

For $a = 0$ and $x = 0$, the following identity results:

(15) $$\sum_{k=0}^{n} \binom{n}{k}B'(k,0) = B'(n,n).$$

This is equivalent to

(16) $$\sum_{k=0}^{n} \binom{n}{k}B_k = B_{n+1}.$$

To my knowledge, identities (10)-(14) were heretofore unknown.

If we ignore the restricting inequality in (4), and substitute $n = 0$, we get $1 = B'(0,0) = B'(-1,-1)$. From this value, we may obtain values of $B'(n,-1)$ for $n \geq -1$ (see Table 2).

Note the following identity:

(17) $$B'(n-1,-1) + B'(n,-1) = B'(n,0) = B_n.$$

TABLE 2. Values of B'(n,-1)

n	$B'(n,-1)$
-1	1
0	0
1	1
2	1
3	4
4	11
5	41
6	162
7	715

Apparently the Bell triangle cannot be extended further because $B(-1,0) = B_{-1}$ which is undefined, by equation (1). Epstein [3] drops the term $0^n/0!$ in equation (1) without explanation and therefore gets $B_0 = 1 - 1/e$, in contradiction with Williams [5], Bell [1], and Rota [4].

The Bell numbers have combinatoric significance in that B_n is the number of ways of factoring a product of n distinct primes. Whether the rest of the numbers in the Bell triangle have any such significance remains to be seen.

REFERENCES

1. E. T. Bell. "Exponential Numbers." *American Math. Monthly* 41 (1934):411-419.
2. Martin Cohn, et al. "On the Number of Partitionings of a Set of n Distinct Objects." *American Math. Monthly* 69 (1962):782-785.
3. Leo Epstein. "A Function Related to the Series e^{e^x}." *J. Math. and Physics* 18 (1939): 153-173.
4. Gian-Carlo Rota. "The Number of Partitions of a Set." *American Math. Monthly* 71 (1964):498-504.
5. G. T. Williams. "Numbers Generated by the Function e^{e^x-1}." *American Math. Monthly* 52 (1945):323-327.

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THE EQUATIONS $z^2 - 3y^2 = -2$ AND $z^2 - 6x^2 = -5$

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The four numbers 2, 4, 12, 420 have the property that the product of any two increased by 1 is a perfect square. The object of this paper is to prove that no positive integer can replace 420.

Any integer N which can replace 420 while preserving this property must satisfy the equations

$$2N + 1 = x^2, \quad 4N + 1 = y^2, \quad 12N + 1 = z^2.$$

Eliminating N, we have

$$z^2 - 3y^2 = -2 \quad \text{and} \quad z^2 - 6x^2 = -5.$$

Now, the equation $z^2 - 3y^2 = -2$ can be written in the form

(1) $$u^2 - 3v^2 = 1$$

where $u = z^2 + 1$, $v = zy$.

Substituting for z^2 in $z^2 - 6x^2 = -5$, we have

(2) $$X^2 = 6u + 24$$

where $X = 6x$.

Hence, to solve the equations of the title, it is sufficient to solve (1) and (2) simultaneously.

Now, all the positive integral solutions of (1) are given by the formula:

(3) $$u_n + \sqrt{3}V_n = (2 + \sqrt{3})^n$$

By (3), we have

$$u_n = \frac{\alpha^n + \beta^n}{2} \quad \text{and} \quad v_n = \frac{\alpha^n - \beta^n}{2\sqrt{3}}$$

where $\alpha = 2 + \sqrt{3}$ and $\beta = 2 - \sqrt{3}$. We have the following equations and congruences:

(4) $u_{-n} = u_n$,

(5) $v_{-n} = v_n$,

(6) $u_{m+n} = u_m u_n + 3v_m v_n$,

(7) $v_{m+n} = u_m v_n + v_m u_n$,

(8) $u_{2n} = 2u_n^2 - 1$,

(9) $v_{2n} = 2u_n v_n$,

(10) $u_{3n} = u_n \cdot f_1(u_n)$,

(11) $v_{3n} = v_n \cdot f_2(u_n)$,

(12) $u_{5n} = u_n \cdot f_3(u_n)$,

(13) $v_{5n} = v_n \cdot f_4(u_n)$,

(14) $u_{7n} = u_n \cdot f_5(u_n)$,

(15) $v_{7n} = v_n \cdot f_6(u_n)$,

(16) $u_{9n} = u_n \cdot f_1(u_n) \cdot f_7(u_n)$,

(17) $v_{9n} = v_n \cdot f_2(u_n) \cdot f_8(u_n)$,

(18) $u_{15n} = u_n \cdot f_1(u_n) \cdot f_3(u_n) \cdot f_9(u_n)$

(19) $v_{15n} = v_n \cdot f_2(u_n) \cdot f_4(u_n) \cdot f_{10}(u_n)$,

(20) $u_{n+2r} \equiv u_n \pmod{v_r}$,

(21) $u_{n+2r} \equiv -u_n \pmod{u_r}$,

where

$f_1(u_n) = 4u_n^2 - 3$,

$f_2(u_n) = 4u_n^2 - 1$,

$f_3(u_n) = 16u_n^4 - 20u_n^2 + 5$,

$f_4(u_n) = 16u_n^4 - 12u_n^2 + 1$

$f_5(u_n) = 64u_n^6 - 112u_n^4 + 56u_n^2 - 7$,

$f_6(u_n) = 64u_n^6 - 80u_n^4 + 24u_n^2 - 1$,

$f_7(u_n) = 64u_n^6 - 96u_n^4 + 36u_n^2 - 3$,

$f_8(u_n) = 64u_n^6 - 96u_n^4 + 36u_n^2 - 1$,

$f_9(u_n) = 256u_n^8 - 448u_n^6 + 224u_n^4 - 32u_n^2 + 1$,

$f_{10}(u_n) = 256u_n^8 - 576u_n^6 + 416u_n^4 - 96u_n^2 + 1$.

We now have the following table of values:

n	u_n	v_n
0	1	0
1	2	1
2	7	4
3	26	15
4	97	56
5	362	209
6	1351	780
7	5042	2911
8	18817	10864
9	70226	40545
10	262087	151316
11	978122	564719
12	3650401	2107560
13	13623482	7865521

We note that both z and y are odd and hence u is even and v is odd. Hence, we have to consider only the odd values of n.

The proof is now accomplished in eleven stages:

(i) (2) is impossible if $n \equiv 3 \pmod 6$.
For, $u_n \equiv 0 \pmod{13}$ and then $X^2 \equiv -2 \pmod{13}$ and since $(-2|13) = -1$, (2) is impossible.

(ii) (2) is impossible if $n \equiv 5 \pmod{10}$.
For, using (20), $u_n \equiv u_5 \pmod{v_5} \equiv 362 \pmod{209} \equiv -1 \pmod{11}$. But then $X^2 \equiv 7 \pmod{11}$ and $(7|11) = -1$ and hence (2) is impossible.

(iii) (2) is impossible if $n \equiv \pm 5 \pmod{14}$.
For, $u_n \equiv u_{\pm 5} \pmod{v_7} \equiv u_5 \pmod{v_7}$, using (4). Now, $71|v_7$, $u_5 \equiv 7 \pmod{71}$ and then $X^2 \equiv -5 \pmod{71}$. Since $(-5|71) = -1$, (2) is impossible.

(iv) (2) is impossible if $n \equiv \pm 3 \pmod{20}$.
For, using (21), $u_n \equiv u_{\pm 3} \equiv \pm u_3 \pmod{u_{10}}$ and then $X^2 \equiv 180$ or $-132 \pmod{7 \cdot 37441}$. Now, since $(180|7) = -1$ and $(-132|37441) = -1$, (2) is impossible.

(v) (2) is impossible if $n \equiv \pm 3, \pm 11, \pm 13 \pmod{28}$.
For, when $n \equiv \pm 11 \pmod{28}$, using (4) and (20) we have $u_n \equiv u_{11} \pmod{v_{14}}$. Now, $2521 | v_{14}$ and $u_{11} \equiv -26 \pmod{2521}$. But then $X^2 \equiv -132 \pmod{2521}$ and since $(-132|2521) = -1$, this is impossible.
When $n \equiv \pm 3, \pm 13 \pmod{28}$, using (4) and (21) we have, $u_n \equiv \pm u_3, \pm u_{13} \pmod{u_{14}}$. Now, $7, 337, 3079 | u_{14}$ and $u_3, u_{13} \equiv 5 \pmod 7$, $u_3 \equiv 26 \pmod{337}$ and $u_{13} \equiv 1986 \pmod{3079}$.
Hence, $X^2 \equiv 24 + 6u_3$, $X^2 \equiv 24 + 6u_{13}$ are impossible modulo 7, $X^2 \equiv 24 - 6u_3$ is impossible modulo 337, and $X^2 \equiv 24 - 6u_{13}$ is impossible modulo 3079.

(vi) (2) is impossible if $n \equiv \pm 11, \pm 13 \pmod{30}$.
For, $u_n \equiv u_{11}, u_{13} \pmod{v_{15}}$. Now $29 | v_{15}$ and $u_{11} \equiv 10 \pmod{29}$ and $u_{13} \equiv 7 \pmod{29}$. Hence, $X^2 \equiv -3 \pmod{29}$ and $X^2 \equiv 8 \pmod{29}$ and since $(-3|29) = -1$, $(8|29) = -1$, both are impossible.

(vii) (2) is impossible if $n \equiv \pm 13 \pmod{42}$.
For, $u_n \equiv u_{13} \pmod{v_{21}}$ and then $X^2 \equiv 24 + 6u_{13} \pmod{v_{21}}$. Now $2017 | v_{21}$ and $X^2 = 1991 \pmod{2017}$, and since $(1991|2017) = -1$, (2) is impossible.

(viii) (2) is impossible if $n \equiv \pm 21 \pmod{70}$. For, $u_n \equiv u_{21} \pmod{v_{35}}$.

$$v_{35} = v_{7\cdot 5} = v_5(8u_5 - 4u_5 - 4u_5 + 1)(8u_5 + 4u_5 - 4u_5 - 1)$$
$$= v_5 \cdot v_7 \cdot 9243361 \cdot 5352481.$$

Also, $u_{21} = u_7(4u_7^2 - 3)$

$$x^2 = 24 + 6u_7(4u_7^2 - 3) \pmod{5352481}$$
$$\equiv -305121648 \pmod{5352481}.$$

Now, $$\left(\frac{-305121648}{5352481}\right) = \left(\frac{2}{5352481}\right)^4 \left(\frac{3}{5352481}\right)\left(\frac{6356701}{5352481}\right) = \left(\frac{1004220}{5352481}\right) = \left(\frac{797}{5352481}\right)$$

$$= \left(\frac{-171}{797}\right) = \left(\frac{113}{171}\right) = \left(\frac{29}{113}\right) = \left(\frac{-3}{29}\right) = -1.$$

Hence, (viii) is impossible.

(ix) (2) is impossible if $n \equiv \pm 29, \pm 31 \pmod{90}$.
For, $u_n \equiv u_{29}, u_{31} \pmod{v_{45}}$. Now $83609 | v_{45}$ and $u_{29} = 2u_{30} - 3v_{30} = 2u_{10}(4u_{10}^2 - 3) - 3v_{10}(4u_{10}^2 - 1) \equiv 9253 \pmod{83609}$. Hence, $X^2 \equiv 55542 \pmod{83609}$ and since $(55542|83609) = -1$, (2) is impossible.
Also, $17 | v_{45}$ and $u_{31} = 2u_{30} + 3v_{30} \equiv 5 \pmod{17}$ and hence $X^2 \equiv 3 \pmod{17}$. Since $(3|17) = -1$, (2) is impossible.

(x) (2) is impossible if $n \equiv \pm 1 \pmod{252}$, $n \neq \pm 1$.
For, we can write $n = \pm 1 + 63k(2\ell + 1)$, where ℓ is an integer and $k = 2^t$, $t \geq 2$.
Then, $u_n \equiv \pm u_{\pm 1 + 63k} \equiv \pm 3v_{63k} \pmod{u_{63k}}$.
Now, $v_{63k} = v_{9\cdot 7k} \equiv v_{7k} \pmod{u_{7k}} \equiv v_k(32u_k^4 - 32u_k^2 + 6)\ [\bmod f_5(u_k)]$

And, $v_{63k} = v_{7\cdot 9k} \equiv -v_{9k} \pmod{u_{9k}} \equiv -2v_k(4u_k^2 - 1)\ [\bmod f_7(u_k)]$

Hence, $X^2 \equiv 24 \pm 18v_k(32u_k^4 - 32u_k^2 + 6)\ [\bmod f_5(u_k)]$
$\equiv 24 \mp 36v_k(4u_k^2 - 1)\ [\bmod f_7(u_k)]$.

First, consider $X^2 \equiv 24 + 18v_k(32u_k^4 - 32u_k^2 + 6)\ [\bmod f_5(u_k)]$.

Now, $$\left(\frac{24 + 18v_k(32u_k^4 - 32u_k^2 + 6)}{f_5(u_k)}\right) = \left(\frac{24 + 18v_k(288v_k^4 + 96v_k^2 + 6)}{1728v_k^6 + 720v_k^4 + 72v_k^2 + 1}\right)$$

$$= \left(\frac{144v_k^4 + 36v_k^2 - 8v_k^2 + 1}{\frac{1}{2}(432v_k^5 + 144v_k^3 + 9v_k + 2)}\right)$$

$$= \left(\frac{36v_k^3 + 24v_k^2 + 6v_k + 2}{144v_k^4 + 36v_k^2 - 8v_k + 1}\right)$$

(continued)

$$= \left(\frac{3}{\tfrac{1}{2}(36v_k^3 + 24v_k^2 + 6v_k + 2)}\right)\left(\frac{228v_k^2 + 19}{\tfrac{1}{2}(36v_k^3 + 24v_k^2 + 6v_k + 2)}\right)$$

$$= (-)\left(\frac{36v_k^3 + 24v_k^2 + 6v_k + 2}{19}\right)$$

Similarly, $\left(\frac{24 - 18v_k(32u_k^4 - 32u_k^2 + 6)}{f_5(u_k)}\right) = \left(\frac{36v_k^3 - 24v_k^2 + 6v_k - 2}{19}\right)$

Next, consider $X^2 \equiv 24 \mp 36v_k(4u_k^2 - 1)$ [mod $f_7(u_k)$].

Now, $\left(\frac{24 - 36v_k(4u_k^2 - 1)}{f_7(u_k)}\right) = \left(\frac{24 - 36v_k(12v_k^2 + 3)}{1728v_k^6 + 864v_k^4 + 108v_k^2 + 1}\right) = \left(\frac{1728v_k^6 + 864v_k^4 + 108v_k^2 + 1}{\tfrac{1}{2}(36v_k^3 + 9v_k - 2)}\right)$

$$= \left(\frac{96v_k^3 + 24v_k + 1}{\tfrac{1}{2}(36v_k^3 + 9v_k - 2)}\right) = \left(\frac{36v_k^3 + 9v_k - 2}{19}\right)$$

Similarly, $\left(\frac{24 + 36v_k(4u_k^2 - 1)}{f_7(u_k)}\right) = (-)\left(\frac{36v_k^3 + 9v_k + 2}{19}\right)$

The residues of v_k, $36v_k^3 \pm 24v_k^2 \pm 6v_k + 2$ and $36v_k^3 + 9v_k \pm 2$ modulo 19 are periodic and the length of the period is 4. The following table gives these residues and the signs of $(24 \pm 18v_k(32u_k^4 - 32u_k^2 + 6)|f_5(u_k))$ and $(24 \mp 36v_k(4u_k^2 - 1)|f_7(u_k))$.

$k = 2^t$	$t = 2$	3	4	5	6
v_k (mod 19)	-1	-4	1	4	-1
$36v_k^3 + 24v_k^2 + 6v_k + 2$ (mod 19)	3	-4	-8	-3	
$36v_k^3 - 24v_k^2 + 6v_k - 2$ (mod 19)	8	3	-3	4	
$36v_k^3 + 9v_k + 2$ (mod 19)	-5	-1	9	5	
$36v_k^3 + 9v_k - 2$ (mod 19)	-9	-5	5	1	
$(24 + 18v_k(32u_k^4 - 32u_k^2 + 6)\|f_5(u_k))$	+1	+1	-1	-1	
$(24 - 36v_k(4u_k^2 - 1)\|f_7(u_k))$	-1	-1	+1	+1	
$(24 - 18v_k(32u_k^4 - 32u_k^2 + 6)\|f_5(u_k))$	-1	-1	+1	+1	
$(24 + 36v_k(4u_k^2 - 1)\|f_7(u_k))$	+1	+1	=1	-1	

From the above table, we see that the congruences $X^2 \equiv 24 + 18v_k(32u_k^4 - 32u_k^2 + 6)$ [mod $f_5(u_k)$] and $X^2 \equiv 24 - 36v_k(4u_k^2 - 1)$ [mod $f_7(u_k)$] cannot hold simultaneously, and the congruences $X^2 \equiv 24 - 18v_k(32u_k^4 - 32u_k^2 + 6)$ [mod $f_5(u_k)$] and $X^2 \equiv 24 + 36v_k(4u_k^2 - 1)$ [mod $f_7(u_k)$] cannot hold simultaneously.

Hence, (2) is impossible.

(xi) $n \equiv \pm 7 \pmod{60}$; $n \neq \pm 7$ is impossible.

For, we can write $n = \pm 7 + 2.15k\ell$, where $k = 2^t$, $t \geq 1$ and ℓ is an odd integer. Then, by applying (21) ℓ times, we have

$$u_n \equiv -u_7 \pmod{u_{15k}} \equiv -5042 \ (\text{mod } u_k \cdot f_1(u_k) \cdot f_3(u_k) \cdot f_9(u_k))$$

Hence, $X^2 \equiv 24 - 6.5042 \equiv -30228 \ (\text{mod } u_k \cdot f_1(u_k) \cdot f_3(u_k) \cdot f_9(u_k))$. Note that when $t = 1$, $u_n \equiv -2 \pmod 7$ and then $X^2 \equiv 5 \pmod 7$ and $(5|7) = -1$.

When $t \geq 2$, we have

$$(-30228|u_k) = (u_k|11)(u_k|229) = (-)(u_k|229) \text{ when } u_k \equiv -4 \pmod{11}$$
$$= (u_k|229) \quad \text{when } u_k \equiv -2 \pmod{11};$$

$$(-30228|f_1(u_k)) = (-)(f_1(u_k)|229);$$

$$(-30228|f_3(u_k)) = (-)(f_3(u_k)|229) \text{ when } u_k \equiv -4 \pmod{11}$$
$$= (f_3(u_k)\ 229) \quad \text{when } u_k \equiv -2 \pmod{11};$$
$$(-30228|f_9(u_k)) = (-)(f_9(u_k)|229).$$

The residues of u_k, $f_1(u_k)$, $f_3(u_k)$, and $f_9(u_k)$ modulo 229 are periodic and the length of the period is 9. The following table gives the values of these residues and the signs of $(-30228|u_k)$, $(-30228|f_1(u_k))$, $(-30228|f_3(u_k))$ and $(-30228|f_5(u_k))$.

$k = 2^t$	$t = 2$	3	4	5	6	7	8	9	10	11
u_k (mod 229)	97	39	64	-53	121	-31	89	40	-7	97
$f_1(u_k)$ (mod 229)	77	127	122	12	-63	177	79	-15	193	
$f_3(u_k)$ (mod 229)	51	-4	-109		12	132	-93			
$f_9(u_k)$ (mod 229)	103				159		58			
	when $u_k \equiv -4 \pmod{11}$									
$(-30228\ u_k)$	-1	+1	-1	-1	-1	+1	+1	+1	+1	
$(-30228\ f_1(u_k))$		+1				+1	+1	-1	-1	
$(-30228\ f_3(u_k))$		-1				-1	+1			
$(-30228\ f_9(u_k))$							-1			
	when $u_k \equiv -2 \pmod{11}$									
$(-30228\ u_k)$	+1	-1	+1	+1	+1	-1	-1	-1	-1	
$(-30228\ f_1(u_k))$	+1		+1	-1	+1					
$(-30228\ f_3(u_k))$	+1		-1		+1					
$(-30228\ f_9(u_k))$	-1				-1					

Hence, (2) is impossible.

Summarizing the results, we see that (1) and (2) can hold for n odd, only for $n = 1$ and $n = 7$, and these values do indeed satisfy with $u = 2$, $v = 1$, $x = 1$, and $u = 5042$, $v = 2911$, $x = 29$. $x = 1$ gives the trivial solution $N = 0$ and $x = 29$ gives the solution $N = 420$.

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GENERATION OF FIBONACCI NUMBERS BY DIGITAL FILTERS

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ABSTRACT

This paper presents some applications of Fibonacci numbers in system and communication theory. Methods of generating Fibonacci sequences and codes by sequential binary filters are given.

INTRODUCTION

The role that Fibonacci numbers play in system theory is worthy of engineering investigations. Fibonacci numbers find their way in algebraic coding theory in communications, linear sequential circuits, and linear digital filters. Although some of these applications are not direct realizations of Fibonacci numbers, they provide the conceptual framework for the related model. For example, the concept of recurrence equation that generates the numbers is utilized to generate difference codes which are used in radar ranging by long-range radars, such as satellite tracking radars and radars that are used for planet's ranging [2]. Another example of Fibonacci numbers is one used to generate a model for population growth in animal and biological colonies. Digital realizations of these models will be given later in the sequel. We will introduce some general applications of Fibonacci numbers and present their digital filter realizations. Then, we present Fibonacci recurrence codes, and give an example of a binary digital sequential circuit to generate these codes.

GENERAL APPLICATIONS

The Z-transforms of discrete time function $y(k)$; $k = 0, 1, 2, \ldots$, is defined by:

(1) $$Z\{y(k)\} = \sum_{k=0}^{\infty} y(k)Z^{-k}.$$

The Z-transform of $y(k + n)$, $n > 0$, is given by:

(2) $$Z\{Y(k + n)\} = Z^n Y(Z) - Z^n \sum_{j=0}^{n-1} Y(j)Z^{-j}.$$

The Z-transform of the Fibonacci equation, after rearrangement,

(3) $$y(k + 1) = y(k) + y(k - 1); \quad y(0) = 0, \quad y(1) = 1,$$

is given by:

(4) $$(Z^2 - Z - 1)Y(Z) = 0$$

The equation

(5) $$Z^2 - Z - 1 = 0$$

is the characteristic equation of the Fibonacci recurrence equation (3). The digital filter that realizes equation (4), and generates the Fibonacci sequence represented by equation (3), is shown in Figure 1.

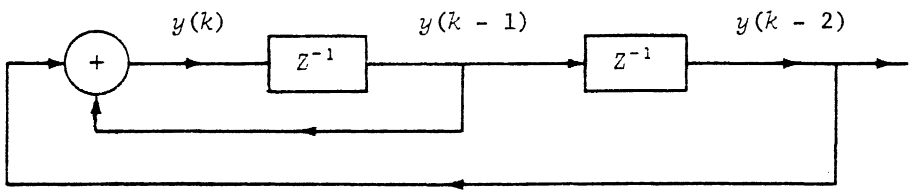

FIGURE 1. Fibonacci Sequence Generator

It is understood that the small box that contains Z^{-1} in Figure 1 represents a unit delay. Unfortunately, the above filter is unstable, since one of the roots has absolute value more than unity. However, this unstable behavior can be of great advantage if we assume that $y(k)$, $k = 0, 1, 2, \ldots$, are elements in a field $GF(p)$ of prime characteristic p. An example of application of the Fibonacci sequences is modeling of population growth of rabbit population by:

(6) $$y(k) = y(k - 1) + y(k - 2) + u(k) \ldots$$

where $y(k)$ represents number of pairs of rabbits at the kth month, and $u(k)$, $k = 0, 1, 2, \ldots$, is a control sequence which, if chosen properly, yields a stable population. The control sequence $u(k)$ may be chosen as feedback linear combination of $y(k - 1)$ and $y(k - 2)$, that is:

(7) $$u(k) = -\beta_1 y(k - 1) - \beta_2 y(k - 2).$$

Substituting (7) in (6) yields the equation:

(8) $$y(k) = (1 - \beta_1)y(k - 1) + (1 - \beta_2)y(k - 2)$$

whose characteristic equation is given by:

(9) $$Z^2 + (\beta_1 - 1)Z + (\beta_2 - 1) = 0.$$

Clearly the roots of the characteristic equation (9) can be assigned arbitrarily by proper choice of β_1 and β_2.

A filter realization of this model is shown in Figure 2. The circles represent multipliers by:

$$\alpha_1 = 1 - \beta_1 \quad \text{and} \quad \alpha_2 = 1 - \beta_2.$$

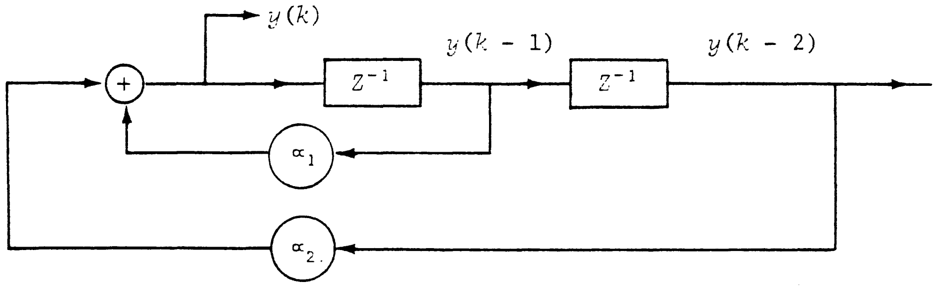

FIGURE 2. Controlled Population Model

DIFFERENCE CODES

Difference code transforms a given m-digit initial sequence $a_0, a_1, \ldots, a_{m-1}$ into an infinitely long sequence $y_0, y_1, y_2, \ldots$, sequentially by the linear difference equation,

$$b_0 y(k) + b_1 y(k-1) + \cdots + b_m y(k-m) = 0; \; b_0, b_m \neq 0;$$

$$k = m, m-1, \ldots, a_0 = y(0), a_1 = y(1), \ldots, a_{m-1} = y(m-1),$$

where $y(k)$ and $b(k)$ are elements of the finite Galois field, $GF(p)$. The left characteristic equation of the difference equation (10) is:

$$C(Z) = b_0 + b_1 Z + \cdots + b_m Z^m. \tag{11}$$

The generating function of this code is given by the formal power series $G(Z)$:

$$G(Z) = y(0) + y(1)Z + y(2)Z^2 + \cdots = \sum_{n=0}^{\infty} y(n)Z^n. \tag{12}$$

It can be shown [2] that each solution

$$y(0), y(1), y(2), \ldots,$$

corresponding to (10) has a generating function

$$G(Z) = \sum_{n=0}^{\infty} y(n)Z^n = \frac{A(Z)}{G(Z)}, \tag{13}$$

where $A(Z)$ is the polynomial in Z with the initial sequence $a_0, a_1, \ldots, a_{m-1}$ as coefficients; that is,

$$A(Z) = a_0 + a_1 Z + \cdots + a_{m-1}Z^{m-1}. \tag{14}$$

The generation function is obtained from (13) by long division over the specified field. For example, over the field of real numbers, the Fibonacci sequence is given as coefficients of the power series $G(Z)$ given by:

$$G(Z) = \frac{1}{1 - Z - Z^2} = 1 + Z + 2Z^2 + 3Z^3 + 5Z^4 + \cdots. \tag{15}$$

It is not difficult to see that difference codes over finited fields are periodic. The Fibonacci sequence over the binary field has the generating function

$$G(Z) = \frac{A(Z)}{G(Z)} = \frac{1}{1 + Z + Z^2} = 1 + Z + Z^3 + Z^4 + Z^6 + Z^7 + Z^9 + \cdots. \tag{16}$$

$A(Z) = 1 + (0)Z$, since the initial code word is given the initial sequence $a_0 = 1$, $a_1 = 0$. Therefore, the difference code given by (16) is periodic, with period 3, and has the form 110110110110... .

Periodic codes of maximal period are of interest in long-range radar ranging, especially those used in satellite tracking. Those codes are generated by difference equations whose characteristic equations are primitive, with respect to the given finite field. The polynomial $C(Z)$ of degree n is primitive over the field $GF(p)$ if $C(Z)$ divides $Z^{(p^n-1)} - 1$ and it divides no polynomial $(Z^t - 1)$ with $t < p^n - 1$. The difference code whose characteristic polynomial is primitive and has degree n, is maximal period code. The maximal period of the code equals $p^n - 1$, where p is the prime characteristic of the field. Over the binary field $GF(2)$, the primitive characteristic polynomial, $C(Z)$, of the Fibonacci equation is given by:

(17) $$C(Z) = 1 + Z + Z^2.$$

$C(Z)$ is primitive of degree 2. Therefore, the code generated by $C(Z)$ is periodic with maximal period 3. Long period difference codes of this type are usually used in satellite communications. As an example, the primitive polynomial $1 + x + x^{22}$ generates a code sequence of a period $2^{22} - 1 \simeq 4{,}194{,}393$.

The Fibonacci code sequence over $GF(2)$ has correlation function $R(\ell) = -1$ for all shifts ℓ except for $\ell = 0$ and multiples of $2^2 - 1$, at which the value of $R(\ell)$ is $2^2 - 1 = 3$. The correlation property is of great importance in the ranging operation of satellite radars.

It has been shown that Fibonacci sequences can be used in coding and communication theory, and can be implemented by binary digital filters. Similar applications can utilize this approach to generate Fibonacci numbers.

REFERENCES

1. Brother U. Alfred. *An Introduction to Fibonacci Discovery*. San Jose, Calif.: The Fibonacci Association, 1965.
2. Garret Birkhoff and Thomas C. Bartee. *Modern Applied Algebra*. New York: McGraw-Hill Book Company, Inc., 1970.
3. James A. Cadzow. *Discrete-Time Systems*. Englewood Cliffs, N.J.: Prentice-Hall, Inc., 1973.

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THE FIBONACCI SERIES IN THE DECIMAL EQUIVALENTS OF FRACTIONS

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SUMMARY

Four numbers below 100, as denominators of fractions, yield decimal equivalents in which the sequence of digits can also be produced by summations of the terms of the Fibonacci series.

Where every Fibonacci term is used, and moving each term one place to the right, the sequence is that for 1/89; using every second term, the sequence is that for 1/71; with every third term, 2/59; and with every fourth term, 3/31.

The larger denominators: 9899, 9701, 9599, 9301, 8899, 8201, 7099, 6301, and 2399, give repeating decimal equivalents which can be obtained by the summations of every Fibonacci term, every second, third, ..., up to every ninth term, in this case moving each successive term two places to the right. Moreover, the numerators associated with these denominators are: 1, 1, 2, 3, 5, 8, 13, 21, and 34, the first nine terms in the Fibonacci series.

Still larger denominators yield Fibonacci decimal equivalents. Using every fourteenth term, and moving each term three places to the right, the sequence for 377/15701 is obtained.

The decimal equivalents for 9/71, 1/109, 1/10099, and others, can be generated from *right* to *left* by a reverse summation of Fibonacci terms.

The Lucas-, Negative Fibonacci-, Tribonacci-, and other series produce sequences of digits in repeating decimals.

INTRODUCTION

The Fibonacci series is thus defined: $F_1 = 1$; $F_2 = 1$; $F_{(n-1)} + F_n = F_{(n+1)}$; and the first several terms are 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ... Recently, Brousseau [1] called attention to the fact that the sequence of digits in the decimal equivalent of 1/89 is developed by a summation of the Fibonacci series where each successive term is moved one place to the right; thus,

```
112358
     13
      21
       34
        55
         89
         144
           233
            377
             ...
11235955056 ...
```

This method of summation has been called "diagonalization" by Kaprekar [2]. A better expression for the summation leading to 1/89 is as follows:

$$1/89 = F_1 \cdot 10^{-2} + F_2 \cdot 10^{-3} + F_3 \cdot 10^{-4} + \cdots + F_n \cdot 10^{-(n+1)}$$

or

$$1/89 = \sum_{n=0}^{\infty} F_n \cdot 10^{-(n+1)} = 0.011235955056\ldots .$$

In 1971, Wlodarski [3] showed that the digital sequence for 2/59 was produced by the diagonalization of every third Fibonacci term, starting with the third term.

This paper shows how the Fibonacci series can generate the repeating decimal equivalents of an infinity of fractions.

DISCUSSION

A general expression for the decimal equivalents of certain fractions derivable from the Fibonacci series is

$$1/N = \sum_{n=0}^{\infty} F_{(an+b)} \cdot 10^{-k(n+1)},$$

where $a = 1, 2, 3, \ldots$, and indicates whether every term is used ($a = 1$), or every second term ($a = 2$), every third term ($a = 3$), ...; where $b = 0, 1, 2, 3, \ldots$, and defines further which term is used to start the diagonalization, and where $k = 1, 2, 3, \ldots$, which controls the number of places that each successive term is moved to the right.

In the application of this expression where $a = 1$, $b = 0$, and $k = 1$, the value for 1/89 is obtained. As b takes other values, 1, 2, 3, 4, ..., the denominator remains 89, but the numerators are 1, 10, 11, 21, 32, 53, ..., which appear in a Fibonacci sequence. The reader may wish to check some of these numerators himself.

Where $a = 2$, $b = 0$, and $k = 1$, the sequence is:

```
0.0138
     21
      55
      144
       377
        987
         ...
-------------------
0.0140845... = 1/71
```

Where $a = 3$, $b = 0$, and $k = 1$, the sequence is:

```
0.028
    34
    144
     610
     2554
     10946
         ...
-------------------
0.0338983... = 2/59
```

Where $a = 4$, which means starting with the fourth term and using every fourth term thereafter, a decimal equivalent is not easily obtained, because the diagonalization does not rapidly converge, although it does definitely not diverge. This fraction is inferred to be 3/31 = 0.09774193... from subsequent considerations.

In the diagonalization where $k = 2$, $a = 1, 2, 3, 4, \ldots$, and $b = 0$, the tediously developed decimal equivalents were for the fractions: 1/9899, 1/9701, 2/9599, 3/9301, 5/8899, and 8/8201. For example, 1/9899 = 0.000101020305081321345... . The ..99 and ..01 terminations of these denominators suggested a classification into two groups. They are so arranged in Table 1, bounded by the dashed lines, and their differences were noted also. These differences appeared to be in a Lucas sequence, and when extended outside the dashed lines, they gave rise to additional denominators. The numerators associated with each of them are also tabulated.

The new numbers thus found, 7099, 6301, and 2399, are the denominators of fractions, the decimal equivalents of which can be found by the diagonalization of every seventh, eighth, and ninth Fibonacci term, each term being moved two places to the right. The respective numerators for these three fractions are 13, 21, and 34.

TABLE 1. Differences between Denominators (k = 2)

GROUP I			GROUP II		
Num.	Denom.	Diff.	Num.	Denom.	Diff.
-	10099		-	9801	
		200			100
1	9899		1	9701	
		300			400
2	9599		3	9301	
		700			1100
5	8899		8	8201	
		1800			2900
13	7099		21	5301	
		4700			7600
34	2399			-2299	
		12300			
	-9901				

It was easy to extend this line of reasoning to the case where $a = 3$, meaning that each term is displaced three places to the right. For example,

$$1/998999 = 0.000001001002003005008013021\ldots\ .$$

In Table 2, the denominators within the dashed lines were found by calculation, and by means of their differences all of the other numbers were determined. The numerators associated with them are also included.

TABLE 2. Differences between Denominators (k = 3)

GROUP I			GROUP II		
Num.	Denom.	Diff.	Num.	Denom.	Diff.
-	1000999		-	998001	
		2000			1000
1	998999		1	997001	
		3000			4000
2	995999		3	993001	
		7000			11000
5	988999		8	982001	
		18000			29000
13	970999		21	953001	
		47000			76000
34	923999		55	877001	
		123000			199000
89	800999		144	678001	
		322000			521000
233	478999		377	157001	
		843000			1364000
-	-364001		-	-1206999	

These tabulations suggested a return to the numbers below 100, arranging them in the same way and noting the differences between them, as in Table 3. By analogy with Tables 1 and 2, the number 31 should be reasonably included as a denominator, with 3 as its numerator. The statement in paragraph 2 of the Summary is thus justified, if not rigorously proved.

At the bottoms of Tables 1, 2, and 3, there are some negative numbers that are developed by the application of successive Lucas differences, but these have not been investigated to see how they might relate to the Fibonacci series. At the top right of Tables 1, 2, and 3 are found the numbers 81, 9801, and 998001, which are 9^2, 99^2, and 999^2, respectively. Their relation to the Fibonacci series is not clear.

TABLE 3. Differences between Denominators (k = 1)

GROUP I			GROUP II		
Num.	Denom.	Diff.	Num.	Denom.	Diff.
-	109		-	81	
		20			10
1	89		1	71	
		30			40
2	59		3	31	
		70			110
-	-11		-	-79	

REVERSE FIBONACCI SERIES

Using the Fibonacci series and starting at the right, moving each successive term one place to the left, a reverse Fibonacci diagonalization is obtained:

```
                         13853211
                        21
                       34
                      55
                     89
                   144
                  233
                 377
              ......
1/109 = 0.0091473......8348623853211
```

where 88 intermediate digits have been omitted.

In the same manner, but moving each successive term two places to the left, the decimal equivalent of 1/10099 would ultimately be obtained, where the terminal digits again form a reverse Fibonacci series, and the repeating decimal portion is of undetermined length:

1/10099 = 0.000091473......2113080503020101

Taking every second Fibonacci term and diagonalizing one place to the left, the result is the decimal equivalent of 9/71:

9/71 = 0.12676053......4507042253521

CONCLUSION

There are infinite families of denominators which have repeating decimal equivalents with digital sequences derivable from the Fibonacci series. The numerators of these denominators, as well as the differences between them, also form Fibonacci sequences.

The interested reader might wish to extend the above abbreviated presentation.

REFERENCES

1. Brother Alfred Brousseau. "Ye Olde Fibonacci Curiosity Shoppe." *The Fibonacci Quarterly* 10, No. 4 (1972):442.
2. D. R. Kaprekar. *J. Univ. Bombay* (1939, 1940).
3. J. Wlodarski. "A Number Problem." *The Fibonacci Quarterly* 9, No. 2 (1971):195.

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GENERALIZATION OF A PROBLEM OF GOULD AND ITS SOLUTION BY A CONTOUR INTEGRAL

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The following research problem was posed by H. W. Gould in [1].

Problem 1: If, for an arbitrary sequence $\{A_n\}_{n=0}^{\infty}$,

$$f(x) = \sum_{n=0}^{\infty} A_n x^n, \quad h(x) = \sum_{n=0}^{\infty} A_n^2 x^n,$$

how are functions f and h related?

The preceding problem is readily generalized as follows.

Problem 2: If, for arbitrary sequences $\{A_n\}_{n=0}^{\infty}$ and $\{B_n\}_{n=0}^{\infty}$,

$$f(x) = \sum_{n=0}^{\infty} A_n x^n, \quad g(x) = \sum_{n=0}^{\infty} B_n x^n, \quad \text{and} \quad h(x) = \sum_{n=0}^{\infty} A_n B_n x^n,$$

how is function h related to functions f and g?

Problem 2 was at least partially solved in a previous paper (viz. [2]), using the techniques of the umbral calculus. However, the "solution" obtained in [2] is expressed as a function of finite difference operators, thereby necessitating caution in its application. The aim of this paper is to obtain a rigorous solution to Problem 2 above, under the assumption that f and g are "sufficiently" analytic. We will find it slightly more tedious, but more far-reaching, to solve the even more general

Problem 3: If, for arbitrary sequences $\{A_n\}_{n=-\infty}^{\infty}$ and $\{B_n\}_{n=-\infty}^{\infty}$, and $z_0 \in C$,

$$f(z) = \sum_{n=-\infty}^{\infty} A_n (z - z_0)^n, \quad g(z) = \sum_{n=-\infty}^{\infty} B_n (z - z_0)^n, \quad h(z) = \sum_{n=-\infty}^{\infty} A_n B_n (z - z_0)^n,$$

how is function h related to functions f and g?

Before proceeding to the main theorem of this paper, which solves Problem 3, we will find it convenient to make a few preliminary definitions and remarks.

Definitions: Let z be an arbitrary point in the complex plane (i.e., the z-plane), and suppose the following Laurent series expansions valid in the annuli indicated:

(1) $$f(z) = \sum_{n=-\infty}^{\infty} A_n (z - z_0)^n, \quad \forall z \ni r_1 < |z - z_0| < R_1,$$

(2) $$g(z) = \sum_{n=-\infty}^{\infty} B_n (z - z_0)^n, \quad \forall z \ni r_2 < |z - z_0| < R_2,$$

where $\max(r_1, r_2) \equiv \rho_1 < \rho_2 \equiv \min(R_1, R_2)$.

We permit $\rho_1 = 0$, $\rho_2 = \infty$. Let

(3) $$D_1 = \{z : r_1 < |z - z_0| < R_1\},$$

(4) $$D_2 = \{z : r_2 < |z - z_0| < R_2\},$$

(5) $$D_3 = \{z : \rho_1^2 < |z - z_0| < \rho_2^2\},$$

(6) $$D_4 = \{z : r_1 r_2 < |z - z_0| < R_1 R_2\}.$$

Also, define the Laurent series

(7) $$h(z) = \sum_{n=-\infty}^{\infty} A_n B_n (z - z_0)^n,$$

which is necessarily valid $\forall z \in D_4$. Given $z \in D_3$, consider another complex plane (say, the w-plane), and define the annulus

(8) $$\Delta(z) = \{w : s_1(z) < |w - z_0| < s_2(z)\}, \text{ where}$$

(9) $$s_1(z) = \max(\rho_1, |z - z_0|/\rho_2), \quad s_2(z) = \min(\rho_2, |z - z_0|/\rho_1).$$

Let Γ be any simple closed contour contained in $\Delta(z)$ (in the w-plane), traversed in the positive direction, and containing the point $w = z_0$ in its interior.

Remarks: Note that $D_1 \cap D_2 \neq \emptyset$, and $D_3 \subseteq D_4$ (since $r_1 r_2 \leq \rho_1^2 < \rho_2^2 \leq R_1 R_2$). Also, if $z \in D_3$, then $\rho_1 < s_2(z)$ and $|z - z_0|/\rho_2 < s_2(z)$, so that $s_1(z) < s_2(z)$, which implies that $\Delta(z) \neq \emptyset$ for all $z \in D_3$.

Theorem: Given (1)-(9), then for all $z \in D_3$,

$$(10) \qquad h(z) = \frac{1}{2i\pi} \oint_\Gamma \frac{f(w) g\left\{z_0 + \dfrac{z - z_0}{w - z_0}\right\}}{w - z_0} dw = \frac{1}{2i\pi} \oint_\Gamma \frac{f\left\{z_0 + \dfrac{z - z_0}{w - z_0}\right\} g(w)}{w - z_0} dw$$

Proof: Because of the symmetry between functions f and g, it suffices to prove only the first relation in (10). Let

$$G(w) = g\left\{z_0 + \frac{z - z_0}{w - z_0}\right\}.$$

Then, by (2), the Laurent series for G, namely

$$(11) \qquad G(w) = \sum_{n=-\infty}^{\infty} B_n \left(\frac{z - z_0}{w - z_0}\right)^n,$$

represents an analytic function in the annulus $\Delta_2(z)$ given by:

$$(12) \qquad \Delta_2(z) = \{w : |z - z_0|/R_2 < |w - z_0| < |z - z_0|/r_2\}.$$

Also, the Laurent series for $f(w)$, given in (1), but replacing z by w, represents an analytic function in the annulus $\Delta_1(z)$ of the w-plane, given by:

$$(13) \qquad \Delta_1(z) = \{w : r_1 < |w - z_0| < R_1\}.$$

For all $z \in D_3$, $r_1 r_2 \leq \rho_1^2 < |z - z_0| < \rho_2^2 \leq R_1 R_2$; hence, $|z - z_0|/R_2 < R_1$ and $|z - z_0|/r_2 > r_1$. Also, $r_1 < R_1$, and $|z - z_0|/R_2 < |z - z_0|/r_2$ (since $r_2 < R_2$). It follows that $\max(r_1, |z - z_0|/R_2) < \min(R_1, |z - z_0|/r_2)$. This, in turn, implies that

$$\Delta_1(z) \cap \Delta_2(z) \neq \emptyset.$$

Next, observe that, for all $z \in D_3$, $s_1(z) = \max(\rho_1, |z - z_0|/\rho_2) = \max\{\max(r_1, r_2), \max(|z - z_0|/R_1, |z - z_0|/R_2)\} = \max(r_1, r_2, |z - z_0|/R_1, |z - z_0|/R_2) \geq \max(r_1, |z - z_0|/R_2)$. Also, $s_2(z) = \min(\rho_2, |z - z_0|/\rho_1) = \min\{\min(R_1, R_2), \min(|z - z_0|/r_1, |z - z_0|/r_2)\} = \min(R_1, R_2, |z - z_0|/r_1, |z - z_0|/r_2) \leq \min(R_1, |z - z_0|/r_2)$. This implies that, for all $z \in D_3$,

$$\Delta(z) \subseteq \{\Delta_1(z) \cap \Delta_2(z)\}.$$

Since $\Gamma \subset \Delta(z)$ and z lies in the interior of Γ, thus the function $f(w)/(w - z_0)$ is continuous on $\Gamma \subset \Delta_1(z)$; moreover, G is analytic on $\Delta(z) \subseteq \Delta_2(z)$. By a well-known theorem of complex analysis, it is therefore legitimate to interchange the integral and summation signs in the following expression.

$$(14) \qquad \frac{1}{2i\pi} \oint_\Gamma \frac{f(w)}{w - z_0} \sum_{n=-\infty}^{\infty} B_n \left(\frac{z - z_0}{w - z_0}\right)^n dw = \sum_{n=-\infty}^{\infty} B_n (z - z_0)^n \frac{1}{2i\pi} \oint_\Gamma \frac{f(w) dw}{(w - z_0)^{n+1}}.$$

But, since $\Gamma \subset \Delta(z) \subseteq \Delta_1(z)$, we may apply the formula for the coefficients of a Laurent series, namely:

$$A_n = \frac{1}{2i\pi} \oint_\Gamma f(w)/(w - z_0)^{n+1} dw.$$

Hence, the right member of (14) is the restriction of $h(z)$ to D_3, a subset of D_4. The left member of (14), by (11), is equal to the first integral expression in (10). This establishes the first equation of (10), and therefore the theorem.

Additional Remarks: Although the result of the theorem has been proven valid for all $z \in D_3$, as given in (5), the series defining $h(z)$ represents an analytic function in the larger domain D_4, as given in (6). Hence, the series in (7) is the *analytic continuation* of the integral expression for h in (10), from D_3 to D_4. If the latter expression yields a "closed" formula for $h(z)$, then this must be the closed form "sum-function" of h as given by (7), and holds for all $z \in D_4$.

The argument proving the preceding theorem may be slightly modified, and is somewhat simplified, if $r_1 = r_2 = 0$, thereby leading to a corresponding result involving Taylor, instead of Laurent series.

Corollary: Suppose f and g are as given in (1) and (2), with $A_{-n} = B_{-n} = 0$ $(n = 1, 2, \ldots)$, i.e.,

$$(15)\qquad f(z) = \sum_{n=0}^{\infty} A_n(z - z_0)^n, \quad z \in D_1 = \{z : |z - z_0| < R_1\},$$

$$(16)\qquad g(z) = \sum_{n=0}^{\infty} B_n(z - z_0)^n, \quad z \in D_2 = \{z : |z - z_0| < R_2\}.$$

Then, for all $z \in D_3 = \{z : |z - z_0| < \rho_2^2\}$ (where ρ_2 has been previously defined):

$$(17)\qquad h(z) \equiv \sum_{n=0}^{\infty} A_n B_n (z - z_0)^n = \frac{1}{2i\pi}\oint_\Gamma \frac{f(w)g\left\{z_0 + \dfrac{z - z_0}{w - z_0}\right\}}{w - z_0}\,dw = \frac{1}{2i\pi}\oint_\Gamma \frac{f\left\{z_0 + \dfrac{z - z_0}{w - z_0}\right\}g(w)}{w - z_0}\,dw,$$

where Γ is as described in the theorem, except that $s_1(z) = |z - z_0|/\rho_2$, $s_2(z) = \rho_2$.

We illustrate the theorem and its corollary with several examples, the first few of which are trivial (but serve to corroborate the results), the last few a bit more interesting.

Example 1: Let $z_0 = 0$, $B_n = 1$ $(n = 0, 1, 2, \ldots)$, $A_{-n} = B_{-n} = 0$ $(n = 1, 2, \ldots)$. Then $g(z) = (1 - z)^{-1}$ for all $z \in D_2$, the open unit disk. For all $z \in D_3$, where $D_3 = \{z : |z| < \rho_2^2\}$, let $\Delta(z) = \{w : |z|/\rho_2 < |w| < \rho_2\}$. We see that $h(z) = f(z)$, trivially, and expect that the corollary yields this result. Naively applying the corollary, we then obtain, $\forall\, z \in D_3$:

$$h(z) = \frac{1}{2i\pi}\oint_\Gamma \frac{f(w)g(z/w)}{w}\,dw = \frac{1}{2i\pi}\oint_\Gamma \frac{f(w)dw}{w - z} = f(w)\Big|_{w=z} = f(z),$$

as expected. By the "additional remark" preceding, the foregoing result is still true for all $z \in D_4 = D_1 = \{z : |z| < R_1\}$.

Example 2: Let $z_0 = 0$, $A_n = a^n$, $B_n = b^n$ $(n = 0, 1, 2, \ldots)$, $A_{-n} = B_{-n} = 0$ $(n = 1, 2, \ldots)$, where $a \neq 0$, $b \neq 0$; without loss of generality, we may assume $|a| \le |b|$. Then $D_1 = \{z : |z| < 1/|a|\}$, $D_2 = \{z : |z| < 1/|b|\}$, $D_3 = \{z : |z| < |b|^{-2}\}$, $D_4 = \{z : |z| < 1/|ab|\}$, and $\Delta(z) = \{w : |bz| < |w| < 1/|b|\}$. It is clear that $f(z) = (1 - az)^{-1}$, $g(z) = (1 - bz)^{-1}$, and $h(z) = (1 - abz)^{-1}$, for all $z \in D_1$, D_2, and D_4, respectively; nevertheless, it will be instructive to derive this from the corollary. Applying the first formula in (17), we have:

$$h(z) = \frac{1}{2i\pi}\oint_\Gamma \frac{dw}{(1 - aw)(w - bz)}, \quad \forall\, z \in D_3.$$

Hence, since the points $w = bz$ and $w = 1/a$ are interior and exterior to Γ, respectively, we find upon applying the Cauchy integral theorem, that $h(z) = (1 - aw)^{-1}\Big|_{w=bz} = (1 - abz)^{-1}$, $\forall\, z \in D_3$. Again using the remark on analytic continuation, we obtain the anticipated result, namely $h(z) = (1 - abz)^{-1}\ \forall\, z \in D_4$.

Example 3: Let $f(z) = f(z,t) = \exp\{\frac{1}{2}t(z - z^{-1})\} = \sum_{n=-\infty}^{\infty} J_n(t)z^n$, the generating function of the Bessel functions of integral order. Similarly, let $g(z) = f(z,u)$. It is known that both series converge and represent analytic functions of z in the domain $D_1 = D_2 = D_3 = D_4 = \{z : 0 < |z| < \infty\}$, i.e., for all finite z except $z = 0$. In the nomenclature of the theorem's conditions, $z_0 = 0$, $r_1 = r_2 = 0$, $R_1 = R_2 = \infty$; hence, $\Delta(z) = \{w : 0 < |w| < \infty\}$. Thus, taking Γ as in the theorem, by the formula for the coefficients of a Laurent series:

(19) $$J_n(t) = \frac{1}{2i\pi}\oint_\Gamma \frac{\exp\{\frac{1}{2}t(w - w^{-1})\}}{w^{n+1}}\,dw,\quad n = 0, \pm 1, \pm 2, \ldots,$$

with a similar formula for $J_n(u)$, valid for all complex t (or u). Applying the theorem, we then have, for all $z \in D_3 = D_4$,

$$h(z) \equiv \sum_{n=-\infty}^{\infty} J_n(t)J_n(u)z^n = \frac{1}{2i\pi}\oint_\Gamma \exp\{\tfrac{1}{2}t(w - w^{-1})\}\exp\{\tfrac{1}{2}u(z/w - w/z)\}\cdot\frac{dw}{w},$$

or

(20) $$h(z) = \frac{1}{2i\pi}\oint_\Gamma \exp\{\tfrac{1}{2}(t - u/z)w\}\exp\{\tfrac{1}{2}(uz - t)w^{-1}\}\cdot\frac{dw}{w}.$$

We now make the substitution

$$w = a\xi, \text{ where } a = (t - uz)^{\frac{1}{2}}(t - u/z)^{-\frac{1}{2}},$$

and restrict z further so that $z \neq t/u$, $z \neq u/t$, which implies $a \in D_1$. Since, in the w-plane and ξ-plane, a is a constant, the above substitution transforms Γ into a topologically equivalent simple closed contour Γ' in the ξ-plane, which is still oriented in the positive direction. Therefore,

(21) $$h(z) = \frac{1}{2i\pi}\oint_{\Gamma'} \exp\{\tfrac{1}{2}b(\xi - \xi^{-1})\}\cdot\frac{d\xi}{\xi},\text{ where } b = (t - uz)^{\frac{1}{2}}(t - u/z)^{\frac{1}{2}}.$$

Comparing this last expression with (19), we see that

(22) $$h(z) = J_0(b).$$

Thus, we have proved the interesting identity

(23) $$\sum_{n=-\infty}^{\infty} J_n(t)J_n(u)z^n = J_0\{(t - uz)^{\frac{1}{2}}(t - u/z)^{\frac{1}{2}}\},\quad \forall z \neq 0.$$

Note that (23) is valid also for the previously excluded values $z = t/u$ and $z = u/t$, provided $t \neq 0$, $u \neq 0$ (by analytic continuation). Therefore, we obtain the following formulas, as special cases of (23):

(24) $$\sum_{n=-\infty}^{\infty} J_n(t)J_n(u)(t/u)^n = 1,$$

(25) $$\sum_{n=-\infty}^{\infty} J_n(t)J_n(u) = J_0(t - u),\quad \forall t,u \neq 0.$$

The identity given in (23) is not in itself new, appearing (in variant form), e.g., in [3].

Example 4: Let $f_m(z)$ be the generating function for the mth powers of the Fibonacci numbers ($m = 1, 2, \ldots$), i.e.,

(26) $$f_m(z) = \sum_{n=0}^{\infty} F_n^m z^n,\text{ valid for all } z \in D_2 = \{z : |z| < \alpha^{-m}\}$$

[in this example, $\alpha = \frac{1}{2}(1 + \sqrt{5})$, $\beta = \frac{1}{2}(1 - \sqrt{5})$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$]. We let $f(z) = f_1(z) = z(1 - z - z^2)^{-1} = 5^{-\frac{1}{2}}\{(1 - \alpha z)^{-1} - (1 - \beta z)^{-1}\}$, and $g = f_m$ in the corollary, with $R_1 = \alpha^{-1}$, $R_2 = \alpha^{-m} = \rho_2$; then $D_3 = \{z : |z| < \alpha^{-2m}\}$, $D_4 = \{z : |z| < \alpha^{-m-1}\}$, and $\Delta(z) = \{w : |z|\alpha^m < |w| < \alpha^{-m}\}\ \forall z \in D_3$. We see readily, from (7), that $h(z) = f_{m+1}(z)$. Choosing Γ in $\Delta(z)$, we note that it contains the points $w = \alpha z$ and $w = \beta z$ in its interior, since $|\beta z| = \alpha^{-1}|z| < \alpha|z| \leq \alpha^m|z| < |w|\ \forall w \in \Gamma$. Applying the corollary, we thus have, for $m = 1, 2, \ldots$:

$$f_{m+1}(z) = \frac{1}{2i\pi}\oint_\Gamma f_m(w)f_1(z/w) \cdot \frac{dw}{w} = \frac{1}{2i\pi}\oint_\Gamma 5^{-\frac{1}{2}}f_m(w)\{(w - \alpha z)^{-1} - (w - \beta z)^{-1}\}dw,$$

which reduces to the elegant recursion

$$f_{m+1}(z) = 1/\sqrt{5}\{f_m(\alpha z) - f_m(\beta z)\}, \tag{27}$$

which is actually valid for all $z \in D_4$, $m = 0, 1, 2, \ldots$.

Of course, (27) may readily be derived using more elementary techniques, but the item of interest here is the method by which it was derived. Without too much difficulty, induction may be used on (27) to derive the partial fraction decomposition of $f_m(z)$, which is given by:

$$f_m(z) = 5^{-\frac{1}{2}m}\sum_{k=0}^{m}(-1)^k\binom{m}{k}(1 - \alpha^{m-k}\beta^k z)^{-1}. \tag{28}$$

This is a variant of a result in [4].

<u>*Example 5*</u>: Recall the generating function of the Legendre polynomials,

$$f(z) = f(z,t) = (1 - 2tz + z^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} P_n(t)z^n. \tag{29}$$

The radius of convergence of this series depends on t and complicates the subsequent computations. Therefore, we will assume that the indicated operations, throughout this example, are legitimate and we will not attempt to justify them. It may be shown that, for appropriately chosen t and z, full rigor may be obtained. Let

$$g(z) = e^z = \sum_{n=0}^{\infty}\frac{z^n}{n!}. \tag{30}$$

Then, for an appropriately chosen contour Γ, the corollary implies that

$$h(z) \equiv \sum_{n=0}^{\infty} P_n(t)\frac{z^n}{n!} = \frac{1}{2i\pi}\oint_\Gamma \frac{e^w dw}{(w^2 - 2tzw + z^2)^{\frac{1}{2}}}. \tag{31}$$

In order to evaluate the last integral, we make the substitution $w = tz + \frac{1}{2}z(1 - t^2)^{\frac{1}{2}}(\xi - \xi^{-1})$. For suitably chosen t, z, and Γ, this mapping transforms Γ into another simple closed contour Γ' with sufficiently desirable properties. Proceeding formally, we obtain, after some simplification,

$$h(z) = \frac{1}{2i\pi}\oint_{\Gamma'} e^{tz}\exp\{\tfrac{1}{2}z(1 - t^2)^{\frac{1}{2}}(\xi - \xi^{-1})\}\frac{d\xi}{\xi}. \tag{32}$$

The quantity e^{tz} may be factored out of the integrand and the remaining expression, compared (again) with (19), yields the following identity:

$$\sum_{n=0}^{\infty} P_n(t)\frac{z^n}{n!} = e^{tz}J_0(z\sqrt{1 - t^2}). \tag{33}$$

The last identity is actually valid for all finite z, since the right-hand member of (33) is clearly an entire function. The identity in (33) is indicated in [5], along with the comment that it is of unknown origin.

The foregoing examples adequately illustrate the applicability of the theorem and its corollary, to obtain a closed form expression for $h(z)$. This may be immediately obvious, or may require an appropriate transformation and/or recognition of known relations, as the previous examples illustrate. If this is possible (and this may not always be the case), a certain degree of ingenuity is required to hit upon the proper transformation. With sufficient imagination and industry, the interested reader will discover other relations of the types illustrated above. The aim of this paper was to obtain a solution of Gould's problem, in closed form or otherwise, and this has been accomplished by the theorem and its corollary.

REFERENCES

1. H. W. Gould. "Some Combinatorial Identities of Bruckman—A Systematic Treatment with Relation to the Older Literature." *The Fibonacci Quarterly* 10, No. 6 (1972):625-626.
2. Paul S. Bruckman. "On Generating Functions with Composite Coefficients." *The Fibonacci Quarterly* 15, No. 3 (1977):269-275.

3. E. T. Whittaker and G. N. Watson. *A Course of Modern Analysis*. Cambridge, 1948, p. 380.
4. John Riordan. "Generating Functions for Powers of Fibonacci Numbers." *Duke Math. J.* 29 (1962):5-12.
5. Earl D. Rainville. *Special Functions*. Chelsea, 1960, p. 165.

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(A) The digital root of 1979 is 8, which also is the sum of the two absent odd digits, 3 and 5. Otherwise, $F_4 + F_5 = F_6$.

$1 \cdot 9 \cdot 7 \cdot 9 = 567$, three consecutive digits in ascending order.

$1^9 \cdot 7^9 = 40353607$, which contains five consecutive digits.

1979 is a cyclic compression of two palindromes—the composite 979 (= 11 • 89) and the prime 919.

(B) $1979_{10} = 118E_{12} = 153X_{11} = 2638_9 = 3673_8 = 5525_7 = 13055_6$
$= 3044_5 = 132323_4 = 2201022_3 = 11110111011_2.$

In base four, the integer is almost smoothly undulating. In base three, the palindromic integer contains the three distinct digits in that base. In base two, the groups of 1's form a decreasing sequence.

(C)
$$\begin{aligned}1979 &= (11)(11)(11) + [(111 - 1)/(1 + 1) - 1](11 + 1)\\ &= 2222 - 222 - 22 + 2/2\\ &= (333 - 3)(3!) - 3/3\\ &= 4(444 + 44 + 4 + 4) - 4 - 4/4\\ &= 5 \cdot 5 \cdot 5 \cdot 5 + 5 \cdot 5 \cdot 5 \cdot 5 + 555 + 5 \cdot 5 \cdot 5 + 55 - 5 - 5/5\\ &= 6 \cdot 6 \cdot 6 \cdot 6 + 666 + 6 + 6 + 6 - 6/6\\ &= 7 \cdot 7 \cdot 7 \cdot 7 - 7 \cdot 7 \cdot 7 - 77 - 7/7 - 7/7\\ &= 888 + 888 + 88 + 88 + 8 + 8 + 8 + 8/8 + 8/8 + 8/8\\ &= (9999 - 999)/9 + 999 - 9 - 9 - 9/9 - 9/9\end{aligned}$$

(D)
$$\begin{aligned}1 + 9 + 7 + 9 &= 26\\ 19 + 97 + 79 + 91 &= 286\\ 197 + 979 + 791 + 919 &= 2886\\ 1979 + 9791 + 7919 + 9197 &= 28886\end{aligned}$$

(E) Here are several of the ways that 1979 can be written using conventional mathematical symbols and one 1, nine 9's, seven 7's, and nine 9's.

$$\begin{aligned}1979 &= 1(999 + 9997/9997) + 9(99 + 779/779) + 7(9 + 9/9) + 9\\ &= 1(999 + 9/9) + 9(99 + 9/9) + 7(9 + 9/9) + 9(99777/99777)\\ &= 19(99 + 99999/99999) + 7(\sqrt{9}\sqrt{9} + 7779/7779) + 9\\ &= 197(9 + 777/777) + \sqrt{9}\sqrt{9}(9999999/9999999)\\ &= 1(999 + 9/9) + \sqrt{9}\sqrt{9}(99 + 7/7) + 7(77/77 + 9) + 9 + 9(999 - 999)\end{aligned}$$

In the last expression, the digit groups are intact and in the order of occurrence in 1979.

(F) 19 • 79 = 1501 is one of eleven composite integers between the primes 1499 and 1511. Consequently, it is the corner element of the following third-order magic square composed of composite elements and having a magic constant of $4512 = 2 \cdot 47 \cdot 48 = 2^5 \cdot 3 \cdot 47$.

1501	1506	1505		$19 \cdot 79$	$2 \cdot 3 \cdot 251$	$5 \cdot 7 \cdot 43$
1508	1504	1500	or	$2^2 \cdot 13 \cdot 29$	$2^5 \cdot 47$	$2^2 \cdot 3 \cdot 5^3$
1503	1502	1507		$3^2 \cdot 167$	$2 \cdot 751$	$11 \cdot 137$

(G)
$$\begin{aligned}1979 &= 1979 + 1 + \sqrt{9} - 7 + \sqrt{9}\\ &= 1979(-1\sqrt{9} + 7 - \sqrt{9})\end{aligned}$$

(continued)

$= 197 \cdot 9 + 197 + 9$
$= 19 \cdot 79 + 1 \cdot 9 \cdot 7 \cdot 9 - 1 - 9 - 79$
$= 1 \cdot 979 + 1 + 97 \cdot 9 - 1 + 97 + 9 + 19 - 7 + 9$

(*H*) $1979 = 2 \cdot 3 - 4 + 5 - 6 + 1978$
$= 3(729) - 4(58 - 6)(1)$
$= 59 + 31 \cdot 62 - (8 - 7)\sqrt{4}$
$= 28 + 5(396) - 4 \cdot 7 - 1$
$= 1 \cdot 4 \cdot 5 + 6(329) - 7 - 8$

$1979 = 1 - 58 + 7! - 94 - 3026$
$= 403(2 + 8 - 9) + 1576$
$= 10 - 4 - 5 + 6 + 7 \cdot 283 - 9$
$= 9 \cdot 201 + 5[38 - 4(7 - 6)]$
$= 1098 + 2 \cdot 473 - 65$

(*I*) $1979 = 43 + 44^2 = 45^2 - 46$

$1979 = F_5 + F_{14} + F_{17} = L_6 + L_9 + L_{13} + L_{15}$

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101 FACES OF 1979

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$0 = 1 + 9 - 7 - \sqrt{9}$
$1 = 1(\sqrt{9}) + 7 - 9$
$2 = 1 - \sqrt{9} + 7 - \sqrt{9}$
$3 = -1 + \sqrt{9} + 7 - (\sqrt{9})!$
$4 = (1 - 9)/(7 - 9)$
$5 = 1 \cdot 9 - 7 + \sqrt{9}$
$6 = -1 + \sqrt{9} + 7 - \sqrt{9}$
$7 = 1 \cdot 9 + 7 - 9$
$8 = 1 + 9 + 7 - 9$
$9 = (-1 + 9 - 7)9$
$10 = -1 \cdot \sqrt{9} + 7 + (\sqrt{9})!$
$11 = 1 \cdot 9 - 7 + 9$
$12 = 1 + 9 - 7 + 9$
$13 = 1 \cdot \sqrt{9} + 7 + \sqrt{9}$
$14 = 1 + 9 + 7 - \sqrt{9}$
$15 = 19 - 7 + \sqrt{9}$
$16 = 1 \cdot \sqrt{9} + 7 + (\sqrt{9})!$
$17 = 1 + \sqrt{9} + 7 + (\sqrt{9})!$
$18 = -1 + 9 + 7 + \sqrt{9}$
$19 = 1 \cdot \sqrt{9} + 7 + 9$
$20 = 1 + 9 + 7 + \sqrt{9}$
$21 = 19 - 7 + 9$
$22 = (1 + \sqrt{9} + 7)(!\sqrt{9})$
$23 = 19 + 7 - \sqrt{9}$
$24 = -1 + 9 + 7 + 9$
$25 = 1 \cdot 9 + 7 + 9$
$26 = 1 + 9 + 7 + 9$
$27 = 1 \cdot \sqrt{9} \cdot 7 + (\sqrt{9})!$
$28 = 1 + \sqrt{9} \cdot 7 + (\sqrt{9})!$
$29 = -1 + \sqrt{9} \cdot 7 + 9$
$30 = 1 \cdot \sqrt{9} \cdot 7 + 9$
$31 = 1 + \sqrt{9} \cdot 7 + 9$
$32 = -1 - 9 + 7(\sqrt{9})!$
$33 = -1 \cdot 9 + 7(\sqrt{9})!$
$34 = 1 - 9 + 7(\sqrt{9})!$
$35 = 19 + 7 + 9$
$36 = -1(\sqrt{9})! + 7(\sqrt{9})!$
$37 = (1 + \sqrt{9})7 + 9$
$38 = 19(-7 + 9)$
$39 = -1 \cdot \sqrt{9} + 7(\sqrt{9})!$
$40 = 1 - \sqrt{9} + 7(\sqrt{9})!$
$41 = -1^9 + 7(\sqrt{9})!$
$42 = 1^9 \cdot 7(\sqrt{9})!$
$43 = 1^9 + 7(\sqrt{9})!$
$44 = -1 + \sqrt{9} + 7(\sqrt{9})!$
$45 = 1 \cdot \sqrt{9} + 7(\sqrt{9})!$
$46 = 1 + \sqrt{9} + 7(\sqrt{9})!$
$47 = -1 + (\sqrt{9})! + 7(\sqrt{9})!$
$48 = 1 \cdot (\sqrt{9})! + 7(\sqrt{9})!$
$49 = 1 + (\sqrt{9})! + 7(\sqrt{9})!$
$50 = -1 + 9 + 7(\sqrt{9})!$
$51 = 1 \cdot 9 + 7(\sqrt{9})!$
$52 = 1 + 9 + 7(\sqrt{9})!$
$53 = -1 - 9 + 7 \cdot 9$
$54 = (-1 + 9)7 - !\sqrt{9}$
$55 = (-1 + 9)7 - !(!\sqrt{9})$
$56 = -1 + 9 \cdot 7 - (\sqrt{9})!$
$57 = 1 \cdot 9 \cdot 7 - (\sqrt{9})!$
$58 = 1 + 9 \cdot 7 - (\sqrt{9})!$
$59 = -1 - \sqrt{9} + 7 \cdot 9$
$60 = -19 + 79$
$61 = 1 - \sqrt{9} + 7 \cdot 9$
$62 = 1 + 9 \cdot 7 - !\sqrt{9}$
$63 = -1 + 9 \cdot 7 + !(!\sqrt{9})$
$64 = -1 + 9 \cdot 7 + !\sqrt{9}$
$65 = (-1 + 9)7 + 9$
$66 = 1 \cdot 9 \cdot 7 + \sqrt{9}$
$67 = 1 + \sqrt{9} + 7 \cdot 9$
$68 = -1 + 9 \cdot 7 + (\sqrt{9})!$
$69 = -1 - 9 + 79$
$70 = -1 \cdot 9 + 79$
$71 = -1 + 9 + 7 \cdot 9$
$72 = -1 - (\sqrt{9})! + 79$
$73 = 1 + 9 + 7 \cdot 9$
$74 = 1 - (\sqrt{9})! + 79$
$75 = -1 - \sqrt{9} + 79$
$76 = -1 \cdot \sqrt{9} + 79$
$77 = 1 - \sqrt{9} + 79$
$78 = 1 - !\sqrt{9} + 79$
$79 = 1^9 \cdot 79$
$80 = -1 + !\sqrt{9} + 79$
$81 = -1 + \sqrt{9} + 79$
$82 = 1 \cdot \sqrt{9} + 79$
$83 = 1 + \sqrt{9} + 79$
$84 = -1 + (\sqrt{9})! + 79$
$85 = 1 \cdot (\sqrt{9})! + 79$
$86 = 1 + (\sqrt{9})! + 79$
$87 = -1 + 9 + 79$
$88 = 1 \cdot 9 + 79$
$89 = 1 + 9 + 79$
$90 = -1 + 97 - (\sqrt{9})!$
$91 = 1 \cdot 97 - (\sqrt{9})!$
$92 = 1 + 97 - (\sqrt{9})!$
$93 = -1 + 97 - \sqrt{9}$
$94 = 1 \cdot 97 - \sqrt{9}$
$95 = 1 + 97 - \sqrt{9}$
$96 = 1 + 97 - !\sqrt{9}$
$97 = 1 \cdot 97[!(!\sqrt{9})]$
$98 = 19 + 79$
$99 = -1 + 97 + \sqrt{9}$
$100 = 1 \cdot 97 + \sqrt{9}$

In each of these expressions, 1, 9, 7, and 9 appear in the same order as they do in the year. The symbol $!x$ represents "sub-factorial x." Thus, $!3 = 2$ and $!2 = 1$.

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DETERMINANTS RELATED TO 1979

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$$\begin{vmatrix}1 & 9\\ 7 & 9\end{vmatrix} = -2 \cdot 3^3 \quad \text{and} \quad \begin{vmatrix}1 & 9\\ 9 & 7\end{vmatrix} = -2 \cdot 37.$$

$$\begin{vmatrix}1 & 9 & 7 & 9\\ 9 & 1 & 9 & 7\\ 7 & 9 & 1 & 9\\ 9 & 7 & 9 & 1\end{vmatrix} = -2^4 \cdot 3^2 \cdot 65 = -\begin{vmatrix}1 & 9 & 7 & 9\\ 9 & 7 & 9 & 1\\ 7 & 9 & 1 & 9\\ 9 & 1 & 9 & 7\end{vmatrix}.$$

$$\begin{vmatrix}1 & 9 & 7 & 9\\ 9 & 7 & 9 & 0\\ 7 & 9 & 0 & 0\\ 9 & 0 & 0 & 0\end{vmatrix} = 3^8 \cdot \begin{vmatrix}1 & 9 & 7 & 9\\ 0 & 1 & 9 & 7\\ 0 & 0 & 1 & 9\\ 0 & 0 & 0 & 1\end{vmatrix}.$$

$$\begin{vmatrix}1 & 9 & 7 & 9\\ 9 & 7 & 9 & 7\\ 7 & 9 & 7 & 9\\ 9 & 7 & 9 & 1\end{vmatrix} = -6^2(9^2 - 7^2) = -2^7 \cdot 3^2.$$

$$D = \begin{vmatrix}1 & 9 & 7 & 9\\ 9 & a & a & 7\\ 7 & a & a & 9\\ 9 & 7 & 9 & 1\end{vmatrix} = (9^2 - 7^2) - 80a = 2^4(2^6 - 5a),$$

so for pertinent values of a, we have

a	0	1	7	9
D	2^{10}	$2^4 \cdot 59$	$2^4 \cdot 29 = 464$	$2^4 \cdot 19$

$$\begin{vmatrix}9 & 1 & 9\\ 7 & x & 7\\ 9 & 1 & 9\end{vmatrix} = 0, \quad \text{and } d = \begin{vmatrix}1 & 9 & 7\\ 9 & b & 9\\ 7 & 9 & 1\end{vmatrix} = 12(81 - 4b),$$

so for pertinent values of b, we have

b	0	1	7	9
d	$2^3 \cdot 3^5$	$2^2 \cdot 3 \cdot 79$	$2^2 \cdot 3 \cdot 59$	$2^2 \cdot 3^3 \cdot 5$

$$-\begin{vmatrix}1 & 2 & 3\\ 8 & 5 & 4\\ 6 & 7 & 9\end{vmatrix} = 1, \quad \begin{vmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 7 & 9 & 8\end{vmatrix} = 9, \quad \begin{vmatrix}1 & 2 & 3\\ 4 & 5 & 8\\ 6 & 7 & 9\end{vmatrix} = 7, \quad \begin{vmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 8 & 9 & 7\end{vmatrix} = 9.$$

The first nine-digit determinant becomes the third upon change of sign and reversal of the second row; the third becomes the second upon a one-step cyclic permutation of the digits 6, 7, 9, 8; and the second becomes the fourth upon reversal of the third row.

$$\begin{vmatrix}1 & 4 & 8\\ 9 & 2 & 7\\ 5 & 6 & 3\end{vmatrix} + \begin{vmatrix}1 & 4 & 8\\ 7 & 2 & 6\\ 5 & 9 & 3\end{vmatrix} + \begin{vmatrix}1 & 4 & 7\\ 8 & 2 & 6\\ 5 & 9 & 3\end{vmatrix} + \begin{vmatrix}1 & 4 & 7\\ 9 & 2 & 6\\ 5 & 8 & 3\end{vmatrix} + \begin{vmatrix}1 & 4 & 7\\ 8 & 2 & 5\\ 6 & 9 & 3\end{vmatrix}\begin{matrix} & \\ 8 & \\ 6 & 9\end{matrix}$$

$$= 348 + 412 + 410 + 404 + 405 = 1979.$$

The first nine-digit determinant becomes the second upon a one-step counter-clockwise rotation of the 6, 7, 9 configuration; the second becomes the third upon interchange of 7 and 8; the third becomes the fourth upon interchange of 8 and 9; the fourth becomes the fifth upon a two-step rotation of the 9, 6, 8, 5 configuration. In the last determinant, the nine digits are in order of magnitude along a main diagonal and the two broken diagonals parallel to it.

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REITERATIVE ROUTINES APPLIED TO 1979

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(A) Sum the digits of the integer.

1 + 9 + 7 + 9 = 26, 2 + 6 = 8, the digital root of 1979.

(B) Compute the absolute difference of the integer and its reverse.

1979	7812	5625	360	297	495	99
9791	2187	5265	063	792	594	99
7812	5625	360	297	495	99	0

Seven operations to reach the inevitable 0.

(C) Add the integer and its reverse.

1979	11770	19481	37972	65945	120901
9791	07711	18491	27973	54956	109021
11770	19481	37972	65945	120901	229922

Six operations to reach a palindrome. Continuing the procedure for 18 more operations produces the palindrome 8813200023188.

(D) The Kaprekar routine. Arrange the digits in descending order, and from it subtract its reverse.

9971	8721	7443	9963	6642	7641
1799	1278	3447	3699	2466	1467
8172	7443	3996	6264	4176	6174

Six operations to reach Kaprekar's constant, the self-replicating 6174.

(E) The Collatz algorithm. If it is odd, triple it and add 1; if it is even, divide it by 2.

1979	530	143	233	1132	911	122
5938	265	430	700	566	2734	61
2969	796	215	350	283	1367	184
8908	398	646	175	850	4102	92
4454	199	323	526	425	2051	46
2227	598	970	263	1276	6154	23
6682	299	485	790	638	3077	70
3341	898	1456	395	319	9232	35
10024	449	728	1186	958	4616	106
5012	1348	364	593	479	2308	53
2506	674	182	1780	1438	1154	160
1253	337	91	890	719	577	80
3760	1012	274	445	2158	1732	40
1880	506	137	1336	1079	866	20
940	253	412	668	3238	433	10
470	760	206	334	1619	1300	5
235	380	103	167	4858	650	16
706	190	310	502	2429	325	8
353	95	155	251	7288	976	4
1060	286	466	754	3644	488	2
			377	1822	244	1

It takes 143 operations to reach the inevitable 1.

(F) 1979 is part of a ten-digit multiplicative bracelet wherein each element is the units' digit of the product of the four preceding digits, namely:

1 9 7 9 7 9 9 3 1 3'1 9 7 9.

(G) 1979 is part of a 1560-digit additive bracelet wherein each element is the units' digit of the sum of the four preceding digits, namely:

19796 13992 33758 33938 33714 ... .

The complete bracelet is included in "A Digital Bracelet for 1967," *The Fibonacci Quarterly* 5 (1967):477-480.

(H) Add the squares of the digits of the integers. $1^2 + 9^2 + 7^2 + 9^2 = 212$. Subsequent terms in the sequence are 9, 81, 65, 61, 37, 58, 89, 145, 42, 20, 4, 16, 37. Six operations to enter an eight-member loop.

(I) Add the cubes of the digits of the integers. $1^3 + 9^3 + 7^3 + 9^3 = 1802$, followed by 521, 134, 92, 737, 713, 371. Seven operations to reach the self-replicating 371.

(J) Add the fourth powers of the digits of the integers. $1^4 + 9^4 + 7^4 + 9^4 = 15524$, then 1523, 723, 2498, 10929, 13139, 6725, 4338, 4514, 1138, 4179, 9219, 13139. Six operations to enter a seven-member loop.

(K) Add the squares of the odd digits to the sum of the even digits. 1979, 212, 5, 25, 27, 51, 26, 8. Seven operations to reach the self-replicating 8.

(L) Add the squares of the even digits to the sum of the odd digits. 1979, 26, 40, 16, 37, 10, 1. Six operations to reach the self-replicating 1.

(M) Add the squares of the composite digits to the sum of the other digits. $1 + 9^2 + 7 + 9^2 = 170$, then 8, 64, 52, 7. Five operations to reach the self-replicating 7.

(N) Add the composite digits to the sum of the squares of the other digits. $1^2 + 9 + 7^2 + 9 = 68$, then 14, 5, 25, 29, 13, 10, 1. Eight operations to reach the self-replicating 1.

(O) For a four-digit integer $abcd$, compute $a^4 + b^3 + c^2 + d$. $1^4 + 9^3 + 7^2 + 9 = 788$, then 415, 70, 49, 25, 9. Six operations to reach the self-replicating 9.

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1979 AND ASSOCIATED PRIMES

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(A) The prime 1979, which contains only one prime digit, is a concatenation of the two primes 19 and 79. Of the seven different two-digit integers that can be formed from the digits of 1979, five are primes. Their sum, 17 + 19 + 71 + 79 + 97 = 283, a prime.

Of the twelve different three-digit integers that can be formed from the digits of 1979, eight are prime. These include two sets consisting of cyclic permutations. The sum of the eight, 197 + 971 + 719 + 199 + 991 + 919 + 179 + 997 = 5172 = 431 • 3 • 4, a palindromic arrangement. Two of the composite integers that are cyclic permutations have factors that are cyclic permutations; that is, 791 = 7 • 113 and 917 = 7 • 131.

Of the 4! permutations of the digits of 1979, five form prime integers: 1979, 1997, 7919, 9719, and 9791.

(B) Since 79 - 19 = 60, both 19 and 79 are members of eleven arithmetic progressions with common differences of d = 1, 2, 3, 4, 5, 6, 10, 12, 15, 20, and 30, respectively. In eight of these, the square 49 is the middle term. Two of these progressions are worthy of note. In

19 31 43 55 67 79

only one term is not a prime, and it is the product of the alternate primes 5 and 11. The other progression

19 25 31 37 43 49 55 61 67 73 79

contains eight primes, two squares, and the product $5 \cdot 11$.

(C)

$2 = 1 - \sqrt{9} + 7 - \sqrt{9}$
$3 = -1 + \sqrt{9} + 7 - (\sqrt{9})!$
$5 = 1 \cdot 9 - 7 + \sqrt{9}$
$7 = 1 \cdot 9 + 7 - 9$
$11 = 1 \cdot 9 - 7 + 9$
$13 = 1\sqrt{9} + 7 + (\sqrt{9})$
$17 = 1 + \sqrt{9} + 7 + (\sqrt{9})!$
$19 = 1\sqrt{9} + 7 + 9$
$23 = 1 + 9 + 7 + (\sqrt{9})!$
$29 = -1 + (\sqrt{9})(7) + 9$
$31 = 1 + (\sqrt{9})(7) + 9$
$37 = (1 + \sqrt{9})7 + 9$
$41 = -1^9 + 7(\sqrt{9})!$
$43 = 1^9 + 7(\sqrt{9})!$
$47 = -1 + (\sqrt{9})! + 7(\sqrt{9})!$
$53 = -1 - 9 + 7 \cdot 9$
$59 = -1 - \sqrt{9} + 7 \cdot 9$
$61 = 1 - \sqrt{9} + 7 \cdot 9$
$67 = 1 + \sqrt{9} + 7 \cdot 9$
$71 = -1 + 9 + 7 \cdot 9$
$73 = 1 + 9 + 7 \cdot 9$
$79 = 1^9 \cdot 79$
$83 = 1 + \sqrt{9} + 79$
$89 = 1 + 9 + 79$
$97 = 1 \cdot 97 \cdot [!(!\sqrt{9})]$

In each of the expressions of the primes < 100, the digits of 1979 are in the same order as in the year. $!x$ is "sub-factorial x." $!3 = 2$ and $!2 = 1$.

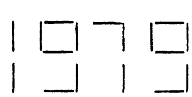
(D) In each of the following sums of distinct primes equal to 1979, the primes are consecutive with the exception of the primes in parentheses.

$1979 = (5) + 983 + 991$
$= (23) + 479 + 487 + 491 + 499$
$= (23) + 311 + 313 + 317 + 331 + 337 + 347$
$= (79) + 223 + 227 + 229 + 233 + 239 + 241 + 251 + 257$
$= (61) + 131 + 137 + 139 + 149 + 151 + 157 + 163 + 167 + 173 + 179 + 181 + 191$
$= (53) + 103 + 107 + 109 + 113 + 127 + 131 + 137 + 139 + 149 + 151 + 157 + 163 + 167 + 173$
$= (23) + 83 + 89 + 97 + 101 + 103 + 107 + 109 + 113 + 127 + 131 + 137 + 139 + 149 + 151 + 157 + 163$
$= (53) + 67 + 71 + 73 + 79 + 83 + 89 + 97 + 101 + 103 + 107 + 109 + 113 + 127 + 131 + 137 + 139 + 149 + 151$
$= (31) + 53 + 59 + 61 + 67 + 71 + 73 + 79 + 83 + 89 + 97 + 101 + 103 + 107 + 109 + 113 + 127 + 131 + 137 + 139 + 149$
$= (3 + 5) + 11 + 13 + 17 + 19 + 23 + 29 + 31 + 37 + 41 + 43 + 47 + 53 + 59 + 61 + 67 + 71 + 73 + 79 + 83 + 89 + 97 + 101 + 103 + 107 + 109 + 113 + 127 + 131 + 137$
$= 2 + 3 + 5 + 7 + 11 + 13 + 17 + 19 + 23 + 29 + 31 + 37 + 41 + 43 + 47 + 53 + 59 + 61 + 67 + 71 + 73 + 79 + 83 + 89 + 97 + 101 + 103 + 107 + 109 + (499)$
$= 2 + 3 + 5 + 7 + 11 + 13 + 17 + 19 + 23 + 29 + 31 + 37 + 41 + 43 + 47 + 53 + 59 + 61 + 67 + 71 + 73 + 79 + 83 + 89 + 97 + (919)$
$= 2 + 3 + 5 + 7 + 11 + 13 + 17 + 19 + 23 + 29 + 31 + 37 + 41 + (1741)$
$= 2 + 3 + 5 + 7 + 11 + 13 + 17 + 19 + 23 + (1879)$
$= 2 + 3 + 5 + 7 + 11 + (1951)$

(E)

$1979 = 3 \cdot 5 + 19 \cdot 43 + 31 \cdot 37$
$= 5 \cdot 7 + 11 \cdot 137 + 19 \cdot 23$
$= 5 \cdot 67 + 11 \cdot 13 + 19 \cdot 79$
$= 5 \cdot 79 + 19 \cdot 23 + 31 \cdot 37$
$= 7 \cdot 11 + 17 \cdot 59 + 29 \cdot 31$

(F) A prime number, 17, of toothpicks can be assembled into



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THE POWERFULL 1979

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(A) $1979 = 990^2 - 989^2$

(B) $1979 = 3^2 + 11^2 + 43^2 = 3^2 + 17^2 + 41^2$

$= 2^2 + 5^2 + 7^2 + 11^2 + 13^2 + 17^2 + 19^2 + 31^2$

(C) $1979 = 5^2 + 27^2 + 35^2$

$= 7^2 + 29^2 + 33^2$

$= 1^2 + 4^2 + 21^2 + 39^2$

$= 3^2 + 5^2 + 24^2 + 37^2$

$= 3^2 + 7^2 + 25^2 + 36^2$

$= 1^2 + 3^2 + 6^2 + 13^2 + 42^2$

$= 1^2 + 4^2 + 5^2 + 16^2 + 41^2$

$= 2^2 + 7^2 + 17^2 + 26^2 + 31^2$

$= 1^2 + 2^2 + 3^2 + 5^2 + 28^2 + 34^2$

$= 1^2 + 3^2 + 4^2 + 5^2 + 22^2 + 38^2$

$= 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 18^2 + 40^2$

$= 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 30^2 + 32^2$

$= 1^2 + 2^2 + 6^2 + 8^2 + 10^2 + 19^2 + 20^2 + 22^2 + 23^2$

$= 3^2 + 4^2 + 6^2 + 7^2 + 8^2 + 9^2 + 11^2 + 12^2 + 13^2 + 14^2 + 15^2 + 16^2 + 17^2 + 18^2$

These expressions, that involve the squares of all positive integers < 44, are just a few examples chosen from the multitude of partitions of 1979 into squares.

(D) $1979 = 2^3 + 3^3 + 6^3 + 12^3$

$= 1^1 + 13^2 + 8^3 + 6^4 + 1^5$

$= 2^0 + 2^1 + 2^3 + 2^4 + 2^5 + 2^7 + 2^8 + 2^9 + 2^{10}$

$= 2^{11} - 2^6 - 2^2 - 2^0$

$= -3^0 + 3^2 + 3^3 - 3^5 + 3^7$

$= 1^3 + 9^3 + 7^3 + 9^3 + 1^1 + 9^2 + 7^1 + 9^2 + 1 \cdot 9 \cdot 7/9$

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AN OBSERVATION CONCERNING WHITFORD'S "BINET'S FORMULA GENERALIZED"

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In [1], Whitford generalizes the Fibonacci sequence by modifying the defining equations of the Fibonacci sequence by letting

$$G_n = \frac{[(1 + \sqrt{p})/2]^n - [(1 - \sqrt{p})/2]^n}{\sqrt{p}} \qquad (n \geq 1).$$

This leads to a sequence whose defining equations are $G_1 = G_2 = 1$,

$$G_{n+2} = G_{n+1} + [(p - 1)/4]G_n \qquad (n \geq 1).$$

One can also use Whitford's Generalization of Binet's formula to obtain a generalization of the Lucas sequence. From [2], $L_n = \alpha^n + \beta^n$ $(n \geq 1)$, where $\alpha = (1 + \sqrt{5})/2$ and $\beta = (1 - \sqrt{5})/2$. By using Whitford's α and β, the Lucas sequence can be generalized by a sequence H_n, where

$$H_n = [(1 + \sqrt{p})/2]^n + [(1 - \sqrt{p})/2]^n \qquad (n \geq 1).$$

Now, since α and β satisfy $x^2 - x - [(p - 1)/4] = 0$,

$$\begin{aligned} H_{n+2} &= \alpha^{n+2} + \beta^{n+2} = \alpha^n(\alpha^2) + \beta^n(\beta^2) = \alpha^n(\alpha + [(p - 1]/4]) + \beta^n(\beta + [(p - 1)/4]) \\ &= \alpha^{n+1} + \beta^{n+1} + [(p - 1)/4](\alpha^n + \beta^n) = H_{n+1} + [(p - 1)/4]H_n. \end{aligned}$$

Furthermore, $H_1 = (1 + \sqrt{p})/2 + (1 - \sqrt{p})/2 = 1$ and

$$H_2 = [(1 + \sqrt{p})/2]^2 + [(1 - \sqrt{p})/2]^2 = (p + 1)/2.$$

Thus, the analog of Whitford's generalization of the Fibonacci sequence is the generalization of the Lucas sequence,

$$H_1 = 1,\ H_2 = (p + 1)/2,\ H_{n+2} = H_{n+1} + [(p - 1)/4]H_n \qquad (n \geq 1).$$

Note that, of course, the Lucas sequence corresponds to the case $p = 5$.

The following table, analogous to Whitford's gives the first ten terms of the sequences corresponding to the first five positive integers of the form $4k + 1$.

p	$\frac{p-1}{4}$	G_1	G_2	G_3	G_4	G_5	G_6	G_7	G_8	G_9	G_{10}
1	0	1	1	1	1	1	1	1	1	1	1
5	1	1	3	4	7	11	18	29	47	76	123
9	2	1	5	7	17	31	65	127	257	511	1025
13	3	1	7	10	31	61	154	337	799	1810	4207
17	4	1	9	13	49	101	297	701	1889	4693	12249

The following are some of the identities satisfied by the sequences H_n and G_n.

(1) $$\lim_{n\to\infty} \frac{H_{n+1}}{H_n} = (1 + \sqrt{p})/2,$$

(2) $$G_{2n} = G_n H_n,$$

(3) $$H_n^2 = H_{2n} + 2[(1 - p)/4]^n,$$

(4) $$H_n = G_{n+1} + [(p - 1)/4]G_{n-1},$$

(5) $$pG_n^2 = H_{2n} - 2[(1 - p)/4]^n.$$

The major change in the generalized identities occurs where $\alpha\beta = -1$ appears in the Fibonacci/Lucas identities, with $\alpha\beta = (1 - p)/4$ in their generalizations.

REFERENCES

1. A. K. Whitford. "Binet's Formula Generalized." *The Fibonacci Quarterly* 15 (1977):21.
2. V. E. Hoggatt, Jr. *Fibonacci and Lucas Numbers*. Boston: Houghton Mifflin, 1969.

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ON THE DISTRIBUTION OF QUADRATIC RESIDUES

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For p an odd prime, each of the integers 1, 2, ..., $p - 1$ is either a quadratic residue or a quadratic nonresidue. In [1], Andrews proves that the number of pairs of consecutive quadratic residues, the number of pairs of consecutive quadratic nonresidues, etc., are the values listed in Table 1. This note is a further investigation of the distribution of the quadratic residues and quadratic nonresidues which will include new proofs of the results in Table 1.

The integers 1, 2, ..., $p - 1$ can be partitioned into disjoint cells, in an alternate fashion, according to whether they are consecutive quadratic residues or quadratic nonresidues.

For example, for $p = 13$, the quadratic residues are 1, 3, 4, 9, 10, 12, which lead to the partition:

1 2 3,4 5,6,7,8 9,10 11 12

(this is much easier to picture when written vertically).

Notation: In this note, "quadratic residue" and "quadratic nonresidue" will be abbreviated by qr and qnr, respectively. For a fixed odd prime p, s will denote the number of singleton cells, e will denote the number of integers which appear as left end points of cells (or right end points since a nonsingleton cell has a left end point and a right end point), and i will denote the number of integers which are interior points in the cells (that is, excluding the end points). Finally, subscripts r and n will denote quadratic residue and quadratic nonresidue, respectively.

TABLE 1

$(p =)$	$4k + 1$	$4k + 3$
qr-qr pairs	$(p - 5)/4$	$(p - 3)/4$
qr-qnr pairs	$(p - 1)/4$	$(p + 1)/4$
qnr-qr pairs	$(p - 1)/4$	$(p - 3)/4$
qnr-qnr pairs	$(p - 1)/4$	$(p - 3)/4$

For example, for $p = 13$: 1, 2, 11, 12 form singletons, 6 and 7 are interior points, and 3, 5, 9 are left end points; $s = 4$, $i = 2$, and $e = 3$.

Theorem 1: The partitioning into cells is symmetric in that if, for example, there are k elements in the first cell, then there are k elements in the last cell, etc.

Proof: For $p = 4k + 1$, x is a qr if and only if $p - x$ is a qr. Therefore, for a cell of k consecutive qr (qnr), there is a corresponding cell of k consecutive qr (qnr). For $p = 4k + 3$, x is a qr if and only if $p - x$ is a qnr. Therefore, for a cell of k consecutive qr (qnr), there is a corresponding cell of k consecutive qnr (qr).

Corollary 1: If the number of cells is odd, then the middle cell must contain an even number of elements.

Proof: If the middle cell contained an odd number of elements, then due to symmetry (the number of elements in cells preceding the middle cell equaling the number of elements in cells following the middle cell), the partition would contain an odd number of elements. But, this would contradict the fact that there are $p - 1$ elements in the partition.

Corollary 2: The first and last cells are singletons if and only if $p \not\equiv \pm 1 \pmod 8$.

Proof: The conclusion follows from the fact that 1 is a qr, 2 is a qr if and only if $p \equiv \pm 1 \pmod 8$, and the partition is symmetric.

The following lemmas, involving the Legendre symbol, are proven in [1]. Lemma 1 also appears as an exercise in [2].

Lemma 1: $\displaystyle\sum_{a=1}^{p-2} \left(\frac{a(a+1)}{p}\right) = -1.$

In Lemma 2, $\left(\frac{0}{p}\right)$ is defined to be 0.

Lemma 2: $\displaystyle\sum_{a=2}^{p} \left(\frac{(a-1)(a+1)}{p}\right) = -1.$

Theorem 2: There are $(p + 1)/2$ cells.

Proof: In the summation in Lemma 1, there are $(p - 3)/2$ plus ones and $(p - 1)/2$ minus ones, since there are $p - 2$ terms with one more minus than plus. Now,

$$\left(\frac{a(a+1)}{p}\right) = -1$$

if and only if a is in one cell and $a + 1$ is in the next cell. Thus, there are $(p - 1)/2 + 1 = (p + 1)/2$ cells.

The result in the next corollary will be extended considerably in a later theorem.

Corollary 3: The partition must contain at least two singletons, that is, $s \geq 2$.

Proof: Suppose each cell contained at least two elements; then, there are at least

$$2\,\frac{(p+1)}{2} = p + 1$$

elements, a contradiction. By Corollary 1, the middle cell is not a singleton; hence, by symmetry, there must be at least two singletons.

Theorem 3: The following equations are identities:

(1) $$s + e = (p + 1)/2,$$

(2) $$e + i = (p - 3)/2,$$

(3) $$s = i + 2,$$

(4) $$s + 2e + i = p - 1.$$

Proof: Part (1) follows from Theorem 2, since each cell is either a singleton or has a left end point. As seen earlier, there are $(p - 3)/2$ plus ones in the summation in Lemma 1. Now,

$$\left(\frac{a(a + 1)}{p}\right) = 1$$

if and only if a and $a + 1$ are in the same cell. Hence, a must be a left end point or an interior point of a cell, and (2) follows. Part (3) follows from the subtraction of part (2) from part (1). Part (4) follows from the fact that the number of left end points equals the number of right end points, and there are $p - 1$ integers in the partition.

A counterpart to the next lemma will follow Theorem 4.

Lemma 3: Let $p = 4k + 1$; then, a is a qnr singleton if and only if a', the inverse of a, is a qnr interior point.

Proof: First note that $a \neq 1,\ p - 1$. The conclusion follows from the fact that

$$\left(\frac{a - 1}{p}\right) = 1,\ \left(\frac{a}{p}\right) = -1,\ \left(\frac{a + 1}{p}\right) = 1, \text{ if and only if}$$

$$\left(\frac{a' - 1}{p}\right) = -1,\ \left(\frac{a'}{p}\right) = -1,\ \left(\frac{a' + 1}{p}\right) = -1, \text{ where}$$

$$\left(\frac{a' - 1}{p}\right) = \left(\frac{(1 - a)a'}{p}\right) = \left(\frac{-1}{p}\right)\left(\frac{a - 1}{p}\right)\left(\frac{a'}{p}\right) = 1 \cdot 1 \cdot (-1) = -1.$$

Theorem 4: The results in Table 1 hold.

Proof: If $p = 4k + 3$, then there are an even number of cells, the first cell qr and the last cell qnr. A qr followed by a qnr occurs only between a cell of qr followed by a cell of qnr. Hence, there are $1/2\frac{(p + 1)}{2}$ pairs of cells of this type, and so $(p + 1)/4$ pairs of qr followed by qnr. A qnr followed by qr occurs only between a cell of qnr followed by a cell of qr. These pairs occur starting with the second cell and ending with the next to the last cell, yielding $1/2\left[\frac{(p + 1)}{2} - 2\right] = \frac{(p - 3)}{4}$ pairs. Recalling the notation and the symmetry discussed in Theorem 1, $e_r = e_n$. Similarly, $i_r = i_n$. From (2) of Theorem 3, $e_r + e_n + i_r + i_n = (p - 3)/2$, which yields $e_r + i_r = e_n + i_n = (p - 3)/4$. Now, a pair of consecutive qr (qnr) occurs only in a nonsingleton cell, and there are precisely as many such pairs as there are qr (qnr) interior points plus one per such cell. This total is precisely $e_r + i_r(e_n + i_n)$.

If $p = 4k + 1$, then there is an odd number of cells, the first and last consisting of qr. This implies that the number of pairs of a qr followed by a qnr (first cell to second cell, third cell to fourth cell, etc.) equals the number of pairs of a qnr followed by a qr (second cell to third cell, fourth cell to fifth cell, etc.). Since these pairs result in $(p - 1)/2$ minus ones in Lemma 1, there are $(p - 1)/4$ pairs of each type. In particular, it follows that

$$e_n + s_n = \frac{(p - 1)}{4}.$$

Now, from Lemma 3, $s_n = i_n$, and so, from (2), $e_r + e_n + i_r + i_n = (p - 3)/2$. Therefore,

$e_r + i_r = (p - 3)/2 - (e_n + s_n) = (p - 5)/4$. Also, $e_n + i_n = e_n + s_n = (p - 1)/4$. And the conclusion follows as in the previous case.

Lemma 4: Let $p = 4k + 3$ and a an element not its own inverse; then, a is a qr singleton if and only if a', the inverse of a, is a qr right end point.

Proof: The conclusion follows from the fact that

$$\left(\frac{a-1}{p}\right) = -1,\ \left(\frac{a}{p}\right) = 1,\ \left(\frac{a+1}{p}\right) = -1, \text{ if and only if}$$

$$\left(\frac{a'-1}{p}\right) = 1,\ \left(\frac{a'}{p}\right) = 1,\ \left(\frac{a'+1}{p}\right) = -1, \text{ where}$$

$$\left(\frac{a'-1}{p}\right) = \left(\frac{(1-a)a'}{p}\right) = \left(\frac{-1}{p}\right)\left(\frac{a-1}{p}\right)\left(\frac{a'}{p}\right) = (-1)(-1)\cdot 1 = 1.$$

Lemma 5: Suppose that $a \neq p - 1, p$; then, in the summation in Lemma 2, $\left(\frac{(a-1)(a+1)}{p}\right) = 1$ if and only if a is a singleton or an interior point.

Proof: The Legendre symbol $\left(\frac{(a-1)(a+1)}{p}\right) = 1$ if and only if $a - 1$ and $a + 1$ are both qr or both qnr. If a is of the same type, then a is an interior point; if not, then a is a singleton.

TABLE 2

$(p =)$	$8k + 1$	$8k + 3$	$8k + 5$	$8k + 7$
s	$(p - 1)/4$	$(p + 5)/4$	$(p + 3)/4$	$(p + 1)/4$
e	$(p + 3)/4$	$(p - 3)/4$	$(p - 1)/4$	$(p + 1)/4$
i	$(p - 9)/4$	$(p - 3)/4$	$(p - 5)/4$	$(p - 7)/4$

Theorem 5: The results in Table 2 hold.

Proof: With the use of Equations (1) and (3) of Theorem 3, the conclusions will follow once the results are established for the number of singleton cells. For the cases $8k + 3$ and $8k + 7$ consider Lemma 4. If $p = 8k + 7$, then the first and the last cells are not singletons since 2 is a qr. Thus, no singleton is its own inverse, and $s = e$ (recall the symmetry). From (1) of Theorem 3, $s = (p + 1)/4$. If $p = 8k + 3$, 1 and $p - 1$ are both singletons not included in Lemma 4; hence, $s = e + 2$. From (1), $s = (p + 5)/4$. For the cases $8k + 1$ and $8k + 5$, consider Lemma 5. If $p = 8k + 1$, neither 1 nor $p - 1$ is a singleton (2 is a qr), and so there are $s + i + 1$ plus ones in the summation in Lemma 5 (the "1" is for the case $a = p$). As in Lemma 2, there are $(p - 3)/2$ plus ones in the summation in Lemma 5 [also $(p - 1)/2$ minus ones and one zero]. Therefore, $s + i + 1 = (p - 3)/2$ and since $s = i + 2$ [part (3) of Theorem 3], $s = (p - 1)/4$.

If $p = 8k + 5$, then 1 and $p - 1$ are singletons not included in Lemma 5; thus, there are $(s - 2) + i + 1$ plus ones. Then, $(s - 2) + i + 1 = (p - 3)/2$ and $s = i + 2$ yield $s = (p + 3)/4$.

It should be noted that Lemma 5 might have been used to prove all cases in Theorem 5. Lemma 4 was used for the two cases to which it applied because it was so easy to apply and the result in Lemma 4 was itself interesting.

REFERENCES

1. George E. Andrews. *Number Theory*. Philadelphia: W. B. Saunders Co., 1971, pp. 128-138.
2. David M. Burton. *Elementary Number Theory*. Boston: Allyn and Bacon, Inc., 1976, p. 202.

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DIVISIBILITY OF BINOMIAL AND MULTINOMIAL COEFFICIENTS BY PRIMES AND PRIME POWERS

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1. ABSTRACT AND INTRODUCTION

Questions on the title subject have been raised and answered many times. However, there does not seem to be a place where all this knowledge is gathered together, other than Dickson (*History of the Theory of Numbers*, Vol. I, Chapter 9). It is my intention to give here a systematic presentation of the subject. Much of the material is known, but there is a moderate amount of new formulations and new results.

The main theorem in the subject is that $p^e || \binom{n}{k}$ if and only if e is the number of carries in the addition $k + (n - k) = n$ when done in p-ary arithmetic. The multinomial analog is that $p^e || n!/\Pi k_j!$ if and only if e is the total amount carried in the p-ary addition $\Sigma k_j = n$. The historical background of these results and its relation to Lucas' result will be discussed.

Then the main theorem is used to investigate the following topics:

a) When does $d | \binom{n}{k}$ for $k = 1, 2, \ldots, n - 1$?

b) When does $d \nmid \binom{n}{k}$ for $k = 0, 1, \ldots, n$?

c) Equalities, lower bounds and upper bounds for e in $p^e || \binom{n}{k}$.

For example, we shall see that $p^s | \binom{p^s}{k}$ iff $(k,p) = 1$ and $p^e | \binom{n}{k}$ implies $p^e \leqq n$.

d) The multinomial analogs of a, b, and c.

e) How often does $p || \binom{n}{k}$ or does $p^2 | \binom{n}{k}$ for $k = 0, 1, \ldots, n$?

f) How often does $d | \binom{n}{k}$ for $n = k, k + 1, \ldots$?

Numerous related questions arise in connection with these topics and some unsolved problems occur. Some other related results are discussed afterward. The contents are described in more detail in Section 3, after introducing notations in Section 2.

2. NOTATIONS AND CONVENTIONS

All letters n, k, e, etc., denote nonnegative integers, with p and q being distinct primes, $r \geqq 2$, and (usually) $d > 1$. In general, k is always the bottom term of some binomial coefficient $\binom{n}{k}$ and is always assumed to satisfy $0 \leqq k \leqq n$. Similarly, if we have $\binom{p^s}{k}$, we assume $0 \leqq k \leqq p^s$. The phrases "all k," "some k," etc., will always imply this, unless otherwise specified.

For $r \geqq 2$, we let $\overline{k}$ denote any r-tuple $(k_1, k_2, \ldots, k_r)$ such that $\Sigma k_j = n$ (or whatever the top term of the r-nomial coefficient concerned is). The phrases "all $\overline{k}$," "some $\overline{k}$," etc., will always imply this, unless otherwise specified. We denote the multionomial (or r-nomial) coefficient $n!/\Pi k_j!$ by $M_r(n,\overline{k})$.

Any n has a unique p-ary representation (or expansion)

$$n = \sum_{i=0}^{m} a_i p^i \quad \text{with} \quad 0 \leqq a_i < p.$$

Occasionally, it is convenient to have $a_m \neq 0$. In that case, we must exclude $n = 0$ or let $0 = 0 \cdot p^0$ with $m = 0$. In most cases, we write the sum indefinitely: $n = \Sigma a_i p^i$. We also denote the p-ary expansion by $(a_m, \ldots, a_1, a_0)$ or by $(\ldots, a_i, \ldots, a_1, a_0)$. We let $k = \Sigma b_i p^i$, $n - k = \Sigma c_i p^i$, and $k_j = \Sigma_i b_{ji} p^i$ be the respective p-ary representations. We refer to the positions as the 0th, 1st, ..., ith, etc., so that the ith position (or place) means the place corresponding to p^i and it has i places to its right.

We use $p^e || n$ for $p^e | n$ and $p^{e+1} \nmid n$. Note that $p^0 || n$ means $p \nmid n$. Square brackets [ ] will denote the greatest integer function. We use the ALGOL symbol ↑ to denote exponentiation when the exponent becomes complicated. E.g., we write $n = \Pi p_i \uparrow e_i$.

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Definition 1: $E(p,n) = e$ if $p^e || n$.

Definition 2: (A) $f(p,n) = E(p,n!)$.

(B) $e(p,n,k) = E\left(p, \binom{n}{k}\right)$.

(C) $e_r(p,n,\overline{k}) = E\left(p, M_r(n,\overline{k})\right)$.

Clearly, E and e stand for exponent and f is used to avoid too many e's and because it is next to e.

Definition 3: (A) $N(n,d)$ is the number of k such that $d \nmid \binom{n}{k}$.

(B) $N_r(n,d)$ is the number of $\overline{k}$ such that $d \nmid M_r(n,\overline{k})$.

When there is no danger of confusion, we may drop references to p and/or r in $M_r(n,\overline{k})$, $f(p,n)$, $e(p,n,k)$, $e_r(p,n,k)$ and $N_r(n,d)$.

All main items (theorems, definitions, lemmas, and propositions) are numbered consecutively. Corollary 16.1 denotes the first corolllary to item 16. (I hope that those readers who have found a Definition 4 located between Theorem 8 and Lemma 2 or who have tried to locate a Definition 3.1.2.4 will find this system a bit easier to follow.)

3. SUMMARY

With the above notations in hand, we can now give a more precise description of the contents of the paper.

Section 4 will present the main theorem that $e(p,n,k)$ is the number of carries in the p-ary addition $k + (n - k)$, and its multidimensional analog that $e_r(p,n,\overline{k})$ is the total amount carried in the p-ary addition $\Sigma k_j = n$. These will be derived from Legendre's classic results. Then we deduce a necessary and sufficient condition for $p \nmid \binom{n}{k}$ and for $p \nmid M_r(n,\overline{k})$.

Section 5 will discuss the history of the results given in Section 4, in their several forms. The connection with Lucas' congruence will be noted.

In Section 6, we shall determine $N(n,p)$, when $N(n,p) = 2$ and when $N(n,d) = 2$. The final result is that $N(n,d) = 2$ if any only if $d = p$ and $n = p^m$ for some p.

In Section 7, we shall consider $N(n,p^e) = n + 1$ and $N(n,d) = n + 1$. The main result is that $N(n,p^e) = n + 1$ if and only if $n = ap^s - 1$ with $1 \leqq a < p^e$. The related question of determining n such that $\left(d, \binom{n}{k}\right) = 1$ for all k is considered.

Section 8 will give a number of results on the exact value of, or lower or upper bounds for, $e(p,n,k)$, depending on n and k. This will lead us to the determination of

$$\text{GCD}\left\{\binom{n\alpha}{k} | (k,n) = 1\right\} \quad \text{and of} \quad \text{LCM}\left\{\binom{n}{k}\right\}$$

and to results such as:

$$p^s | \binom{p^s}{k} \text{ iff } (k,p) = 1; \; p^e | \binom{n}{k} \text{ implies } p^e \leqq n; \text{ and } \frac{n}{(n,k)} | \binom{n}{k}.$$

(This section is large and contains many diverse things. I can only give an idea in this short summary.)

In Section 9, we shall find multinomial analogs for most of the results of Sections 6, 7, and 8. The main result of Section 7 is radically different when $r \geqq 3$:

$$N_r(n,d) = \binom{n + r - 1}{r - 1}$$

has only finitely many solutions and all have $n < d$.

Section 10 will cover a number of results on the number of k such that $p || \binom{n}{k}$ and $p^2 || \binom{n}{k}$.

Section 11 will deal with problems on the density of n such that $d | \binom{n}{k}$, for $n = k$, $k + 1, \ldots$. The basic result is the theorem of Zabek which gives the period of $\binom{n}{k}$ (mod d) and hence shows that the density being considered does exist. We shall see that the density is $\geqq d^{-1}$, with equality iff $d = p^e$ and $k = p^m$ for some prime p.

In Section 12, a few related topics that occur in the literature are discussed.

The references are intended to be reasonably exhaustive (but not too exhausting).

4. THE MAIN THEOREMS

We first state and sketch two well-known results of Legendre.

Lemma 4: $f(p,n) = f(n) = \sum_{j \geqq 1} [n/p^j]$.

Lemma 5: $f(n) = (n - \Sigma a_i)/(p - 1)$.

Sketch of Proofs: For the first, observe that $[n/p^j]$ counts the number of terms in $n!$ that are divisible by p^j. A term which is exactly divisible by p^e will be counted exactly e times in the sum, once by each $[n/p^j]$ with $1 \leq j \leq e$. For the second, observe that

$$[n/p^j] = a_j + a_{j+1}p + \cdots + a_m p^{m-j};$$

collect terms and simplify. ■

Lemma 4 may be found, usually in more detail, in [41, p. 10; 2, p. 50; 4, p. 25; 22, p. 41; 23, p. 86; 28, p. 342; 38, p. 46; 39, p. 7; 42, p. 90; 44, p. 47; 46, p. 79; 49, p. 117; 50, p. 113; 58, p. 131; 66, p. 99; 67, p. 17]. Lemma 5 may be found in [41, p. 12; 2, p. 55; 4, p. 26; 22, p. 42; 38, p. 49; 39, p. 8; 60; 66, p. 103].

Note that when $p = 2$, Lemma 5 becomes $f(n) = n - \Sigma a_i$ and that Σa_i is simply the number of ones in the binary expansion of n [4, p. 26; 25, p. 158; 38, p. 49].

Theorem 6: $e(p,n,k) = e(n,k)$ is the number of carries in the p-ary addition $k + (n - k)$.

Proof: Applying Lemma 5 to k, $n - k$, and n, we have:

(1) $$e(n,k) = f(n) - f(k) - f(n - k);$$

(2) $$= \Sigma(b_i + c_i - a_i)/(p - 1).$$

Now consider the p-ary addition. Set $\varepsilon_i = 1$ if there is a carry from the ith place and set $\varepsilon_i = 0$ if not. (Let $\varepsilon_{-1} = 0$.) Then

(3) $$a_i + p\varepsilon_i = b_i + c_i + \varepsilon_{i-1}.$$

Hence $\Sigma(b_i + c_i - a_i) = p\Sigma\varepsilon_i - \Sigma\varepsilon_{i-1} = (p - 1)\Sigma\varepsilon_i$ and so $e(n,k) = \Sigma\varepsilon_i$ is the number of carries. ■

Corollary 6.1: $e(n,k)$ is the number of borrows in the p-ary subtraction $n - k$.

Corollary 6.2: For $p = 2$, $e(n,k) = \Sigma b_i + \Sigma c_i - \Sigma a_i$ and Σa_i is the number of ones in the binary representation of n, etc.

Theorem 7: $e_r(p,n,\overline{k}) = e(n,\overline{k})$ is the total amount carried in the p-ary addition $\Sigma k_j = n$.

Proof: Proceeding as before, we get

(4) $$e(n,\overline{k}) = \Sigma_i(\Sigma_j b_{ji} - a_i)/(p - 1)$$

and we have

(5) $$a_i + p\varepsilon_i = \Sigma_j b_{ji} + \varepsilon_{i-1},$$

where ε_i is the amount carried and may be greater than one. Hence, $e(n,\overline{k}) = \Sigma\varepsilon_i$ is the total amount carried. ■

Corollary 7.1: For $p = 2$, $e(n,\overline{k}) = \Sigma_j\Sigma_i b_{ji} - \Sigma_i a_i$.

Proposition 8: $p \nmid \binom{n}{k}$ if and only if $0 \leq b_i \leq a_i$ for all i.

Proof: We have that $p \nmid \binom{n}{k}$ iff $e(n,k) = 0$ iff $a_i = b_i + c_i$ for all i iff $0 \leq b_i \leq a_i$ for all i. (Note that $0 \leq b_i \leq a_i$ for all i implies that $0 \leq k \leq n$.) ■

Proposition 9: $p \nmid M(n,\overline{k})$ if and only if $\Sigma_j b_{ji} = a_i$ for all i.

5. HISTORICAL NOTES

Lemma 4 is due to Legendre [7, p. 263, item 2; 41, p. 10] but is only rarely attributed to him [14; 22, p. 41; 50, p. 113]. Lemma 5 is also due to Legendre [7, p. 263, item 2; 41, p. 12] and is sometimes attributed to him [1; 2, p. 55; 14; 36; 38, p. 49; 60]. Carlitz [3, p. 305] cites Bachmann [2] for Lemma 5, but this is presumably not intended as a primary reference. In general, number theorists all know these results are due to Legendre, especially Lemma 4, but they don't seem to write it down in textbooks. (None of the other sources I have mentioned give any reference for these results. Personally, I think this is a shame.)

Kummer [7, p. 270, item 71; 40, p. 115] gives most of Theorem 6, but he does not identify $\Sigma\varepsilon_i$ as the number of carries. To me, this identification is an important step; it clarifies the equations (3) and it reduces the whole question to simple p-ary arithmetic. I have found only two references to Theorem 6 in its present form, namely Knuth [38, p. 68], who gives it as a problem and cites Kummer, and Simmons [59], who mentions only the case appropriate to Proposition 8. Bachmann [2, p. 60] shows that $\Sigma b_i + \Sigma c_i = \Sigma a_i + (p - 1)\Sigma\varepsilon_i$, but not in a context of binomial coefficients.

On the other hand, Glaisher [17, pp. 353, 357] specifically states Corollary 6.1, although Dickson's reference [7, p. 273, item 92] does not mention it. Dickson gives no references to either Theorem 6 or Corollary 6.1 in their present forms.

Dickson [7, p. 273, item 93; 6, p. 378] has essentially obtained Theorem 7, but without identifying the ε_i or forming their sum or even stating the result. Fray [14, p. 473] notes this and states the result, but does not identify the ε_i as carries.

Modern authors have used Kummer's original form [3, p. 302; 14, p. 470] or other forms, sometimes simply equivalent and sometimes not. If one puts Lemma 4 directly into equation (1), we have one such form:

$$e(n,k) = \sum_{j \geq 1} [n/p^j] - [k/p^j] - [(n-k)/p^j]. \tag{6}$$

See [1; 9; 63]. Another form is simply equation (2) as it stands. See [7, p. 272, item 79; 53; 57]. Corollary 6.2 occurs in [33]. Some complicated forms occur in [24; 26; 36; 43], the first two being related to Glaisher's form, Corollary 6.1.

Theorem 6 is complementary to Lucas' result:

$$\binom{n}{k} \equiv \Pi\binom{a_i}{b_i} \pmod{p},$$

where we set $\binom{a}{b} = 0$ for $b > a$. See [7, p. 271, items 76 and 77; 3; 11; 14; 15; 16; 38, p. 68]. Clearly, Proposition 8 also follows easily from this result. Dickson [7, p. 273, item 90; 5, p. 76[has generalized Lucas' result to multinomial coefficients and derived Proposition 9 from it. Numerous authors have given Proposition 8, usually as a consequence of Lucas' result; see [11; 15; 17, p. 357; 53; 59]. Proposition 9 has been given less often [7, p. 273, item 90; 5; 14, p. 473].

6. WHEN DOES $N(n,d) = 2$?

The topic of this section is to determine when $d \mid \binom{n}{k}$ for $k = 1, 2, \ldots, n-1$. We are only interested in $d > 1$ and $n \geqq 1$. Then $d \nmid \binom{n}{0}$ and $d \nmid \binom{n}{n}$ so we always have $N(n,d) \geqq 2$ and $d \mid \binom{n}{k}$ for $1 \leq k \leq n-1$ is equivalent to $N(n,d) = 2$.

Proposition 10: $N(n,p) = \Pi(a_i + 1)$.

Proof: This follows easily from Proposition 8. See also [3; 11; 53]. ■

Corollary 10.1: Setting $p = 2$, the number of odd binomial coefficients in the nth row = $N(n,2) = 2\uparrow(\Sigma a_i)$. (See also [7, p. 274, item 98; 16, p. 156].)

Corollary 10.2: $N(n,p) = 2$ if and only if $n = p^m$. (See also [11; 12; 53].)

Corollary 10.3: Again setting $p = 2$, $\binom{n}{k}$ is even for $1 \leq k \leq n-1$ if and only if $n = 2^m$.

Proposition 11: For $n > 1$, $N(n,pq) > 2$.

Proof: If $N(n,pq) = 2$, then $N(n,p) = N(n,q) = 2$. This would imply that $n = p^m = q^\mu$, which is impossible. Since $N(n,pq) \geqq 2$, we must have $N(n,pq) > 2$. ■

Theorem 12: For $n > 1$, $N(n,p^2) > 2$.

Proof: If $N(n,p^2) = 2$, then $N(n,p) = 2$ and so $n = p^m$. The p-ary expansion of n is $(1, 0, 0, \ldots, 0)$. Let $k = (0, 1, 0, \ldots, 0)$, so $n - k = (0, p-1, 0, \ldots, 0)$. Clearly there is only one carry in the addition of k and $n - k$, so $p^2 \nmid \binom{n}{k}$ and $N(n,p^2) > 2$. See also [12]. ■

Theorem 13: For $d > 1$ and $n \geqq 1$, we have $N(n,d) = 2$ if and only if $n = 1$ or d is a prime p with $n = p^m$ and $m > 0$.

Proof: For $n = 1$, everything is trivial. Let $n > 1$, and suppose $N(n,d) = 2$. By Proposition 11, d cannot have two distinct prime factors. By Theorem 12, d cannot have a square prime factor. Hence $d = p$ and Corollary 10.2 gives us $n = p^m$ and $m > 0$ follows since $n > 1$. The converse is given by Corollary 10.2. ■

Theorem 13 can be rephrased as saying that the GCD of $\binom{n}{k}$ for $1 \leq k \leq n-1$ can only be a prime p, and then iff $n = p^m$ [7, p. 274, item 98], or as saying that $(a+b)^n \equiv a^n + b^n \pmod{d}$ can hold iff $n = 1$ or $d = p$ with $n = p^m$.

7. WHEN DOES $N(n,d) = n + 1$?

The theme of this section is to partially determine when $d \nmid \binom{n}{k}$ for all k, i.e., when is $N(n,d) = n + 1$, and to solve the related question of when $\left(d, \binom{n}{k}\right) = 1$ for all k. In this section, $n = 0$ is permissible.

Theorem 14: For $e \geq 1$, $N(n,p^e) = n + 1$ if and only if $n = ap^s - 1$ with $1 \leq a < p^e$.

Proof: For $n = 0$, everything is trivial, so consider $n > 0$ and suppose $N(n,p^e) = n + 1$. Assume that $n = (a_m, a_{m-1}, \ldots, a_0)$ with $a_m \neq 0$. We claim that $a_i = p - 1$ for $i \leq m - e$. Consider $k = (a_m - 1, p - 1, p - 1, \ldots, p - 1)$. Then $0 \leq k \leq n$. Consider the addition of k and $n - k$. If $a_i < p - 1$, then there must be a carry from the ith position, which creates carries up to one from the $(m - 1)$st position, making a total of $m - i$ carries, so $p^{m-i} \mid \binom{n}{k}$. So if $N(n,p^e) = n + 1$, then $p^e \nmid \binom{n}{k}$, hence $a_i < p - 1$ implies $m - i < e$, or $i > m - e$, as claimed.

Let $s = \min\{i \mid a_i \neq p - 1\}$, so that $s > m - e$ and $m - s < e$. Then $n = (a_m, \ldots, a_s, p - 1, \ldots, p - 1)$ with $a_s < p - 1$. Hence, we have $n = (a_m p^{m-s} + \cdots + a_s)p^s + (p^s - 1) = (\alpha + 1)p^s - 1$, where $0 \leq \alpha < p^e - 1$. So $n = ap^s - 1$ with $1 \leq a < p^e$, by setting $a = \alpha + 1$.

Conversely, if $n = ap^s - 1$ with $1 \leq a < p^e$, we let $a - 1 = (\alpha_{e-1}, \ldots, \alpha_0)$. Then $n = (a - 1)p^s + (p^s - 1) = (\alpha_{e-1}, \ldots, \alpha_0, p - 1, \ldots, p - 1)$. For any k, the subtraction $n - k$ can have at most $e - 1$ borrows; hence,

$$p^e \nmid \binom{n}{k} \text{ for all } k. \blacksquare$$

Corollary 14.1: $N(n,p) = n + 1$ if and only if $n = ap^s - 1$, with $1 \leq a < p$. See [7, p. 274, item 98; 11; 38, p. 483; 53].

Corollary 14.2: Setting $p = 2$, all the binomial coefficients in the nth row are odd if and only if $n = 2^s - 1$. See [16, p. 156; 38, p. 69].

The exact determination of when $N(n,d) = n + 1$ appears intractable for d not a prime power. For example: $N(4, 12) = 5$, but $N(4, 3) = 4$ and $N(4, 4) = 3$. However, we can say the following.

Proposition 15: For any $d > 1$, there are infinitely many n such that $N(n,d) = n + 1$.

Proof: Let $p \mid d$. Then $N(n,p) = n + 1$ implies $N(n,d) = n + 1$, so we can let $n = ap^s - 1$ with $1 \leq a < p$. $\blacksquare$

This result is of particular interest, since it fails for multinomial coefficients with $r \geq 3$.

For the related problem of finding n such that d is relatively prime to each $\binom{n}{k}$, we have an easy solution. If d is a prime power, say $d = p^e$, then the problem is equivalent to finding $N(n,p) = n + 1$ and Corollary 14.1 applies. Otherwise, we have the following.

Theorem 16: Let d have at least two prime divisors. Then there are only a finite number of n such that $\left(d, \binom{n}{k}\right) = 1$ for all k.

Proof: Without loss of generality, we may assume d is square-free and we set $d = \prod p_i$ with $p_1 < p_2 < \ldots$. Then $\left(d, \binom{n}{k}\right) = 1$ for all k if and only if $p_i \nmid \binom{n}{k}$ for all i and k, i.e., $N(n,p_i) = n + 1$ for all i. From Corollary 14.1, we must have $n + 1 = a_i \cdot p_i \uparrow s_i$ with $1 \leq a_i < p_i$. Now $a_1 < p_1 < p_2$, hence $p_2 \nmid (n + 1)$ and so $n + 1 = a_2 < p_2$ and so $n \leq p_2 - 2$. $\blacksquare$

In fact, the proof gives a determination of all such n as all numbers of the form $n = a \cdot p_1^s - 1$ with $1 \leq a < p_1$ and $a \cdot p_1^s < p_2$, since all such numbers have $N(n,p_i) = n + 1$ by Corollary 14.1.

Most of the results of these last two sections are known, but are usually derived via Lucas' result. Theorems 12 and 14 do not follow from Lucas' result, but Theorem 12 can be and has been derived by ad hoc arguments. Theorem 14 does appear to be new, although its corollaries are not. I have not seen Proposition 14 or Theorem 16 before, but their proofs do not require anything new.

8. SOME INEQUALITIES ON $e(p,n,k)$

First we shall consider a few exact determinations of $e(p,n,k) = e(n,k)$. These lead into a number of lower bounds. Combining the lower bounds for various primes will give assertions of divisibility such as $(k,n) = 1$ implies $n \mid \binom{n}{k}$. Then we consider a few upper bounds. Recall that $e(n,k) = e$ is equivalent to $p^e \| \binom{n}{k}$, hence $e(n,k) \geqq e$ is equivalent to $p^e | \binom{n}{k}$ and $e(n,k) \leqq e$ is equivalent to $p^{e+1} \nmid \binom{n}{k}$.

Proposition 17: If $n = kp^s$, then $e(n,k) = \Sigma c_i/(p - 1)$.

Proof: We have that $\Sigma b_i = \Sigma a_i$, so the result follows from equation (2). ■

Corollary 17.1: Setting $p = 2$ and $s = 1$, we have $e(2k,k) = \Sigma b_i$ is the number of ones in the binary representation of k (or $2k$).

Corollary 17.2: For $k \geqq 1$, $\binom{2k}{k}$ is even; $2 \| \binom{2k}{k}$ if and only if $k = 2^m$; $4 | \binom{2k}{k}$ unless $k = 2^m$.

Theorem 18: Let $n = p^m$ and let $p^t \| k$. (If $k = 0$, set $t = m$.) Then $p^{m-t} \| \binom{n}{k}$.

Proof: We have $n = (1, 0, 0, \ldots, 0)$ and $k = (b_m, \ldots, b_t, 0, 0, \ldots, 0)$, hence there are exactly $m - t$ carries in adding k and $n - k$. ■

Corollary 18.1: $p^m | \binom{p^m}{k}$ if and only if $(k,p) = 1$.

Corollary 18.2: For $0 < k < p^{u+1}$, we have $p^s | \binom{p^{s+u}}{k}$.

Proof: Let $p^t \| k$, so $t \leqq u$. By the theorem, $p^s | p^{s+u-t} \| \binom{p^{s+u}}{k}$. (This corollary will be needed in Section 11.) ■

Corollary 18.3: For $0 < k < p^m$, we have $p | \binom{p^m}{k}$, i.e., $N(p^m,p) = 2$. (See Corollary 10.2.)

We have already moved into considering lower bounds with the above corollaries. We now examine lower bounds more directly.

Theorem 19: Let $p^s | n$ and $p^t \| k$. If $t \leqq s$, then $p^{s-t} | \binom{n}{k}$.

Proof: The argument is a slight modification of that of Theorem 18. ■

Corollary 19.1: If $p^s | n$ and $(k,p) = 1$, then $p^s | \binom{n}{k}$.

Corollary 19.2: For $\nu \geqq 1$, we have $\frac{n}{(n,k)} | \binom{n\nu}{k}$.

Proof: Consider any prime p and let $p^s \| n$ and $p^t \| k$. If $t \geqq s$, then $p \nmid \frac{n}{(n,k)}$ and is irrelevant. If $t < s$, then $p^{s-t} \| \frac{n}{(n,k)}$ and $p^{s-t} | \binom{n\nu}{k}$ by the theorem. ■

Corollary 19.3: If $(k,n) = 1$ and $\nu \geqq 1$, then $n | \binom{n\nu}{k}$.

Corollaries 18.1 and 19.3 (with $\nu = 1$) partially resolve the question of when does $n | \binom{n}{k}$. This problem was posed by Hausmann in 1954 [29] and no answer has been published. The first case not covered by the corollaries is $10 | \binom{10}{4}$. See also [47, p. 86; 7, p. 265, items 18 and 21; 2, p. 62; 4, p. 28; 22, p. 45; 46, p. 82). Gould [20] attributes Corollary 19.2 (with $\nu = 1$) to Hermite, apparently on the basis of [7, p. 272, item 85], while Bachmann [2, p. 62] assigns it to Catalan. However, Dickson [7, p. 265, item 18] makes it clear that the result is due to Schonemann. Gupta [25] has studied the parity of the ratio $\frac{(n,k)}{n}\binom{n}{k}$ and asserts that his method applies to the study of its divisibility by any prime.

Corollary 19.4: For $(b,k) = 1$, we have $(ak + b) | \binom{ak+b}{k}$; in particular, $(ak + 1) | \binom{ak+1}{k}$, and $((a - 1)k + 1) | \binom{ak}{k}$. Setting $a = 2$ gives $k + 1 | \binom{2k}{k}$.

The ratios $\binom{2k}{k}/(k + 1) = \binom{2k+1}{k}/(2k + 1)$ are known as Catalan or Segner numbers (although due to Euler). They occur often in combinatorial problems, particularly as the number of ways of associating $k + 1$ terms. See [27, p. 25; 38, pp. 239, 531-533; 45, pp. 140-152; 52, p. 101, and elsewhere (see his index); 69, p. 154; 21; 48], the last two giving numerous other references.

Theorem 20: For $n > 1$, let $n = \Pi p_i \uparrow e_i$ and let $p_i \uparrow f_i \| \nu$. Then $\mathrm{GCD}\left\{\binom{n\nu}{k} \mid (k,n) = 1\right\} = \Pi p_i \uparrow (e_i + f_i)$. (Recall that $0 \leq k \leq n\nu$, by convention.)

Proof: We have $(p_i \uparrow (e_i + f_i)) \mid n\nu$ and $(k,n) = 1$ implies $(k,p_i) = 1$, so that $(p_i \uparrow (e_i + f_i)) \mid \binom{n\nu}{k}$ for all such k by Corollary 19.1. Hence $\Pi p_i \uparrow (e_i + f_i) \mid \mathrm{GCD}$. On the other hand, $\mathrm{GCD} \mid \binom{n\nu}{1} = n\nu$ and $(p_i \uparrow (e_i + f_i)) \| n\nu$, so no higher power of p_i can divide the GCD. Further, the only other primes which can enter into the GCD are primes p such that $p \mid \nu$ and $p \neq p_i$ for each i. Consider such a prime p and let $p^e \| \nu$, so $p^e \| n\nu$ and $p^e \neq n\nu$ (since $n > 1$). Hence $n\nu = (a_m, \ldots, a_e, 0, 0, \ldots, 0)$ with $a_e \neq 0$. Setting $k = p^e = (0, \ldots, 1, 0, 0, \ldots, 0)$, we have $0 \leq k \leq n$, $(k,n) = 1$, and $p \nmid \binom{n}{k}$ by Proposition 8. Hence, $p \nmid \mathrm{GCD}$. ∎

The case $n = 2$ is solved in [64] using a special argument only suitable for $n = 2$ instead of the second half of the above proof.

The next theorem is complementary to Theorem 19.

Theorem 21: Let $p^s \| n + 1$ and let $p^t \mid k + 1$. If $t \geq s$, then $p^{t-s} \mid \binom{n}{k}$.

Proof: We have $n = (a_m, \ldots, a_s, p-1, p-1, \ldots, p-1)$ with $a_s \neq p - 1$ and $k = (b_m, \ldots, b_t, p-1, p-1, \ldots, p-1)$. Hence $n - k$ has at least $t - s$ borrows. ∎

Corollary 21.1: If $(k+1, n+1) = 1$, then $k + 1 \mid \binom{n}{k}$.

Corollary 21.2: $k + 1 \mid \binom{2k}{k}$. (See also Corollary 19.4.)

The proofs of Theorems 19 and 21 can be somewhat generalized to give the following two results.

Proposition 22: Let $p^s \mid n - \alpha$ and $p^t \| k - \alpha$ where $0 \leq \alpha < p^t$ and $t \leq s$. Then $p^{s-t} \mid \binom{n}{k}$.

Proposition 23: Let $p^s \| n + \alpha$ and $p^t \mid k + \alpha$ where $0 < \alpha \leq p^s$ and $t \geq s$. Then $p^{t-s} \mid \binom{n}{k}$.

Note that Proposition 22, with $\alpha = 0$, is Theorem 19 and that Proposition 23, with $\alpha = 0$, is Theorem 21. However, these are the only two simple applications of the propositions.

Now we consider some upper bounds on $e(p,n,k) = e(n,k)$. We now assume that $n = (a_m, \ldots, a_0)$ has $a_m \neq 0$, i.e., $p^m \leq n < p^{m+1}$. For $n = 0$, we take $m = 0$.

Theorem 24: Let $p^t \mid k$. (For $k = 0$, set $t = m$.) Then $e(n,k) \leq m - t$.

Proof: We have $n = (a_m, \ldots, a_t, \ldots, a_0)$ and we have $k = (b_m, \ldots, b_t, 0, 0, \ldots, 0)$. Hence, there can be at most $m - t$ borrows in $n - k$. ∎

Note that Theorems 19 and 24 imply Theorem 18.

Corollary 24.1: For $n > 0$ and any k, $e(n,k) \leq m$. Hence, $p^e \mid \binom{n}{k}$ implies $p^e \leq p^m \leq n$.

See [1; 9; 63] for proofs using equation (6) and [57] for a proof using equation (2). The special case, that $p^e \mid \binom{2k}{k}$ implies $p^e \leq 2k$, occurs often in prime number theory [23, p. 103; 28, p. 342; 42, p. 105; 44, p. 60; 46, p. 165; 58, p. 133].

Corollary 24.1 can also be derived as a consequence of Theorem 14, as $p^e \mid \binom{n}{k}$ implies $N(n,p^e) < n + 1$ and the least such n is the least n not of the form $ap^e - 1$ with $1 \leq a < p^e$, which is p^e.

Corollary 24.2: For $1 < k < n - 1$, $\binom{n}{k}$ is never a prime power. (See [30; 57; 63].)

Proof: If $\binom{n}{k} = p^e$, then $p^e \mid \binom{n}{k}$, hence $p^e \leq n = \binom{n}{1} < \binom{n}{k}$. ∎

Erdös and others [10 and its references and its review] have considered the question of whether $\binom{n}{k}$ can be a power for $1 < k < n - 1$. For $3 < k < n - 3$, Erdös has shown that $\binom{n}{k}$ is never a power, but the situation for $k = 2$ and $k = 3$ does not yet appear to be fully resolved.

The next theorem is the complement of Theorem 24.

Theorem 25: Assume $p^m \leq n + 1 < p^{m+1}$ and $p^s \| n + 1$. Then, for any k, $e(n,k) \leq m - s$ and equality can hold.

Proof: Write $n + 1 = \alpha p^s$, where $p^{m-s} \leqq \alpha < p^{m+1-s}$. Then $n = (\alpha - 1)p^s + (p^s - 1) = (a_m, \ldots, a_s, p - 1, p - 1, \ldots, p - 1)$. Hence, there can be at most $m - s$ carries for any k. Note that $a_m = 0$ may occur, but only if $m = s$ and $\alpha = 1$. Also note that $a_s \neq p - 1$. If $m = s$, then equality holds for any k. If $m > s$, then equality holds for $k = (b_m, \ldots, b_s, \ldots, b_0)$ if and only if $a_s < b_s < p$, $a_i \leqq b_i < p$ for $s < i < m$, and $0 \leqq b_m < a_m$. Such k are readily found. ■

Corollary 25.1: $\mathrm{LCM}\left\{\binom{n}{k}\right\} = \frac{1}{n+1} \Pi_p p\uparrow[\log_p(n + 1)] = \frac{1}{n+1} \Pi_p p\uparrow\left[\frac{\log(n + 1)}{\log p}\right]$.

Meynieux [43] has considered this LCM.

Corollary 25.2: $\mathrm{Max}_k\{e(n,k)\} = e$ if and only if $n = ap^s - 1$ with $p \nmid a$ and $p^e \leqq a < p^{e+1}$.

The form of Corollary 25.2 is clearly reminiscent of Theorem 14. In fact, Theorems 14 and 25 each imply the other. Theorems 24 and 25 do not seem to have generalizations similar to Propositions 22 and 23. The reader may convince himself that Theorems 19, 21, 24, and 25 give all the bounds on $e(n,k)$ which arise in the four cases when n ends in more (or less) zeros (or p - 1's) than k.

9. MULTINOMIAL ANALOGS

In this section, we shall obtain multinomial analogs for most of the results of Sections 6, 7, and 8. In many cases, the analog is straightforward or only requires some greater care in the statement, e.g., the condition $(k,n) = 1$ must be replaced by $\mathrm{GCD}\{k_j\} = 1$. If the reader has forgotten the conventions for the multinomial case, he should review Section 2, Theorem 7 and Proposition 9. We shall place the number(s) of the binomial analog(s) in parentheses after the results in this section.

First, we need the following basic combinatorial fact.

Lemma 26: A nonnegative integer n can be partitioned into an ordered sum of r nonnegative integers in $\binom{n + r - 1}{r - 1}$ ways.

For proofs, see [27, p. 5; 58, p. 402]. This is the same as the number of ways of distributing n objects into r distinct cells [51, p. 92], which is the same as the number of n-combinations of r things, with repetition [45, p. 59; 51, p. 6].

Corollary 26.1: There are $\binom{n + r - 1}{r - 1}$ r-nomial coefficients of rank n.

Proposition 27 (10): $N_r(n,p) = \Pi\binom{a_i + r - 1}{r - 1}$.

Corollary 27.1 (10.1): Setting $p = 2$, the number of odd r-nomial coefficients of rank n is $N_r(n,2) = r\uparrow(\Sigma a_i)$.

Corollary 27.2 (10.2): $N_r(n,p) = r$ if and only if $r = p^m$.

The r-nomial coefficients contain $\binom{r}{2}$ copies of the binomial coefficients, corresponding to setting all but two k_j's equal to zero, and they contain r bounding axes of ones, corresponding to setting all but one k_j equal to zero. We shall refer to these bounding axes as the edges. Consequently, for $n \geqq 1$, we have $N_r(n,d) \geqq r$ and $N_r(n,d) = r$ implies that $N(n,d) = N_2(n,d) = 2$.

Corollary 27.3 (10.3): Again setting $p = 2$, $M(n,\bar{k})$ is even except at the edges if and only if $n = 2^m$.

Theorem 28 (11, 12, 13): For $n \geqq 1$ and $d > 1$, the following are equivalent:

(a) $N_r(n,d) = r$.
(b) $N_2(n,d) = 2$.
(c) Either $n = 1$ or d is a prime p with $n = p^m$ and $m > 0$.

Proof: (a) implies (b) by the discussion above. (b) implies (c) by Theorem 13. (c) implies (a) by Corollary 27.2. ■

Now we ask when can $N_r(n,d) = \binom{n + r - 1}{r - 1}$. This implies that $N_2(n,d) = n + 1$, but not conversely. For example, consider $r = 3$ and $p = 2$. Let $n = 3 = 2^2 - 1$, so that $N_2(n,p) = n + 1$. But $2 \left| \frac{3!}{1!1!1!} \right.$, and so $N_3(n,p) \neq \binom{n + r - 1}{r - 1}$. In fact, for $r \geqq 3$, this question has a radically different solution than for $r = 2$.

Theorem 29 (14, 15): For $d > 1$ and $r \geqq 3$, $N_r(n,d) = \binom{n+r-1}{r-1}$ implies that $n < d$.

Proof: Let $n_1 = n - k_1$ and $n_2 = n_1 - k_2$. Then we have

$$M(n,\overline{k}) = \frac{n!}{k_1!k_2!\ \dots\ k_r!} = \binom{n}{k_1}\binom{n_1}{k_2}\frac{n_2!}{k_3!\ \dots\ k_r!}. \tag{7}$$

By varying the k_j's, we can let n_1 and k_2 be any integers such that $0 \leqq k_2 \leqq n_1 \leqq n$. In particular, we can take $k_2 = 1$. Hence, $N_r(n,d) = \binom{n+r-1}{r-1}$ implies that $d \nmid n_1$ for $n_1 = 2, 3, \dots, n$. Hence, $n < d$. ∎

Corollary 29.1 (14.1): For $r \geqq 3$, $N_r(n,p) = \binom{n+r-1}{r-1}$ if and only if $0 \leqq n < p$.

Corollary 29.2 (14.2): For $p = 2$ and $r \geqq 3$, all r-nomial coefficients of rank n are odd if and only if $n = 0$ or $n = 1$.

The exact determination of when $N_r(n,d) = \binom{n+r-1}{r-1}$ seems awkward, but may perhaps be easier for $r \geqq 3$ than for $r = 2$.

Corollary 29.3 (16): If $r \geqq 3$ and $d = \Pi p_i \uparrow e_i$ with $p_1 < p_2 < \dots$, then $(d, M(n,\overline{k})) = 1$ for all $\overline{k}$ if and only if $0 \leqq n < p_1$.

The converse of Theorem 29 need not hold, even for $r = 3$. Let $r = 3$, $d = p^e = 9$, and $n = 6$. Then $9 \mid \frac{6!}{2!2!2!}$, so $N_r(n,d) \neq \binom{n+r-1}{r-1}$. I shall discuss this more fully at the end of the section.

Now we consider inequalities for $e_r(p,n,\overline{k}) = e(n,\overline{k})$. Proposition 17 can be generalized in several ways, but I shall give only two.

Proposition 30 (17.1): Let $r = p$ and let all $k_j = k = \Sigma b_i p^i$, so that $n = pk$. Then $e(pk,\overline{k}) = \Sigma b_i$ is the sum of the digits in the p-ary expansion of k (or n).

Corollary 30.1 (17.2): For $k \geq 1$, we have $p \mid (pk)!/(k!)^p$ and $p \| (pk)!/(k!)^p$ if and only if $n = p^m$.

Theorem 31 (17.1): Let $k_j = k = \Sigma b_i p^i$ for all j, so that $n = rk$. Then $e(rk,\overline{k}) \geq f(r) \cdot \Sigma b_i$.

Proof: Consider the addition in p-ary arithmetic. In the ith place, we have $\Sigma_j b_{ji} + \varepsilon_{i-1} = rb_i + \varepsilon_{i-1}$. This produces a carry to the $(i+1)$st place of at least $[rb_i/p] + [\varepsilon_{i-1}/p]$ and this produces a carry to the $(i+2)$nd place of at least $[rb_{i+1}/p] + [rb_i/p^2] + [\varepsilon_{i-1}/p^2]$, etc. Hence,

$$\Sigma\varepsilon_i \geqq \Sigma_i\left(\sum_{j\geqq 1}[rb_i/p^j]\right) = \Sigma_i f(rb_i) \geqq \Sigma_i b_i f(r). \ ∎$$

Corollary 31.1 (17.2): If $\Sigma b_i \geq \alpha$ for all primes $p \leq r$, then $(r!)^\alpha \mid (rk)!/(k!)^r$. (Note that $\alpha \geqq 1$.) See [7, p. 266, item 28; 2, p. 57; 42, p. 92; 46, p. 81; 66, p. 103].

The argument used in Theorem 18 fails to generalize to the multinomial case because a carry can now have a value greater than one. In general, this fact prevents us from obtaining any useful upper bounds. However, we do have some nice lower bounds.

Theorem 32 (19): Let $p^s \mid n$ and $(p\uparrow t_j) \| k_j$. Set $t = \min\{t_j\}$. If $t \leq s$, then $p^{s-t} \mid M(n,\overline{k})$.

Proof: Suppose, without loss of generality, that $t = t_1$. Then $p^{s-t} \mid \binom{n}{k_1} \mid M(n,\overline{k})$, using equation (7). ∎

Corollary 32.1 (19.1): If $p^s \mid n$ and $(k_j,p) = 1$ for some j, then $p^s \mid M(n,\overline{k})$.

Corollary 32.2 (19.2): For $\nu \geq 1$, we have $\frac{n}{\mathrm{GCD}\{t_j\}} \mid M(n\nu,\overline{k})$. See [7, p. 265, item 18].

Corollary 32.3 (19.3): If $\mathrm{GCD}\{t_j\} = 1$ and $\nu \geq 1$, then $n \mid M(n\ ,\overline{k})$. See [16, p. 103; 22, p. 46; 46, p. 82]. Obviously the question of when does $n \mid M(n,\overline{k})$ is even more unsolved than $n \mid \binom{n}{k}$.

Corollary 32.4 (19.4): $rk + 1 \mid (rk+1)!/(k+1)!(k!)^{r-1}$, hence $k + 1 \mid (rk)!/(k!)^r$.

One can write down numerous similar consequences of 32.3.

Proposition 33 (20): For $n > 1$, let $n = \Pi p_i \uparrow e_i$ and let $p_i \uparrow f_i || \nu$. Then $\mathrm{GCD}\{M(n,\overline{k}) | \mathrm{GCD}\{k_j\} = 1\} = \Pi p_i \uparrow (e_i + f_i)$.

Theorem 34 (21): Let $p^s || n + 1$ and let $(p \uparrow t_j) | (k_j + 1)$. Set $t = \max\{t_j\}$. If $t \geqq s$, then $p^{t-s} | M(n,\overline{k})$.

Proof: As for Theorem 32. ∎

Corollary 34.1 (21.1): If $(k_j + 1, n + 1) = 1$ for some j, then $k_j + 1 | M(n,\overline{k})$.

Corollary 34.2 (21.2): $k + 1 | (rk)!/(k!)^r$. (See 32.4.)

Versions of Propositions 22 and 23 can be stated, but do not seem useful. I have not been able to obtain any useful upper bounds, but one can still obtain the analog of 24.2.

Proposition 35 (24.2): If $1 < k_j < n - 1$ for some j, then $M(n,\overline{k})$ is not a prime power.

Proof: From equation (7) and symmetry, we have that $\binom{n}{k_j} | M(n,\overline{k})$ for all j. From Corollary 24.2, if $M(n,\overline{k})$ is a prime power, we must have $k_j = 0$, 1, $n - 1$ or n for each j. ∎

I have not seen any work on the general problem of whether $M(n,\overline{k})$ can be a power.

We have obtained Proposition 35, which is the analog of Corollary 24.2, but we have not obtained a multinomial analog of Theorem 24 or of Corollary 24.1. In fact, since $9 | \frac{6!}{2!2!2!}$, the obvious analog of 24.1 does not hold. In [61], I have given a method for finding the least n such that $p^e | M(n,\overline{k})$ for some $\overline{k}$, i.e., the least n such that $N_r(n,p^e) \neq \binom{n + r - 1}{r - 1}$. For $p \geqq r$, the method gives the following simple result. For $e \geqq 1$, let $e = s(r - 1) + \beta$ with $0 < \beta \leqq r - 1$. Then the least n such that $N_r(n,p^e) \neq \binom{n + r - 1}{r - 1}$ is $n = \beta p^{s+1}$.

The exact determination of when $N_r(n,p^e) = \binom{n + r - 1}{r - 1}$ appears to be very messy.

10. DETERMINATION OF $N(n,p^2)$, ETC.

We now return to the ordinary binomial case and use the main Theorem 6 to determine the number of k such that $p || \binom{n}{k}$. This number is simply $N(n,p^2) - N(n,p)$, so that we can then determine $N(n,p^2)$, since $N(n,p)$ is known from Proposition 10.

Theorem 36: $N(n,p^2) - N(n,p) = N(n,p) \sum \left(\frac{p}{a_i + 1} - 1\right)\left(1 - \frac{1}{a_{i+1} + 1}\right)$.

Proof: As remarked above, the left-hand side is the number of k such that $p || \binom{n}{k}$, that is, such that $k + (n - k)$ has exactly one carry. If this carry occurs at the ith place, we have that there is exactly one carry if and only if $a_i < b_i < p$, $0 \leqq b_{i+1} < a_{i+1}$ and $0 \leqq b_j \leqq a_j$ for $j \neq i$, $i + 1$. There are

$$(p - a_i - 1)a_{i+1} \prod_{j \neq 1, i+1} (a_j + 1) = N(n,p)\left(\frac{p}{a_i + 1} - 1\right)\left(1 - \frac{1}{a_{i+1} + 1}\right)$$

ways of doing this. Adding this for all i gives the theorem. ∎ See [3, p. 303; 53].

Corollary 36.1: Let $p = 2$ and let w be the number of pairs $(a_{i+1},a_i) = (1,0)$ in the binary representation of n. Then $N(n,4) - N(n,2) = N(n,2)w/2$ and $N(n,4) = N(n,2)(1 + w/2)$.

The argument of the theorem can be extended to obtain the following results, which we only state.

Proposition 37: $N(n,p^3) - N(n,p^2) = N(n,p) \sum \left(\frac{p}{a_i + 1} - 1\right)\left(\frac{p + 1}{a_{i+1} + 1} - 1\right)\left(1 - \frac{1}{a_{i+2} + 1}\right)$

$$+ N(n,p) \sum_{i+1<j} \left(\frac{p}{a_i + 1} - 1\right)\left(1 - \frac{1}{a_{i+1} + 1}\right)\left(\frac{p}{a_j + 1} - 1\right)\left(1 - \frac{1}{a_{j+1} + 1}\right).$$

See [53].

Corollary 37.1: Let $p = 2$. In the binary expansion of n, let w_1 be the number of triples $(a_{i+2}, a_{i+1}, a_i) = (1,0,0)$; let w_2 be the number of triples $(a_{i+2}, a_{i+1}, a_i) = (1,1,0)$; and let w_3 be the number of quadruples $(a_{j+1}, a_j, a_{i+1}, a_i) = (1,0,1,0)$ with $j > i + 1$. Then $N(n,8) - N(n,4) = N(n,2)(w_1 + (w_2 + w_3)/4)$.

A multinomial analog for Theorem 36 seems very difficult to express. One must determine the number of ways $p + a_i = \Sigma_j b_{ji}$ subject to $0 \leqq b_{ji} < p$.

11. RESULTS FOR k FIXED, n VARYING

Thus far, we have been concerned with k (or n and k) varying. Now we hold k fixed and let n vary; that is, we look at the diagonals of Pascal's triangle, rather than at the rows. We no longer have a finite set of values for n and so we cannot reasonably ask for the number of n with some property, say $p \nmid \binom{n}{k}$. However, one can ask for the density of such n. The basic theorem for this study is due to Zabek [70, p. 42] and determines the period of the sequence $\binom{n}{k}$ (mod p^e) as $n = k, k + 1, \ldots$. We give the proof of Trench [65], somewhat simplified by use of our previous results. In this section, we shall always take $k > 0$, except in one discussion.

Theorem 38: Let $k = \sum_{i=0}^{m} b_i p^i = (b_m, \ldots, b_0)$ with $b_m \neq 0$. (That is, $p^m \leqq k < p^{m+1}$.) Then the sequence of residues $\binom{n}{k}$ (mod p^e) for $n = k, k + 1, \ldots,$ is periodic with minimal period p^{m+e}.

Proof: Let $x = p^{m+e}$. Then $\binom{n+x}{k}$ is a polynomial $f(x)$ of degree k. Let $\Delta f(x) = f(x + 1) - f(x)$ be the usual forward difference operator and let $\Delta^j f(x)$ be the iterates. For $f(x) = \binom{n+x}{k}$, we have $\Delta f(x) = \binom{n+x+1}{k} - \binom{n+x}{k} = \binom{n+x}{k-1}$ and $\Delta^j f(x) = \binom{n+x}{k-j}$. By Newton's formula,

$$f(x) = \sum_{j=0}^{k} \Delta^j f(0)\binom{x}{j} = \sum_{j=0}^{k} \binom{n}{k-j}\binom{x}{j}.$$

Now $j \leqq k < p^{m+1}$, so Corollary 18.2 gives us $p^e \mid \binom{p^{e+m}}{j}$, i.e., $p^e \mid \binom{x}{j}$, for $0 < j \leqq k$. Hence $f(x) = \binom{n+x}{k} \equiv \binom{n}{k}$ (mod p^e) and so $x = p^{m+e}$ is a period.

Now let $n = p^{m+e} + k - p^m = (1, 0, \ldots, 0, b_m - 1, b_{m-1}, \ldots, b_0)$ and let $n_1 = n + p^{m+e+1} = (1, 1, 0, \ldots, 0, b_m - 1, b_{m-1}, \ldots, b_0)$. Examining the subtractions $n - k$ and $n_1 - k$ shows that $p^e \| \binom{n}{k}$ while $p^{e-1} \| \binom{n_1}{k}$, hence p^{m+e-1} is not a period and so p^{m+e} is the minimal period. ∎ See also [14, p. 479].

Corollary 38.1: For $d > 1$, let $d = \Pi p_i \uparrow e_i$. For each i, let $p_i \uparrow (m_i + 1) > k \geqq p_i \uparrow m_i$. Then $\binom{n}{k}$ (mod d) is periodic with minimal period $\Pi p_i \uparrow (m_i + e_i)$.

Definition 39: Given $d > 1$, let $d^* = d^*(k,d)$ be the minimal period of $\binom{n}{k}$ (mod d) as given in Corollary 38.1.

Note that $d^*(k,d)$ is (weakly) multiplicative in d by virtue of Corollary 38.1. Further, $d = d^*$ if and only if $p_i > k$ for each i. If d has r distinct prime factors, then $d^* > k^r$.

Definition 40: Let

$A(k,d)$ be the number of residue classes n (mod d^*) such that $d \nmid \binom{n}{k}$;

$B(k,d)$ be the number of residue classes n (mod d^*) such that $d \mid \binom{n}{k}$;

$C(k,d)$ be the number of residue classes n (mod d^*) such that $\left(d, \binom{n}{k}\right) = 1$; and let

$A^*(k,d) = A(k,d)/d^*$; $B^*(k,d) = B(k,d)/d^*$; $C^*(k,d) = C(k,d)/d^*$ be the corresponding densities.

Proposition 41:

(a) $B(k,d) = d^* - A(k,d)$; $B^*(k,d) = 1 - A^*(k,d)$.
(b) $B(k,d)$, $C(k,d)$, $B^*(k,d)$ and $C^*(k,d)$ are (weakly) multiplicative in d.

(c) $C(k,p) = A(k,p)$; $C^*(k,p^e) = C^*(k,p) = A^*(k,p)$; $C(k,p^e) = p^{e-1}C(k,p)$.

Theorem 42: For $k = \sum_{i=0}^{m} b_i p^i$ with $b_m \neq 0$, we have $A(k,p) = \prod_{i=0}^{m} (p - b_i)$ and so $A^*(k,p) = \prod(1 - b_i/p)$.

Proof: From Proposition 8, we know that $p \nmid \binom{n}{k}$ if and only if $b_i \leq a_i < p$ for each i. Since $\binom{n}{k}$ is periodic (mod p) with period p^{m+1}, we need only consider $0 \leq i \leq m$, so there are $\prod_{i=0}^{m} (p - b_i)$ choices for n (mod p^{m+1}). Hence, $A^*(k,p) = \prod_{i=0}^{m} (p - b_i)/p = \prod(1 - b_i/p)$, where the last product is indefinite, since $i > m$ gives $1 - b_i/p = 1$. ■

We note that we can now determine $C(k,d)$ and $C^*(k,d)$.

Corollary 42.1: For $p = 2$, $A^*(k,2) = 1/(2\uparrow\Sigma b_i) = 1/N(k,2)$.

One may interpret $A^*(0,d) = 1$, for $d > 1$, which agrees with the formula for A^* in Theorem 42. Conversely, $A^*(k,p) = 1$ can only occur for $k = 0$. So, for $k > 0$, the maximal value of $A^*(k,p)$ is $1 - 1/p$.

Corollary 42.2: For $k > 0$, we have $A^*(k,p) \leq 1 - 1/p$, i.e., $B^*(k,p) \geq 1/p$, with equality if and only if $k = p^m$.

Corollary 42.3: For $k > 0$ and m as above, we have $A^*(k,p) \geq 1/p^{m+1}$, i.e., $B^*(k,p) \leq 1 - 1/p^{m+1}$, with equality if and only if $k = p^{m+1} - 1$.

In fact, since $\binom{k}{k} = 1$, we always know at least one residue class $n \equiv k \pmod{p^{m+1}}$ such that $p \nmid \binom{n}{k}$. From the Corollary, this is the only one when $k = p^{m+1} - 1$. For example:

$$2 \nmid \binom{n}{3} \text{ if and only if } n \equiv 3 \pmod 4.$$

We can extend the above inequalities by some simple analysis.

Proposition 43: $B(k,d) \geq k$.

Proof: Consider the k values: $n = d \cdot k! + i$, for $i = 0, 1, \ldots, k - 1$. Then $d \mid \binom{n}{k}$ for all these n. Further, $k < d^*$, so these values are all distinct (mod d^*). ■

Corollary 43.1:

(a) $B^*(k,p^e) \geq k/p^{m+e}$.
(b) $B^*(k,p^e) \geq 1/p^e$ with equality only if $k = p^m$.
(c) $B^*(k,d) \geq 1/d$ with equality only if $d = p^e$, $k = p^m$.

Proposition 44: $B(p^m,p^e) = p^m$.

Proof: We have $k = p^m = (0, \ldots, 0, 1, 0, \ldots, 0)$. Consider $n \equiv (a_{m+e-1}, \ldots, a_m, \ldots, a_0)$. Then $p^e \mid \binom{n}{k}$ if and only if $a_m = a_{m+1} = a_{m+2} = \cdots = a_{m+e-1} = 0$. There are exactly p^m such values. ■

Corollary 44.1:

(a) $B^*(k,p^e) = 1/p^e$ if and only if $k = p^m$.
(b) $B^*(k,d) = 1/d$ if and only if $d = p^e$ and $k = p^m$.

Proposition 45: $B^*(k,p^e) \leq 1 - 1/p^{m+1}$ with equality if and only if $e = 1$ and $k = p^{m+1} - 1$.

Proof: First we have $B^*(k,p^e) \leq B^*(k,p) \leq 1 - 1/p^{m+1}$ by Corollary 42.3. If equality holds, it must also hold on the right and so $k = p^{m+1} - 1 = (0, p - 1, \ldots, p - 1)$. Consider $n = (p - 1, 0, p - 1, \ldots, p - 1)$. Then $p \| \binom{n}{k}$. Hence, for $e \geq 2$, $B^*(k,p^e) < B^*(k,p^2) < B^*(k,p) \leq 1 - 1/p^{m+1}$. ■

I have not been able to find the appropriate form of this result for $B^*(k,d)$. However, for $C^*(k,d)$, we do have a result.

Proposition 46: Let $d = \Pi p_i \uparrow e_i$, let $p_i \uparrow (m_i + 1) > k \geq p_i \uparrow m_i$ and let $d' = \Pi p_i$. Then we have $C^*(k,d) = C^*(k,d') = \Pi C^*(k,p_i) = \Pi A^*(k,p_i) \geq \Pi 1/(p_i(m_i + 1)) = 1/d^*(k,d')$ with equality if and only if $d = p^e$ and $k = p^{m+1} - 1$. See [59; 8].

Proposition 47: $A^*(k,p^2) - A^*(k,p) = A^*(k,p)\sum\left(\frac{p}{p - b_i} - 1\right)\left(1 - \frac{1}{p - b_{i+1}}\right).$

Corollary 47.1: Let $p = 2$ and let w be the number of pairs $(b_{i+1}, b_i) = (0,1)$ in the binary expansion of k. Then

$$A^*(k,4) - A^*(k,2) = A^*(k,2)w/2$$

and

$$A^*(k,4) = A^*(k,2)(1 + w/2).$$

Most of the material in this section, after Zabek's Theorem (Theorem 38), seems to be new, and I feel that there is room for improvement and extension of it. I am not sure what the proper multinomial analogs are.

12. OTHER RESULTS IN THE LITERATURE

In this section, I shall discuss a number of topics related to the subject of this paper, but either too complex or too distant to consider in full detail.

The pattern of the binomial coefficients divisible by an integer d is rather pretty. S. Rösch has published three articles on these patterns [54; 55; 56], the latter two using colors. I sometimes find these, or similar, patterns useful in visualizing theorems.

Fine [11] has shown that the density of binomial coefficients divisible by a prime p is one. One can prove this fairly easily using Proposition 10. On the basis of numerical evidence, Rösch conjectured [54; 56] that the density of coefficients divisible by any integer d is one. Using Theorem 6, I have shown this [62] by showing that p^e divides "almost all" binomial coefficients, using four different senses of "almost all." These include showing that $N(n,p^e)/(n + 1)$ and $A^*(k,p^e)$ both converge in mean to zero.

Sylvester, Schur, and then Erdös [9] have shown that for $n > 2k$, there is a prime p dividing $\binom{n}{k}$ with $p > k$. I do not see that the material of this paper is useful in attacking this type of problem, despite the apparent connection.

Lucas' congruence, mentioned in Section 5, has been generalized by Kazandzidis [36, p. 3] and I have given a simple proof in [60]. The result is that

$$\binom{n}{k} \equiv (-p)^e \prod \frac{a_i!}{b_i!c_i!} \pmod{p^{e+1}}$$

where $e = e(p,n,k)$. This extends readily to multinomial coefficients and to arbitrary ratios of factorials. The analogous result for $n!$ was given by Stickelberger [7, p. 263, items 4, 7, 8; 38, p. 50]:

$$n! \equiv (-p)^f \Pi a_i! \pmod{p^{f+1}}$$

where $f = f(p,n)$.

A problem which has been extensively studied is when a ratio of factorials is an integer. If $\Sigma n_j = \Sigma k_j = n$, then the ratio $\Pi n_j!/\Pi k_j!$ can be expressed as a ratio of multinomial coefficients and we can apply Theorem 7. Another approach is to extend the concept of $p^e||a$ to $p^e||a/b$, allowing $e < 0$. If we set each $n_j = \Sigma_i a_{ji}p^i$, we can obtain

$$e = e(\overline{n},\overline{k}) = \left(\sum_{j,i} b_{ji} - \sum_{j,i} a_{ji}\right)/(p - 1)$$

by arguing as in Theorem 7. Hence, in this case where $\Sigma n_j = \Sigma k_j$, then the ratio $\Pi n_j!/\Pi k_j!$ is an integer iff $\Sigma a_{ji} \leqq \Sigma b_{ji}$ for every prime p.

The problem of when does $n\left|\binom{n}{k}\right.$ can be rephrased in this form as: When is $\frac{(n - 1)!1!}{k!(n - k)!}$ an integer? Hence, the above discussion gives an answer to this problem, but not a very satisfactory one. Dickson [7, pp. 295-269] gives a number of other forms, e.g., the following are always integers:

$$\frac{(2a)!(2b)!}{a!b!(a + b)!} \quad \text{and} \quad \frac{(4a)!(4b)!}{a!b!(2a + b)!(a + 2b)!}.$$

See also [2, p. 63; 4, p. 27; 22, p. 45; 42, p. 92; 46, p. 81; 66, p. 103].

A number of authors have considered generalized binomial coefficients [13; 14; 18; 19; 20; 31; 32; 35; 68] defined by

$$\binom{n}{k}_A = \frac{A_nA_{n-1} \cdots A_1}{A_k \cdots A_1A_{n-k} \cdots A_1}, \text{ with } \binom{n}{0}_A = \binom{n}{n}_A = 1.$$

In general, even if the A_i are integers, $\binom{n}{k}_A$ may not be integers. Remarkably, if $A_n = F_n$

is the nth Fibonacci number (with $F_1 = F_2 = 1$), then the generalized binomial ("Fibonomial") coefficients are integers (see [31]). One also has generalized multinomial coefficients.

I have only seen one paper which treats the divisibility of such coefficients by primes and prime powers, namely Fray [14]. In it, he considers the case when

$$A_n = q^n - 1 \text{ (or } A_n = (q^n - 1)/(q - 1))$$

which gives the q-binomial coefficients of Jackson [34; see the references of 68]. He obtains analogs of Lemma 5, Kummer's form of Theorem 6, Dickson's unstated form of Theorem 7, Proposition 9, Lucas' result, Proposition 10, and Theorem 38. He also observes and states the results for the ordinary case. He establishes that for any n, the least d^* such that

$$\binom{n + d^*}{k} \equiv \binom{n}{k} \pmod{p^e} \text{ for } 0 \leq k \leq n \text{ is } d^* = d^*(n,p^e),$$

a result which is in a somewhat different direction than Theorem 38.

Gould [20] mentions the generalized and the Fibonomial forms of Corollary 19.2 (with $\nu = 1$).

13. ADDENDUM

While this draft was being prepared and typed, several items became available to me. These include some articles which I had previously only known via references, reviews, or memory, and some articles which have only just appeared. This addendum will briefly discuss these articles and the changes to be made in a later version of this paper. The references [A1], etc., refer to the addendum to the references.

Gould [19] gives more detailed information and references on generalized binomial coefficients than I have indicated in Section 12. He remarks that the q-binomial coefficients date back to Causs and Cauchy, prior to Jackson.

Gould has now published [A1], the paper announced in [20]. He again attributes Corollary 19.2 (with $\nu = 1$) to Hermite, referring to [7, p. 272]. He attributes the multinomial analog to Ricci [A4], although it is due to Schönemann [7, p. 264, item 18]. He also considers the following equivalent form of Corollary 19.2 (with $\nu = 1$):

$$\frac{n - k + 1}{(n + 1,k)} \Big| \binom{n}{k}.$$

He gives simple proofs based on $(n,k) = na + kb$. He gives a number of variations and special cases of this type of divisibility relation and extends many of them to Fibonomial coefficients.

Gupta [24] also shows the form of Theorem 6 given in [6] of Section 2, part of Corollary 14.1, and part of Theorem 13. His paper [A2] is an earlier and alternate version of [25].

Sato [A4] has obtained the results of Stickelberger and Kazanzidis discussed in Section 12.

14. REFERENCES

1. H. L. Abbot, P. Erdos, & D. Hanson. "On the Number of Times an Integer Occurs as a Binomial Coefficient." *American Math. Monthly* (to appear).
2. P. Bachmann. *Niedere Zahlentheorie*. Vol. I. Leipzig: B. G. Teubner, 1902.
3. L. Carlitz. "The Number of Binomial Coefficients Divisible by a Fixed Power of a Prime." *Rend. Circ. Mat. Palermo* (II) 16 (1967):229-320. MR 40, #2554.
4. R. D. Carmichael. *The Theory of Numbers and Diophantine Analysis*. New York: Dover, 1959.
5. L. E. Dickson. "The Analytic Representation of Substitutions of a Prime Number of Letters with a discussion of the Linear Group." *Ann. of Math.* 11 (1896-97):65-120.
6. L. E. Dickson. "Theorems on the Residues of Multinomial Coefficients with Respect to a Prime Modulus." *Quarterly J. Pure Appl. Math.* 33 (1901-2):378-384.
7. L. E. Dickson. *History of the Theory of Numbers*. Vol. I. New York: Chelsea, 1952.
8. D. Drazin. "Complements and Comments." *American Math. Monthly* 77 (1970):1078-1079.
9. P. Erdös. "A Theorem of Sylvester and Schur." *J. London Math. Soc.* 9 (1934):282-288.
10. P. Erdös. "On a Diophantine Equation." *J. London Math. Soc.* 26 (1951):176-178. MR 12, p. 804.
11. N. J. Fine. "Binomial Coefficients Modulo a Prime." *American Math. Monthly* 54 (1947): 589-592. MR 9, p. 331.
12. O. M. Fomenko. "Sur quelques propriétés des coefficients binomiaux." *Mathesis* 69 (1960):291-293. MR 25, #5030.
13. G. Fontené. "Généralisation d'une formule connue." *Nouvelles Annales de Mathématiques* (4) 15 (1915):112.

14. R. D. Fray. "Congruence Properties of Ordinary and q-Binomial Coefficients." *Duke Math. J.* 34 (1967):467-480. MR 35, #4151.
15. R. D. Fray. "Solution of Problem E2205: Consequences of a Lucas Congruence." (Proposed by S. M. Farber, D. W. Walkup, and R. J. B. Wets.) *American Math. Monthly* 77 (1970):889-890.
16. J. W. L. Glaisher. "On the Residue of a Binomial-Theorem Coefficient with Respect to a Prime Modulus." *Quarterly J. Pure Appl. Math.* 30 (1899):150-156.
17. J. W. L. Glaisher. "On the Residue with Respect to p^{n+1} of a Binomial-Theorem Coefficient Divisible by p^n." *Quarterly J. Pure Appl. Math.* 30 (1899):349-360.
18. H. W. Gould. "The Bracket Function, q-Binomial Coefficients and Some New Stirling Number Formulas." *The Fibonacci Quarterly* 5 (1967):401-423. MR 37, #1262.
19. H. W. Gould. "The Bracket Function and Fontené-Ward Generalized Binomial Coefficients with Applications to Fibonomial Coefficients." *The Fibonacci Quarterly* 7 (1969):23-40, 55. MR 39, #4021.
20. H. W. Gould. "A New Primality Criterion of Mann and Shanks and Its Relation to a Theorem of Hermite." *Notices Amer. Math. Soc.* 16 (1971):551-552. Abstract 71T-A75.
21. H. W. Gould. *Research Bibliography of Two Special Number Sequences.* Mathematica Monongaliae, No. 12. Department of Mathematics, West Virginia University, Morgantown, W. Va., 1971. MR 43, #4755.
22. H. Griffin. *Elementary Theory of Numbers.* New York: McGraw-Hill, 1954.
23. E. Grosswald. *Topics From the Theory of Numbers.* New York: Macmillan, 1966.
24. H. Gupta. "On the p-Potency of $G(n,r)$." *Proc. Indian Acad. Sci.* Sect. A 1 (1935):620-622.
25. H. Gupta. "On a Problem in Parity." *Indian J. Math.* 11 (1969):157-163. MR 41, #5283.
26. H. Gupta. "Reviewer's Remarks." *Math. Reviews* 40 (1970):467, #2554.
27. M. Hall, Jr. *Combinatorial Theory.* Waltham, Mass.: Blaisdell, 1967.
28. G. H. Hardy & E. M. Wright. *An Introduction to the Theory of Numbers.* 4th ed. Oxford: Oxford University Press, 1960.
29. B. A. Hausmann. "Problem E1145." *American Math. Monthly* 61 (1954):712.
30. F. Hering. "Eine Beziehung zwischen Binomialkoeffizienten und Primzahlpotenzen." *Arch. Math.* (Basel) 19 (1968):411-412. MR 38, #1010.
31. V. E. Hoggatt, Jr. "Fibonacci Numbers and Generalized Binomial Coefficients." *The Fibonacci Quarterly* 5 (1967):383-400. MR 37, #6193.
32. V. E. Hoggatt, Jr., & D. A. Lind. "A Power Identity for Second-Order Recurrent Sequences." *The Fibonacci Quarterly* 4 (1966):274-282. MR 34, #128.
33. F. T. Howard. "A Combinatorial Problem and Congruences for the Rayleigh Function." *Proc. Amer. Math. Soc.* 26 (1970):574-580. MR 42, #1756.
34. F. H. Jackson. "q-Difference Equations." *American J. Math.* 32 (1910):305-314.
35. D. Jarden & T. Motzkin. "The Product of Sequences with a Common Linear Recursion Formula of Order 2." *Riveon Lematematika* 3 (1949):25-27. (Hebrew; English summary.) MR 10, p. 698. Reprinted in English in: D. Jarden, "Recurring Sequences," *Riveon Lematematika* (Jerusalem) (1958):42-45.
36. G. S. Kazandzidis (Γ. Σ. Καζαντζιδου). "On a Congruence and on a Practical Method for Finding the Highest Power of a Prime Which Divides the Binomial Coefficient $\binom{A}{B}$" [Περι μιασ ισοτιμιασ και περι ενοσ πρακτικου κανονοσ δια χην ευρεσιν τησ ανωτατησ δυναμεωσ του πρωτου p τησ διαιρουσησ τον διωνυμικον συντελεστην $\binom{A}{B}$]. *Bull. Soc. Math. Grèce* (NS) 6 (1965):358-360. MR 34, #7440.
37. G. S. Kazandzidis. "Congruences on the Binomial Coefficients." *Bull. Soc. Math. Grèce* (NS) 9 (1968):1-12. MR 42, #182.
38. D. E. Knuth. *The Art of Computer Programming.* Vol. I: *Fundamental Algorithms.* World Student Series Edition. Reading, Mass.: Addison-Wesley, 1972.
39. M. Kraitchik. *Introduction a la théorie des nombres.* Paris: Gauthier-Villars, 1952.
40. E. E. Kummer. "Uber die Ergänzungssätze zu den allgemeinen Reciptocitätsgesetzen." *J. Reine Angew. Math.* 44 (1852):93-146.
41. A. M. Legendre. *Théorie des nombres.* Vol. I. 3rd ed. Paris: Didot Freres, 1830.
42. W. J. LeVeque. *Topics in Number Theory.* Vol. I. Reading, Mass.: Addison-Wesley, 1956.
43. R. Meynieux. "Sur le plus petit commun multiple des coefficients du polynôme $(1 + z)^n$ et celui de certains de ces coefficients." *C. R. Acad. Sci. Paris* Sér A 271 (1970):861-864. MR 42, #5892.
44. T. Nagell. *Introduction to Number Theory.* 2nd ed. New York: Chelsea, 1964.

45. I. Niven. *Mathematics of Choice*. New York: Random House, 1965. (New Mathematical Library, No. 15.)
46. I. Niven & H. S. Zuckerman. *An Introduction to the Theory of Numbers*. New York: John Wiley & Sons, 1960.
47. C. S. Ogilvy. *Tomorrow's Math*. 2nd ed. New York: Oxford University Press, 1972.
48. E. T. Ordman. "Algebraic Characterization of Some Classical Combinatorial Problems." *American Math. Monthly* 78 (1971):961-970.
49. O. Ore. *Invitation to Number Theory*. New York: Random House, 1967. (New Mathematical Library, No. 20.)
50. H. Rademacher. *Lectures on Elementary Number Theory*. New York: Blaisdell, 1972.
51. J. Riordan. *An Introduction to Combinatorial Analysis*. New York: John Wiley & Sons, 1958.
52. J. Riordan. *Combinatorial Identities*. New York: John Wiley & Sons, 1968.
53. E. G.-Rodeja F. "Una Propiedad de los Coefficientes Binomicos." *Rev. Mat. Hisp.-Amer.* (4) 24 (1964):250-253. MR 31, #78.
54. S. Rösch. "Expedition in unerforschtes Zahlenland." *Neues Universum* 79 (1962):93-98.
55. S. Rösch. "Farbenlehre, auf die Mathematik angewandt." 1964. (This is either a pamphlet or a reprint from an unidentifiable journal.) Available from the author, 633 Wetzlar, Philosophenweg 2, West Germany.
56. S. Rösch. "Neues vom Pascal-Dreieck." *Bild der Wissenschaft* 9 (1965):758-762.
57. H. Scheid. "Die Anzahl der primfaktoren in $\binom{n}{k}$. *Arch. Math.* (Basel) 20 (1969):581-582. MR 41, #146.
58. W. Sierpinski. *Elementary Theory of Numbers*. Translated by A. Hulanicki. Warszawa: Państwowe Wydawnictwo Naukowe, 1964.
59. G. J. Simmons. "Some Results Concerning the Occurrence of Specified Prime Factors in $\binom{n}{r}$. *American Math. Monthly* 77 (1970):510-511. MR 42, #4476.
60. D. Singmaster. "Notes on Binomial Coefficients—I: A Generalization of Lucas' Congruence" (to appear).
61. D. Singmaster. "Notes on Binomial Coefficients—II: The Least n Such That p^e Divides an r-Nomial Coefficient in the nth Plane" (to appear).
62. D. Singmaster. "Notes on Binomial Coefficients—III: Any Integer Divides Almost All Binomial Coefficients" (to appear).
63. W. Stahl. "Bemerkung zu einer Arbeit von Hering." *Arch. Math.* (Basel) 20 (1969):500. MR 41, #145.
64. St. Olaf College Students. "Solution of Problem E2227." (Proposed by N. S. Mendelsohn.) *American Math. Monthly* 78 (1971):201.
65. W. F. Trench. "On Periodicities of Certain Sequences of Residues." *American Math. Monthly* 67 (1960):652-656. MR 23A, #2365.
66. J. V. Uspensky & M. A. Heaslet. *Elementary Number Theory*. New York: McGraw-Hill, 1939.
67. I. M. Vinogradov. *An Introduction to the Theory of Numbers*. Translated by H. Popova. London: Pergamon Press, 1955.
68. M. Ward. "A Calculus of Sequences." *American J. Math.* 58 (1936):255-266.
69. M. B. Wells. *Elements of Combinatorial Computing*. Oxford: Pergamon Press, 1971.
70. S. Zabek. "Sur la périodicité modulo m des suites de nombres $\binom{n}{k}$. *Ann. Univ. Mariae Curie-Sklodowska* Sect. A 10 (1956):37-47. MR 20, #1653.

15. ADDENDUM TO REFERENCES

A1. H. W. Gould. "A New Primality Criterion of Mann and Shanks and Its Relation to a Theorem of Hermite with Extension to Fibonomials." *The Fibonacci Quarterly* 10 (1972):355-365, 372.
A2. H. Gupta. "On the Parity of $(n + m - 1)!(n,m)/n!m!$." *Res. Bull. Panjab Univ.* (N.S.) 20 (1969):571-575. MR 43, #3201.
A3. G. Ricci. "Sui coefficienti binomiali e polinomiali. Una dimostrazione del teorema di Staudt-Clausen sui numeri di Bernoulli." *Giorn. Mat. Battaglini* 69 (1931):9-12.
A4. S. Sato. "Some Properties on p-Adic Expansions of Natural Numbers." *Res. Bull. Fac. Ed. Oita Univ.* 3 (1970):1-4. MR 44, #145.

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A MATRIX GENERATION OF FIBONACCI IDENTITIES FOR F_{2nk}

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A series of identities involving even-subscripted Fibonacci numbers and binomial coefficients are derived in this paper by means of a sequence of special 2 x 2 matrices. We begin with the simplest case.

Let $R = \begin{pmatrix} 3 & 1 \\ -1 & 0 \end{pmatrix}$ and the characteristic equation, of course, is $x^2 - 3x + 1 = 0$, which is related to the recursion formula for the alternate Fibonacci numbers. By induction, one can easily establish that, for all integers n,

$$R^n = \begin{pmatrix} F_{2n+2} & F_{2n} \\ -F_{2n} & -F_{2n-2} \end{pmatrix},$$

and, if the auxiliary matrix $S = \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}$, then

$$R^nS = \begin{pmatrix} F_{2n+3} & F_{2n+1} \\ -F_{2n+1} & -F_{2n-1} \end{pmatrix},$$

where F_n is the nth Fibonacci number defined by $F_{n+1} = F_n + F_{n-1}$, $F_1 = F_2 = 1$. Since R satisfies its own characteristic equation, $R^2 - 3R + I = 0$ or $(R + I)^2 = 5R$, which leads to

(1) $$R^m(R + I)^{2n} = 5^nR^{n+m},$$

(2) $$R^m(R + I)^{2n}S = 5^nR^{n+m}S,$$

(3) $$R^m(R + I)^{2n+1} = 5^nR^{n+m}(R + I),$$

(4) $$R^m(R + I)^{2n+1}S = 5^nR^{n+m}(R + I)S.$$

We use the binomial theorem to rewrite equation (1) and equate elements in the upper right from equations (1) and (2), which gives us

$$\sum_{k=0}^{2n} \binom{2n}{k} R^{k+m} = 5^nR^{n+m},$$

(1′) $$\sum_{k=0}^{2n} \binom{2n}{k} F_{2k+2m} = 5^nF_{2n+2m},$$

(2′) $$\sum_{k=0}^{2n} \binom{2n}{k} F_{2k+2m+1} = 5^nF_{2n+2m+1}.$$

Similarly, from equations (3) and (4), we can obtain

(3′) $$\sum_{k=0}^{2n+1} \binom{2n+1}{k} F_{2k+2m} = 5^n(F_{2n+2m+2} + F_{2n+2m}) = 5^nL_{2n+2m+1},$$

(4′) $$\sum_{k=0}^{2n+1} \binom{2n+1}{k} F_{2k+2m+1} = 5^n(F_{2n+2m+3} + F_{2n+2m+1}) = 5^nL_{2n+2m+2},$$

where L_n is the nth Lucas number defined by $L_{n+1} = L_n + L_{n-1}$, $L_1 = 1$, $L_2 = 3$.

The equations above can be simplified still further. Equations (1′) and (2′) can be combined by letting $p = 2m$ in (1′) and $p = 2m + 1$ in (2′), and noting that p takes on any integral value, we write, finally,

$$\sum_{k=0}^{2n} \binom{2n}{k} F_{2k+p} = 5^n F_{2n+p}.$$

Similarly, equations (3′) and (4′) can be combined into the single identity

$$\sum_{k=0}^{2n+1} \binom{2n+1}{k} F_{2k+p} = 5^n L_{2n+1+p}.$$

As an interesting special case, let $p = -(2n + 1)$ in the above equation, and use the index replacement $n - k$ for k, yielding

$$\sum_{k=0}^{2n+1} \binom{2n+1}{k} F_{2k-(2n+1)} = 2\left\{\sum_{k=0}^{n} \binom{2n+1}{n-k} F_{2k-1}\right\} = 5^n L_0$$

or

$$\sum_{k=0}^{n} \binom{2n+1}{n-k} F_{2k-1} = 5^n,$$

a result given by S. G. Guba in [2].

Returning to the characteristic polynomial of R, since $R^2 - 3R + I = 0$, $(R - I)^2 = R$, which leads to

(5) $$R^m(R - I)^{2n} = R^{n+m},$$

(6) $$R^m(R - I)^{2n}S = R^{n+m}S,$$

(7) $$R^m(R - I)^{2n+1} = R^{n+m}(R - I),$$

(8) $$R^m(R - I)^{2n+1}S = R^{n+m}(R - I)S.$$

Proceeding as before and equating elements in the upper right for the four matrix equations above, we have

(5′) $$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k} F_{2k+2m} = F_{2n+2m},$$

(6′) $$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k} F_{2k+2m+1} = F_{2n+2m+1},$$

(7′) $$\sum_{k=0}^{2n+1} (-1)^{k+1} \binom{2n+1}{k} F_{2k+2m} = (F_{2n+2m+2} - F_{2n+2m}) = F_{2n+2m+1},$$

(8′) $$\sum_{k=0}^{2n+1} (-1)^{k+1} \binom{2n+1}{k} F_{2k+2m+1} = (F_{2n+2m+3} - F_{2n+2m+1}) = F_{2n+2m+2}.$$

Again, equations (5′) and (6′) can be combined by taking $p = 2m$ in (5′) and $p = 2m + 1$ in (6′), and letting p be any integer in the resulting identity,

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k} F_{2k+p} = F_{2n+p}.$$

Similarly, combining (7′) and (8′) leads to

$$\sum_{k=0}^{2n+1} (-1)^{k+1} \binom{2n+1}{k} F_{2k+p} = F_{2n+1+p}.$$

The two identities above can be streamlined even more by taking $q = 2n$ in the first and $q = 2n + 1$ in the second, leading to

$$\sum_{k=0}^{q} (-1)^{k+q} \binom{q}{k} F_{2k+p} = F_{q+p},$$

which holds for all integers $q \geq 0$ and for any integer p. The special case $p = -q$ yields

$$\sum_{k=0}^{q} (-1)^{k+1} \binom{q}{k} F_{q-2k} = 0.$$

In order to distinguish between matrices in our sequence, let us call the R matrix just developed R_2. The next matrix of interest is

$$R_4 = \begin{pmatrix} 7 & 1 \\ -1 & 0 \end{pmatrix}.$$

The following matrix identities are easily established by mathematical induction. The proofs are given in the general case so are here omitted for the sake of brevity. We exhibit, for any integer n,

$$R_4^n S_0 = \begin{pmatrix} F_{4n+4} & F_{4n} \\ -F_{4n} & -F_{4n-4} \end{pmatrix} \text{ for } S_0 = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix};$$

$$R_4^n S_1 = \begin{pmatrix} F_{4n+5} & F_{4n+1} \\ -F_{4n+1} & -F_{4n-3} \end{pmatrix} \text{ for } S_1 = \begin{pmatrix} 5 & 1 \\ -1 & -2 \end{pmatrix};$$

$$R_4^n S_2 = \begin{pmatrix} F_{4n+6} & F_{4n+2} \\ -F_{4n+2} & -F_{4n-2} \end{pmatrix} \text{ for } S_2 = \begin{pmatrix} 8 & 1 \\ -1 & 1 \end{pmatrix};$$

$$R_4^n S_3 = \begin{pmatrix} F_{4n+7} & F_{4n+3} \\ -F_{4n+3} & -F_{4n-1} \end{pmatrix} \text{ for } S_3 = \begin{pmatrix} 13 & 2 \\ -2 & -1 \end{pmatrix}.$$

Since R_4 must satisfy its characteristic equation, $R^2 - 7R + I = 0$ or $(R - I)^2 = 5R$, leading to

(9) $$R^m(R - I)^{2n} = 5^n R^{m+n},$$

(10) $$R^m(R - I)^{2n+1} = 5^n R^{m+n}(R - I).$$

The binomial expansion of matrix equation (9) yields

$$\sum_{j=0}^{2n} (-1)^j \binom{2n}{j} R^{j+m} = 5^n R^{m+n}.$$

Multiplication on the right by the auxiliary matrix S_s, chosen from the four listed above, and then equating elements in the upper right yields

(9′) $$\sum_{j=0}^{2n} (-1)^j \binom{2n}{j} F_{4(j+m)+s} = 5^n F_{4(m+n)+s}, \quad s = 0, 1, 2, 3.$$

On the other hand, equation (10) can be expanded as

$$\sum_{j=0}^{2n+1} (-1)^{j+1} \binom{2n+1}{j} R^{j+m} = 5^n (R^{m+n+1} - R^{m+n}).$$

By appropriate S matrices, for $s = 0, 1, 2, 3$, we have

$$\sum_{j=0}^{2n+1} (-1)^{j+1} \binom{2n+1}{j} F_{4(j+m)+s} = 5^n (F_{4(m+n+1)+s} - F_{4(m+n)+s}).$$

But, the latter two terms can be factored, using identities given by I. D. Ruggles in [1]:

(A) $$F_{n+p} - F_{n-p} = L_n F_p \text{ if } p \text{ is even } (F_n L_p \text{ if } p \text{ is odd}),$$

(B) $$F_{n+p} + F_{n-p} = F_n L_p \text{ if } p \text{ is even } (L_n F_p \text{ if } p \text{ is odd}).$$

Here, applying identity (A), we get

$$F_{(4(m+n)+s+2)+2} - F_{(4(m+n)+s+2)-2} = L_{4(m+n)+s+2} F_2.$$

Thus, for $s = 0, 1, 2, 3$,

$$\sum_{j=0}^{2n+1} (-1)^{j+1}\binom{2n+1}{j} F_{4(j+m)+s} = 5^n L_{4(m+n)+s+2}. \tag{10'}$$

Equations (9′) and (10′) can be written in a slightly simpler form by taking $p = 4m + s$. Since there is no restriction on m, there is no restriction on the integer p in the two resulting identities below:

$$\sum_{j=0}^{2n} (-1)^j \binom{2n}{j} F_{4j+p} = 5^n F_{4n+p},$$

$$\sum_{j=0}^{2n+1} (-1)^{j+1}\binom{2n+1}{j} F_{4j+p} = 5^n L_{4n+2+p}.$$

Returning to the characteristic equation for R_4, in a completely similar manner we can obtain

$$R^m(R + I)^{2n} = 3^{2n}R^{n+m}, \tag{11}$$

$$R^m(R + I)^{2n+1} = 3^{2n}R^{n+m}(R + I). \tag{12}$$

Following the previous pattern of equating elements in the upper right in the matrix equations obtained from the binomial expansions of (11) and (12) and multiplying by auxiliary matrices S_s, we are led eventually to

$$\sum_{j=0}^{2n} \binom{2n}{j} F_{4(j+m)+s} = 3^{2n} F_{4(n+m)+s}, \quad s = 0, 1, 2, 3; \tag{11'}$$

$$\begin{aligned}\sum_{j=0}^{2n+1} \binom{2n+1}{j} F_{4(j+m)+s} &= 3^{2n}(F_{4(m+n+1)} + F_{4(m+n)+s}) \\ &= 3^{2n+1} F_{4(m+n)+s+2}, \quad s = 0, 1, 2, 3,\end{aligned} \tag{12'}$$

where in (12′) we applied identity (B). Again, let us write the equations above more compactly, taking $p = 4m + s$ and noting that no restrictions on m implies no restrictions on p, as

$$\sum_{j=0}^{2n} \binom{2n}{j} F_{4j+p} = 3^{2n} F_{4n+p},$$

$$\sum_{j=0}^{2n+1} \binom{2n+1}{j} F_{4j+p} = 3^{2n+1} F_{4n+2+p}.$$

Notice that, by taking $q = 2n$ in the first and $q = 2n + 1$ in the second, we may combine the two identities above into the more general identity,

$$\sum_{j=0}^{q} \binom{q}{j} F_{4j+p} = 3^q F_{2q+p}.$$

The special case $p = -2q - 1$ yields

$$\sum_{j=0}^{q} \binom{q}{j} F_{2q+1-4j} = 3^q,$$

and similar equations arise for the special cases $p = -2q + 1$ and $p = -2q + 2$.

In the above identities, the general elements of R_4^n were written in the form of a quotient; that is, the element in the upper left of R_4^n was $F_{4n+4}/3$. While looking for a general form using a sum of Lucas of Fibonacci numbers we are led by observation of the starting values given to the following expression for the element r_n in the upper left of R_4^n:

$$\begin{aligned} r_1 &= 7 = L_4, \\ r_2 &= 48 = L_8 + 1, \\ r_3 &= 329 = L_{12} + L_4, \\ r_4 &= 2255 = L_{16} + L_8 + 1, \end{aligned}$$

$$r_5 = 15456 = L_{20} + L_{12} + L_4,$$

$$r_n = \sum_{j=0}^{[(n-1)/2]} L_{4(n-2j)} + \begin{cases} 0 \text{ if } n \text{ is odd} \\ 1 \text{ if } n \text{ is even,} \end{cases}$$

where $[x]$ is the greatest integer less than or equal to x. A proof can be made by mathematical induction. Observe that the expression for r_n holds for $n = 1, 2, 3, 4, 5$. Since R satisfies its own characteristic equation, $R^{k+1} = 7R^k - R^{k-1}$, and the elements in the upper left of these matrices must satisfy $r_{k+1} = 7r_k - r_{k-1}$. Assume that the expression for r_n holds for all n up through k. Then, if k is odd,

$$\begin{aligned} r_{k+1} &= 7\left(\sum_{j=0}^{[(k-1)/2]} L_{4(k-2j)}\right) - \sum_{j=0}^{[(k-2)/2]} L_{4(k-1-2j)} - 1 \\ &= \sum_{j=0}^{[(k-2)/2]} (7L_{4(k-2j)} - L_{4(k-1-2j)}) + 7L_4 - 1 \\ &= \sum_{j=0}^{[k/2]} L_{4(k+1-2j)} + 1, \end{aligned}$$

where we noted that $[k/2] = [(k-1)/2]$, $7L_p - L_{p-4} = L_{p+4}$, and $48 = L_8 + 1$. Similarly, if k is even, since $[(k-1)/2] = [(k-2)/2]$ and $7 = L_4$,

$$r_{k+1} = \sum_{j=0}^{[(k-2)/2]} (7L_{4(k-2j)} - L_{4(k-1-2j)}) + 7 = \sum_{j=0}^{[k/2]} L_{4(k+1-2j)}.$$

Then, equating elements in the upper left for R_4^{2k} and R_4^{2k+1} gives us

$$F_{4(2k+1)} = 3\sum_{j=0}^{k-1} L_{4(2k-2j)} + 3,$$

$$F_{4(2k+2)} = 3\sum_{j=0}^{k} L_{4(2k+1-2j)}.$$

From equation (9), $(R - I)^{2n} = 5^n R^n$. Considering the cases $n = 2k$ and $n = 2k + 1$ and equating elements in the upper left, one obtains

$$\sum_{j=0}^{4k} (-1)^j \binom{4k}{j} F_{4j+4} = 3 \cdot 5^n \left(\sum_{j=0}^{k-1} L_{4(2k-2j)} + 1\right),$$

$$\sum_{j=0}^{4k+2} (-1)^j \binom{4k+2}{j} F_{4j+4} = 3 \cdot 5^n \left(\sum_{j=0}^{k} L_{4(2k+1-2j)}\right).$$

Similarly, from equation (11) with $m = 0$, we find

$$\sum_{j=0}^{4k} \binom{4k+2}{j} F_{4j+4} = 3^{2n+1}\left(\sum_{j=0}^{k-1} L_{4(2k-2j)} + 1\right),$$

$$\sum_{j=0}^{4k+2} \binom{4k+2}{j} F_{4j+4} = 3^{2n+1}\left(\sum_{j=0}^{k} L_{4(2k+1-2j)}\right).$$

A third expression for R_4^n was obtained with the element in the upper left given by

$$\sum_{j=0}^{[n/2]} (-1)^j \binom{n-j}{j} L_4^{n-2j}.$$

A proof of the general case will follow, so we will proceed only to use the above form. Equating elements in the upper left of R_4 leads to

$$F_{4n+4} = 3\sum_{j=0}^{[n/2]}(-1)^j\binom{n-j}{j}L_4^{n-2j},$$

or, for the cases $n = 2k$ and $n = 2k + 1$, in that order, to

$$\sum_{j=0}^{k}(-1)^j\binom{2k-j}{j}L_4^{2k-2j} = \sum_{j=0}^{k-1}L_{4(2k-2j)} + 1,$$

$$\sum_{j=0}^{k}(-1)^j\binom{2k+1-j}{j}L_4^{2k+1-2j} = \sum_{j=0}^{k}L_{4(2k+1-2j)}.$$

Now, to exhibit the pattern in general, if

$$R_{2k} = \begin{pmatrix} L_{2k} & 1 \\ -1 & 0 \end{pmatrix},$$

then

$$R_{2k}^n = \frac{1}{F_{2k}}\begin{pmatrix} F_{(2n+2)k} & F_{2nk} \\ -F_{2nk} & -F_{(2n-2)k} \end{pmatrix}.$$

This result has already been observed for $k = 1$ and $k = 2$, and is easily established by induction. Notice that

$$R_{2k} = \frac{1}{F_{2k}}\begin{pmatrix} F_{4k} & F_{2k} \\ -F_{2k} & 0 \end{pmatrix},$$

Since it is well known that $F_{4k} = F_{2k}L_{2k}$. We assume that R_{2k}^n has the above form; then

$$R_{2k}^n R = \frac{1}{F_{2k}}\begin{pmatrix} F_{(2n+2)k}L_{2k} - F_{2nk} & F_{(2n+2)k} \\ -F_{2nk}L_{2k} + F_{(2n-2)k} & -F_{2nk} \end{pmatrix}.$$

But, by the ubiquitous identity (B),

$$F_{2nk+2k+2} + F_{2nk+2k-2k} = F_{2nk+2k}L_{2k},$$

$$F_{2nk+2k} + F_{2nk-2k} = F_{2nk}L_{2k},$$

so that the matrix above has the desired form for R_{2k}^{n+1}. Thus, by mathematical induction, R_{2k}^n has the form prescribed above for all $n > 0$.

Observe that R^{-n} is given by

$$R^{-n} = \frac{1}{F_{2k}}\begin{pmatrix} -F_{(2n-2)k} & -F_{2nk} \\ F_{2nk} & F_{(2n+2)k} \end{pmatrix} = \frac{1}{F_{2k}}\begin{pmatrix} F_{(-2n+2)k} & F_{-2nk} \\ -F_{-2nk} & -F_{(-2n-2)k} \end{pmatrix}$$

and direct multiplication yields

$$R^nR^{-n} = \frac{1}{F_{2k}^2}\begin{pmatrix} -F_{(2n-2)k}F_{(2n+2)k} + F_{2nk}^2 & 0 \\ 0 & F_{2nk}^2 - F_{(2n-2)k}F_{(2n+2)k} \end{pmatrix}.$$

Since $\det(R_{2k}) = 1$, $\det(R_{2k}^n) = 1^n$, so that

$$F_{nk}^2 - F_{(2n+2)k}F_{(2n-2)k} = F_{2k}^2,$$

and we see that $R^nR^{-n} = I$ as well as exhibiting yet another identity arising from the prolific matrices R_{2k}. Also, since $F_{-k} = (-1)^{k+1}F_k$,

$$R_{2k}^0 = \frac{1}{F_{2k}}\begin{pmatrix} F_{2k} & F_0 \\ F_0 & -F_{-2k} \end{pmatrix} = I.$$

Hence, R_{2k}^n has the form given above for all integral exponents n.

The remaining piece of machinery needed is a general expression for the auxiliary S matrices which will raise the subscripts of R_{2k}^n. The matrix

$$S_s = \begin{pmatrix} F_{2k+s} & F_s \\ -F_s & -F_{s-2} \end{pmatrix}$$

adds s to each subscript for elements of R_{2k}^n, as seen by

$$R_{2k}^n S_s = \frac{1}{F_{2k}} \begin{pmatrix} F_{2nk+2k}F_{2k+s} - F_{2nk}F_s & F_{2nk+2k}F_s - F_{2nk}F_{s-2k} \\ -F_{2nk}F_{2k+s} + F_{2nk-2k}F_s & -F_{2nk}F_s + F_{2nk-2k}F_{s-2k} \end{pmatrix}$$

$$= \begin{pmatrix} F_{2nk+2k+s} & F_{2nk+s} \\ -F_{2nk+s} & -F_{2nk-2k+s} \end{pmatrix},$$

where the two matrices can be shown equal element by element. Each case can be demonstrated by judicious use of the known formula

$$F_n F_m - F_{n-k}F_{m+k} = (-1)^{n-k}F_k F_{m+k-n}.$$

Before leaving the matrix S_s, it is interesting to notice that

$$S_1^s = F_{2k}^{s-1}S_s \quad \text{and} \quad S_1 = F_{2k}\sqrt[2k]{R_{2k}}.$$

One more bit of information will allow us to give our most general results. The even-subscripted Lucas numbers have the following curious properties:

$$\left.\begin{aligned} L_{4n} + 2 &= L_{2n}^2, \\ L_{4n} - 2 &= 5F_{2n}^2, \\ L_{4n+2} + 2 &= 5F_{2n+1}^2, \\ L_{4n+2} - 2 &= L_{2n+1}^2. \end{aligned}\right\}$$

We demonstrate the first. If $\alpha = (1 + \sqrt{5})/2$, $\beta = (1 - \sqrt{5})/2$, then $L_n = \alpha^n + \beta^n$. Thus,

$$L_{2n}^2 = (\alpha^{2n} + \beta^{2n})^2 = \alpha^{4n} + \beta^{4n} + 2\alpha^{2n}\beta^{2n} = L_{4n} + 2,$$

since $\alpha\beta = -1$. The other three can be proved just as neatly.

Now the characteristic equation of R_{2k} gives us

$$R_{2k}^2 - L_{2k}R_{2k} + I = 0 \quad \text{or} \quad (R_{2k} \pm I)^2 = (L_{2k} \pm 2)R_{2k},$$

leading to the following by considering properties of even-subscripted Lucas numbers and raising each equation to the nth power:

(15) $$R_{4q}^m(R_{4q} + I)^{2n} = L_{2q}^{2n}R_{4q}^{n+m},$$

(16) $$R_{4q}^m(R_{4q} + I)^{2n+1} = L_{2q}^{2n}R_{4q}^{n+m}(R_{4q} + I),$$

(17) $$R_{4q}^m(R_{4q} - I)^{2n} = 5^nF_{2q}^{2n}R_{4q}^{n+m},$$

(18) $$R_{4q}^m(R_{4q} - I)^{2n+1} = 5^nF_{2q}^{2n}R_{4q}^{n+m}(R_{4q} - I),$$

(19) $$R_{4q+2}^m(R_{4q+2} + I)^{2n} = 5^nF_{2q+1}^{2n}R_{4q+2}^{n+m},$$

(20) $$R_{4q+2}^m(R_{4q+2} + I)^{2n+1} = 5^nF_{2q+1}^{2n}R_{4q+2}^{n+m}(R_{4q+2} + I),$$

(21) $$R_{4q+2}^m(R_{4q+2} - I)^{2n} = L_{2q+1}^{2n}R_{4q+2}^{n+m},$$

(22) $$R_{4q+2}^m(R_{4q+2} - I)^{2n+1} = L_{2q+1}^{2n}R_{4q+2}^{n+m}(R_{4q+2} - I).$$

For each equation above, we will write the binomial expansion, multiply by the auxiliary matrix S_s, and equate elements in the upper right, leading to the correspondingly numbered equations below. For equations (15′) through (18′), $s = 0, 1, 2, \ldots, 4q - 1$; and for equations (19′) through (22′), $s = 0, 1, 2, \ldots, 4q + 1$.

(15′)
$$\sum_{j=0}^{2n} \binom{2n}{j} F_{4q(j+m)+s} = L_{2q}^{2n} F_{4q(n+m)+s}$$

(16′)
$$\sum_{j=0}^{2n+1} \binom{2n+1}{j} F_{4q(j+m)+s} = L_{2q}^{2n}(F_{4q(n+m+1)+s} + F_{4q(n+m)+s}) = L_{2q}^{2n+1} F_{4q(n+m)+2q+s}$$

(17′)
$$\sum_{j=0}^{2n} (-1)^j \binom{2n}{j} F_{4q(j+m)+s} = 5^n F_{2q}^{2n} F_{4q(n+m)+s}$$

(18′)
$$\sum_{j=0}^{2n+1} (-1)^{j+1} \binom{2n+1}{j} F_{4q(j+m)+s} = 5^n F_{2q}^{2n}(F_{4q(n+m+1)+s} - F_{4q(n+m)+s}) = 5^n F_{2q}^{2n+1} L_{4q(n+m)+2q+s}$$

(19′)
$$\sum_{j=0}^{2n} \binom{2n}{j} F_{(4q+2)(j+m)+s} = 5^n F_{2q+1}^{2n} F_{(4q+2)(n+m)+s}$$

(20′)
$$\sum_{j=0}^{2n+1} \binom{2n+1}{j} F_{(4q+2)(j+m)+s} = 5^n F_{2q+1}^{2n}(F_{(4q+2)(n+m+1)+s} + F_{(4q+2)(n+m)+s}) = 5^n F_{2q+1}^{2n+1} L_{(4q+2)(n+m)+2q+1+s}$$

(21′)
$$\sum_{j=0}^{2n} (-1)^j \binom{2n}{j} F_{(4q+2)(j+m)+s} = L_{2q+1}^{2n} F_{(4q+2)(n+m)+s}$$

(22′)
$$\sum_{j=0}^{2n+1} (-1)^{j+1} \binom{2n+1}{j} F_{(4q+2)(j+m)+s} = L_{2q+1}^{2n}(F_{(4q+2)(n+m+1)+s} - F_{(4q+2)(n+m)+s}) = L_{2q+1}^{2n+1} F_{(4q+2)(n+m)+2q+1+s}$$

In each case, the proper Ruggles' identity (A) or (B) was applied.

Equations (15′) through (22′) can be rewritten in more compact forms which better display their properties. In equations (15′) through (18′) take $p = 4qm + s$ and in equations (19′) through (22′) take $p = (4q + 2)m + s$. Notice that since there are no restrictions on m and since s takes on any value from 0 through $4p - 1$ or $4q + 1$, respectively, p can be any integer. In the combined identity below, notice that equation (15′) is the case $r = 2n$ and (16′) the case $r = 2n + 1$:

(23)
$$\sum_{j=0}^{r} \binom{r}{j} F_{(2q)(2j)+p} = L_{2q}^{r} F_{2qr+p}.$$

Equations (17′) and (19′), respectively, lead to

$$\sum_{j=0}^{2n} (-1)^j \binom{2n}{j} F_{(2q)(2j)+p} = 5^n F_{2q}^{2n} F_{(2q)(2n)+p},$$

$$\sum_{j=0}^{2n} \binom{2n}{j} F_{(2q+1)(2j)+p} = 5^n F_{2q+1}^{2n} F_{(2q+1)(2n)+p},$$

which can be combined into the more general identity

$$\sum_{j=0}^{2n} (-1)^{j(t+1)} \binom{2n}{j} F_{2jt+p} = 5^n F_t^{2n} F_{2nt+p}.$$

Similarly, equations (18′) and (20′) can be condensed to the identity

$$\sum_{j=0}^{2n+1} (-1)^{(j+1)(t+1)}\binom{2n+1}{j}F_{2jt+p} = 5^n F^{2n+1} L_{(2n+1)t+p},$$

which becomes (18′) when $t = 2q$ and (20′) when $t = 2q + 1$.

Equations (21′) and (22′) lead to

$$\text{(24)} \qquad \sum_{j=0}^{r} (-1)^{r+j}\binom{r}{j}F_{(2q+1)(2j)+p} = L_{2q+1}^{r}F_{(2q+1)r+p},$$

which is (21′) when $r = 2n$ and (22′) when $r = 2n + 1$.

Finally, equations (23) and (24) taken together provide

$$\sum_{j=0}^{r} (-1)^{(r+j)t}\binom{r}{j}F_{2jt+p} = L_t^r F_{tr+p},$$

which is (23) when $t = 2q$ and (24) when $t = 2q + 1$.

Returning to the matrix R_{2k}^n, the element in its upper left can be shown to be

$$r_n = \sum_{j=0}^{[n/2]} (-1)^j\binom{n-j}{j}L_{2k}^{n-2j},$$

which form readily becomes apparent by computing the first few powers of R_{2k}. Notice that the binomial coefficients used appear on rising diagonals of Pascal's triangle. A proof by mathematical induction is outlined below. First, if $n = 1$, the expression becomes L_{2k}, the element in the upper left of R_{2k}, and if $n = 0$, we find $r_0 = 1$, the element in the upper left of $R_{2k}^0 = I$. From the characteristic equation of R_{2k}, the elements r_{p+1}, r_p, and r_{p-1} must satisfy $r_{p+1} = L_{2k}r_p - r_{p-1}$. Assume that r_p and r_{p-1} have the form given above. Then,

$$\begin{aligned} r_{p+1} &= \sum_{j=0}^{[p/2]} (-1)^j\binom{p-j}{j}L_{2k}^{p-2j} - \sum_{j=0}^{[(p-1)/2]} (-1)^j\binom{p-1-j}{j}L_{2k}^{p-1-2j} \\ &= \sum_{j=0}^{[p/2]} (-1)^j\binom{p-j}{j}L_{2k}^{p+1-2j} - \sum_{j=1}^{[(p+1)/2]} (-1)^{j-1}\binom{p-j}{j-1}L_{2k}^{p+1-2j} \\ &= \sum_{j=0}^{[(p+1)/2]} (-1)^j\binom{p+1-j}{j}L_{2k}^{p+1-2j} \end{aligned}$$

by the recursion relation for binomial coefficients and by carefully considering the end terms. Since r_{p+1} has the prescribed form whenever r_p and r_{p-1} do, r_n has the form given above for all integers $n \geq 0$.

Equating elements in the upper left for the matrix R_{2k}^n yields

$$\text{(25)} \qquad F_{2k}\sum_{j=0}^{[n/2]} (-1)^j\binom{n-j}{j}L_{2k}^{n-2j} = F_{(n+1)2k}.$$

Using equations (15), (17), (19), and (21) with $m = 0$ and equating elements in the upper left,

$$\text{(15″)} \qquad \sum_{k=0}^{2n}\sum_{j=0}^{[k/2]} (-1)^j\binom{2n}{k}\binom{k-j}{j}L_{4p}^{k-2j} = L_{2p}^{2n}\sum_{j=0}^{[n/2]} (-1)^j\binom{n-j}{j}L_{4p}^{n-2j},$$

$$\text{(17″)} \qquad \sum_{k=0}^{2n}\sum_{j=0}^{[k/2]} (-1)^{j+k}\binom{2n}{k}\binom{k-j}{j}L_{4p}^{k-2j} = 5^n F_{2p}^{2n}\sum_{j=0}^{[n/2]} (-1)^j\binom{n-j}{j}L_{4p}^{n-2j},$$

(19″) $$\sum_{k=0}^{2n}\sum_{j=0}^{[k/2]}(-1)^j\binom{2n}{k}\binom{k-j}{j}L_{4p+2}^{k-2j} = 5^nF_{2p+1}^{2n}\sum_{j=0}^{[n/2]}(-1)^j\binom{n-j}{j}L_{4p+2}^{n-2j},$$

(21″) $$\sum_{k=0}^{2n}\sum_{j=0}^{[k/2]}(-1)^{j+k}\binom{2n}{k}\binom{k-j}{j}L_{4p+2}^{k-2j} = L_{2p+1}^{2n}\sum_{j=0}^{[n/2]}(-1)^j\binom{n-j}{j}L_{4p+2}^{n-2j}.$$

Returning to the first expression given for R_{2k}^n, in which the element in the upper right is F_{2nk}/F_{2k}, a second proof can be given which utilizes Chebyshev polynomials. A special group of Chebyshev polynomials of the second kind are defined here by $u_0(\lambda) = 0$, $u_1(\lambda) = 1$, $u_{n+1}(\lambda) = 2\lambda u_n(\lambda) - u_{n-1}(\lambda)$. [Commonly, the starting values of the same series are taken as $u_0(\lambda) = 1$, $u_2(\lambda) = 2\lambda$.] Consider the known relationship:

$$\frac{x}{1 - 2\lambda x + x^2} = \sum_{n=0}^{\infty} u_n(\lambda)x^n.$$

However, as with H. W. Gould [3], for $\alpha = (1 + \sqrt{5})/2$, $\beta = (1 - \sqrt{5})/2$, we have, by summing the geometric series,

$$\frac{1}{1 - \alpha^{2k}x} - \frac{1}{1 - \beta^{2k}x} = \sum_{n=0}^{\infty}\alpha^{2kn}x^n - \sum_{n=0}^{\infty}\beta^{2kn}x^n,$$

which can be rewritten on both sides to yield

$$\frac{(\alpha^{2k} - \beta^{2k})x}{1 - (\alpha^{2k} + \beta^{2k})x + (\alpha\beta)^{2k}x^2} = \sum_{n=0}^{\infty}(\alpha^{2kn} - \beta^{2kn})x^n.$$

Since $\alpha\beta = -1$, $\alpha^n + \beta^n = L_n$, and $(\alpha^n - \beta^n)/(\alpha - \beta) = F_n$,

$$\frac{x}{1 - L_{2k}x + x^2} = \sum_{n=0}^{\infty}\frac{(\alpha^{2nk} - \beta^{2nk})/(\alpha - \beta)x^n}{(\alpha^{2k} - \beta^{2k})/(\alpha - \beta)} = \sum_{n=0}^{\infty}\frac{F_{2nk}}{F_{2k}}x^n$$

for $k \neq 0$. But, we also have, when $\lambda = L_{2k}/2$,

$$\frac{x}{1 - L_{2k}x + x^2} = \sum_{n=0}^{\infty}u_n(L_{2k}/2)x^n,$$

which implies that $u_n(L_{2k}/2) = F_{2nk}/F_{2k}$, $k \neq 0$.

Similar results are obtainable for the Fibonacci polynomials defined by $f_0(\lambda) = 0$, $f_1(\lambda) = 1$, $f_{n+1}(\lambda) = f_n(\lambda) + f_{n-1}(\lambda)$, which lead to

$$\frac{x}{1 - \lambda x - x^2} = \sum_{n=0}^{\infty} f_n(\lambda)x^n$$

and

$$f_n(L_{2k+1}) = F_{(2k+1)n}/F_{2k+1}.$$

A matrix having a Chebyshev polynomial as its characteristic polynomial is

$$R = \begin{pmatrix} 2\lambda & 1 \\ -1 & 0 \end{pmatrix},\quad R^n = \begin{pmatrix} u_{n+1}(\lambda) & u_n(\lambda) \\ -u_n(\lambda) & -u_{n-1}(\lambda) \end{pmatrix},$$

while for the Fibonacci polynomials such a matrix is

$$F = \begin{pmatrix} \lambda & 1 \\ 1 & 0 \end{pmatrix},\quad F^n = \begin{pmatrix} f_{n+1}(\lambda) & f_n(\lambda) \\ f_n(\lambda) & f_{n-1}(\lambda) \end{pmatrix}.$$

[Notice that, when $\lambda = 1$, $f_n(\lambda) = F_n$.]

By substituting $\lambda = L_{2k}/2$ in the above matrix R, we obtain

$$R_{2k} = \frac{1}{F_{2k}}\begin{pmatrix} F_{(2n+2)k} & F_{2nk} \\ -F_{2nk} & -F_{(2n-2)k} \end{pmatrix}.$$

Also, substituting $\lambda = L_{2k}/2$ into $u_{n+1}(\lambda) = 2\lambda u_n(\lambda) - u_{n-1}(\lambda)$ yields the expression for the general element in the upper left of R_{2k}^n as given in equation (25).

Since we could also show that

$$f_{n+1}(L_{2k+1}) = \sum_{j=0}^{[n/2]} \binom{n-j}{j} L_{2k+1}^{n-2j}$$

by substituting $\lambda = L_{2k+1}$ into the recursion formula for the Fibonacci polynomials, and since also $f_{n+1}(L_{2k+1}) = F_{(2k+1)(n+1)}/F_{2k+1}$, we can generalize equation (25) to the following:

$$F_{(n+1)p}/F_p = \sum_{j=0}^{[n/2]} (-1)^{j(p+1)}\binom{n-j}{j} L_p^{n-2j}, \quad p \neq 0,$$

which was a problem posed by H. H. Fern [4].

REFERENCES

1. I. D. Ruggles. "Some Fibonacci Results Using Fibonacci-Type Sequences." *The Fibonacci Quarterly* 1 (1963):75-80.
2. S. G. Guba. "Problem #174." *Matematika V Skole*. Issue #4 (July-August, 1965):73. (As reported by H. W. Gould.)
3. H. W. Gould. "Generating Functions for Products of Powers of Fibonacci Numbers." *The Fibonacci Quarterly* 1 (1963):1-16.
4. H. E. Fern in a letter to V. E. Hoggatt, Jr.

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ANTIMAGIC PENTAGRAMS WITH LINE SUMS IN ARITHMETIC PROGRESSION, $\Delta = 3$

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A pentagram or five-pointed star can be formed by extending the sides of a regular pentagon until they meet. This figure consists of five equal line segments that form a closed path. Each line intersects every other line, so that there are four intersections or vertices on each line, and two lines at each vertex.

A magic pentagram is formed by distributing ten elements on the vertices of a pentagram in such a way that the sum of the four elements (quartet) on one line equals each of the other four line sums. It has been shown [1, 2, 3, 4, 5] that no magic pentagram can be formed with the first ten positive integers.

An antimagic pentagram is one with five different line sums. Those formable with the first ten positive integers are formidably numerous. We restrict our search to those with five line sums in arithmetic progression and a common difference, $\Delta = 3$. In the sum of the five line sums, each element appears twice, so $5[2a + 4(3)]/2 = 2(55)$. Hence, the progression must be 16, 19, 22, 25, and 28.

The partitions of the five terms of this progression into four elements each < 11 are exhibited in Table 1. To make the table compact, 10 is recorded as X. Designate any quartet with a sum of x as an x-quartet. For the purposes of this discussion, two integers are said to be complementary if their sum is 11. Two quartets are complementary and two pentagrams are complementary if their corresponding elements are complementary.

To construct an antimagic pentagram, we start with the 16-quartet ($\underline{1}$, $\underline{2}$, $\underline{3}$, X) and seek a 19-quartet with which it has exactly one element in common, such as ($\underline{3}$, $\underline{7}$, 4, $\underline{5}$). A 22-quartet with exactly one element in common with each of these is ($\underline{2}$, $\underline{5}$, 6, $\underline{9}$). A 25-quartet with exactly one element in common with each of these three quartets is ($\underline{1}$, $\underline{7}$, 8, $\underline{9}$). The unduplicated elements, which are not underscored, in these four quartets form the 28-quartet (4, 6, 8, X). These five quartets can be distributed on the vertices of a pentagram with their line sums intact, as in Figure 1. Proceeding in this fashion to exhaust Table 1, we find 94 distributions exist in complementary pairs as, for example, in Figures 1 and 2.

TABLE 1

PARTITIONS OF LINE SUMS, $\Delta = 3$

16	19	22	25	28
1 2 3 X	1 2 6 X	1 2 9 X	1 5 9 X	1 8 9 X
1 2 4 9	1 2 7 9	1 3 8 X	1 6 8 X	2 7 9 X
1 2 5 8	1 3 5 X	1 4 7 X	1 7 8 9	3 6 9 X
1 2 6 7	1 3 6 9	1 4 8 9	2 4 9 X	3 7 8 X
1 3 4 8	1 3 7 8	1 5 6 X	2 5 8 X	4 5 9 X
1 3 5 7	1 4 5 9	1 5 7 9	2 6 7 X	4 6 8 X
1 4 5 6	1 4 6 8	1 6 7 8	2 6 8 9	4 7 8 9
2 3 4 7	1 5 6 7	2 3 7 X	3 4 8 X	5 6 7 X
2 3 5 6	2 3 4 X	2 3 8 9	3 5 7 X	5 6 8 9
	2 3 5 9	2 4 6 X	3 5 8 9	
	2 3 6 8	2 4 7 9	3 6 7 9	
	2 4 5 8	2 5 6 9	4 5 6 X	
	2 4 6 7	2 5 7 8	4 5 7 9	
	3 4 5 7	3 4 5 X	4 6 7 8	
		3 4 6 9		
		3 4 7 8		
		3 5 6 8		
		4 5 6 7		

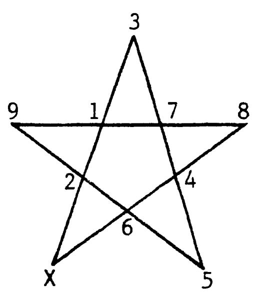

FIGURE 1

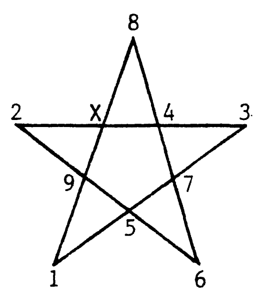

FIGURE 2

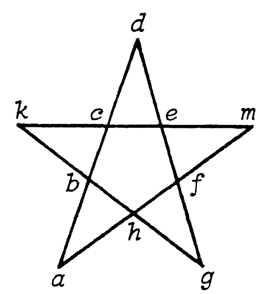

FIGURE 3

One of each complementary pair is listed in Table 2. To facilitate ready construction of any antimagic pentagram from its tabular entry, the vertices have been lettered continuously as in Figure 3. The 16-quartet (a, b, c, d) and the 19-quartet (d, e, f, g) are immediately evident in the table, while the 22-quartet (g, h, b, k), the 25-quartet (k, c, e, m), and the 28-quartet (m, f, h, a) are easily identified. The pentagrams are listed in the order of the appearance of the 16-quartets in Table 1. The asterisks (*) designate the distributions wherein the consecutive digits 2, 3, 4, 5, 6 appear in some order on the vertices of the constituent pentagon.

The distribution of the elements as recorded in Table 2 was made so that in progressing clockwise about the five-line closed path of the pentagram, the line sums would be in increasing order of magnitude. "Essentially, a particular element can appear only in one of two positions—a starpoint or a pentagon vertex. For any quartet, if one element is positioned, the other members can be permuted into 3! orders. Thus any quartet can appear on its line in exactly 2(3!) or 12 orders, not counting reflections. It follows from the tightly interwoven relationship of the quartets that every basic pattern on the pentagram can appear in 12 different guises, all having the same five quartets" [6].

The family of 12 antimagic pentagrams to which the first pentagram in Table 2 belongs is given in Table 3, with 2 in the restricted positions. The 16-, 19-, 22-, 25-, and 28-quartets may be represented by A, B, C, D, and E, respectively. The clockwise and counterclockwise orders of the quartets along each pentagram's closed path are shown in Table 3. The orders in the family comprise all the cyclic permutations of the five quartets.

TABLE 2

ANTIMAGIC PENTAGRAMS WITH LINE SUMS IN A. P., Δ = 3

a	*b*	*c*	*d*	*e*	*f*	*g*	*h*	*k*	*m*	*a*	*b*	*c*	*d*	*e*	*f*	*g*	*h*	*k*	*m*
X	2	1	3	7	4	5	6	9	8	8	2	5	1	6	9	3	7	X	4
*X	2	3	1	5	6	7	4	9	8	8	2	5	1	9	3	6	X	4	7
*X	2	3	1	6	4	8	5	7	9	8	2	5	1	9	6	3	X	7	4
X	2	3	1	8	6	4	7	9	5	8	5	2	1	6	9	3	4	X	7
X	2	3	1	9	4	5	8	7	6	8	5	2	1	9	3	6	7	4	X
X	3	1	2	7	4	6	5	8	9	6	1	2	7	4	5	3	8	X	9
X	3	1	2	8	5	4	6	9	7	6	2	1	7	5	4	3	8	9	X
*X	3	2	1	6	5	7	4	8	9	6	2	7	1	5	4	9	8	3	X
X	3	2	1	9	5	4	7	8	6	6	2	7	1	5	X	3	8	9	4
4	2	9	1	3	X	5	8	7	6	6	7	2	1	X	5	3	8	4	9
4	2	9	1	5	6	7	X	3	8	7	1	6	2	5	3	9	8	4	X
9	1	2	4	5	3	7	6	8	X	7	1	6	2	5	9	3	8	X	4
9	1	2	4	7	5	3	8	X	6	7	6	1	2	5	8	4	3	9	X
*9	4	2	1	5	6	7	3	8	X	7	6	1	2	X	4	3	8	5	9
9	4	2	1	7	8	3	5	X	6	8	1	3	4	7	6	2	9	X	5
9	4	2	1	8	3	7	6	5	X	8	4	3	1	7	9	2	6	X	5
9	4	2	1	X	5	3	8	7	6	8	4	3	1	X	6	2	9	7	5
5	2	8	1	3	6	9	7	4	X	5	1	3	7	4	6	2	9	X	8
5	2	8	1	6	9	3	X	7	4	5	7	3	1	X	6	2	9	4	8
5	8	2	1	6	9	3	4	7	X	7	1	5	3	6	8	2	9	X	4
5	8	2	1	9	6	3	7	4	X	7	5	1	3	X	4	2	9	6	8
*8	2	5	1	3	6	9	4	7	X	4	1	5	6	3	8	2	9	X	7
8	2	5	1	3	9	6	4	X	7	4	1	6	5	2	9	3	8	X	7
8	2	5	1	6	3	9	7	4	X										

TABLE 3

ANTIMAGIC PENTAGRAM FAMILY WITH COMMON LINE ELEMENTS

										Sequences of Sums	
a	*b*	*c*	*d*	*e*	*f*	*g*	*h*	*k*	*m*	Clockwise	Counterclockwise
X	2	1	3	7	4	5	6	9	8	*ABCDE*	*AEDCB*
X	2	3	1	7	8	9	6	5	4	*ADCBE*	*AEBCD*
3	2	X	1	8	7	9	5	6	4	*ADCEB*	*ABECD*
3	2	1	X	8	4	6	5	9	7	*AECDB*	*ABDCE*
1	2	3	X	4	8	6	9	5	7	*AECBD*	*ADBCE*
1	2	X	3	4	7	5	9	6	8	*ABCED*	*ADECB*
2	3	X	1	8	9	7	5	4	6	*ADBEC*	*ACEBD*
2	3	1	X	8	6	4	5	7	9	*AEBDC*	*ACDBE*
2	X	3	1	7	9	8	6	4	5	*ADEBC*	*ACBED*
2	X	1	3	7	5	4	6	8	9	*ABEDC*	*ACDEB*
2	1	3	X	4	6	8	9	7	5	*AEDBC*	*ACBDE*
2	1	X	3	4	5	7	9	8	6	*ABDEC*	*ACEDB*

It is not customary to count rotations and reflections of configurations as separate arrangements. With this qualification, there are 2(47)(12) or 1128 distinct antimagic pentagrams with line sums forming an arithmetic progression that has a common difference of 3.

There are other antimagic pentagrams with line sums in arithmetic progressiong having common differences of 1 [7], 2 [8], and 4 [9].

REFERENCES

1. Michael R. W. Buckley & Frank Rubin. "Solution of Problem 385: Do Pentacles Exist?" *Journal of Recreational Mathematics* 10 (1977-78):288-289.

2. N. M. Dongre. "More about Magic Star Polygons." *American Math. Monthly* 78 (1971):1025.
3. Harry Langman. *Play Mathematics*. New York: Hafner, 1962. Pp. 80-83.
4. Ian Richards. "Impossibility." *Math. Magazine* 48 (1975):249-262.
5. C. W. Trigg. "Solution of Problem 113." *Pi Mu Epsilon Journal* 3 (1960):119-120.
6. Charles W. Trigg. "Ten Elements on a Pentagram." *EUREKA* (Canada) 3 (1977):5-6.
7. Charles W. Trigg. "Antimagic Pentagrams with Consecutive Line Sums." *Journal of Recreational Mathematics* 10 (1977-78):169-173.
8. Charles W. Trigg. "Antimagic Pentagrams with Consecutive Even Line Sums" (submitted to *Mathematics Teacher*).
9. Charles W. Trigg. "Antimagic Pentagrams with Line Sums in Arithmetic Progression" (submitted to *Math. Magazine*).

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TWO FAMILIES OF TWELFTH-ORDER MAGIC SQUARES

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A family of 24,769,797,950,537,728 twelfth-order magic squares can be generated from the basic 9-digit third-order magic square (1) of Figure 1 and the 880 basic fourth-order magic squares.

First, add 9 to each element of square (1) to form square (2) in Figure 1, and repeat the operation until the fifteen derived squares of Figure 1 have been formed. Each of these squares is magic and remains magic in eight orientations: the square itself, its rotations through 90°, 180°, and 270°, and the mirror images of these four.

	(1)			*(2)*			*(3)*			*(4)*	
8	1	6	17	10	15	26	19	24	35	28	33
3	5	7	12	14	16	21	23	25	30	32	34
4	9	2	13	18	11	22	27	20	31	36	29

	(5)			*(6)*			*(7)*			*(8)*	
44	37	42	53	46	51	62	55	60	71	64	69
39	41	43	48	50	52	57	59	61	66	68	70
40	45	38	49	54	47	58	63	56	62	72	65

	(9)			*(10)*			*(11)*			*(12)*	
80	73	78	89	82	87	98	91	96	107	100	105
75	77	79	84	86	88	93	95	97	102	104	106
76	81	74	85	90	83	94	99	92	103	108	101

	(13)			*(14)*			*(15)*			*(16)*	
116	109	114	125	118	123	134	127	132	143	136	141
111	113	115	120	122	124	129	131	133	138	140	142
112	117	110	121	126	119	130	135	128	139	144	137

FIGURE 1. Sixteen 3-by-3 Magic Squares

To construct twelfth-order magic squares, divide a 12-by-12 grid into sixteen 3-by-3 grids, thus forming a 4-by-4 grid of grids. Label this 4-by-4 grid with the elements of one of the basic fourth-order magic squares, such as the familiar pandiagonal square:

1	14	4	15
8	11	5	10
13	2	16	3
12	7	9	6

In each small 3-by-3 grid place the 3-by-3 square, in any of its eight orientations, that has the same identification number as the grid. In forming the twelfth-order square in Figure 2, a different orientation has been given to each of the 3-by-3 squares in the first two rows of the 4-by-4 grid. The same procedure has been followed in filling th last two rows. Indeed, the orientations are such that the square in Figure 2 is pandiagonal. (In a pandiagonal square, the elements in every row, column, and diagonal, broken and unbroken, have the same sum.) Its magic constant is 870.

8	1	6	121	120	125	29	36	31	132	133	128
3	5	7	126	122	118	34	32	30	127	131	135
4	9	2	119	124	123	33	28	35	134	129	130
69	64	71	98	93	94	40	45	38	83	88	87
79	68	66	91	95	99	39	41	43	90	86	82
65	72	67	96	97	92	44	37	42	85	84	89
116	109	114	13	12	17	137	144	139	24	25	20
111	113	115	18	14	10	142	140	138	19	23	27
112	117	110	11	16	15	141	136	143	26	21	22
105	100	107	62	57	58	76	81	74	47	52	51
106	104	102	55	59	63	75	77	79	54	50	46
101	108	103	60	61	56	80	73	78	49	48	53

FIGURE 2. *A Pandiagonal 12-by-12 Magic Square*

Since each of the 880 fourth-order basic magic squares can be used as a foundation (labelling) square, and each small grid can be filled in eight ways, $8^{16}(880)$ or 247 69797 95053 77280 distinct twelfth-order magic squares (exclusive of rotations and reflections) can be constructed in this manner from the first 144 positive integers.

A larger family of twelfth-order magic squares can be constructed by first taking any nine fourth-order magic squares (repetition permitted) from the 880 squares listed by Benson and Jacoby [1]. In Figure 3, the squares have been ordered by their upper left elements.

The first square in Figure 3 is square (1) in Figure 4. To each element of the second square add 16 to form square (2), to each element of square three add $2 \cdot 16$ to form square (3), and continue the process until the addition of $8 \cdot 16$ to the elements of the ninth square forms square (9).

To construct the twelfth-order magic squares, divide a 12-by-12 grid into nine 4-by-4 grids, thus forming a 3-by-3 grid of grids. Label this 3-by-3 grid with the corresponding elements of the basic third-order magic square (1) in Figure 1. In each 4-by-4 grid place the derived square from Figure 4, in any of its eight orientations, that has the same identification number as the gird. The result is the twelfth-order magic square in Figure 5.

Since any of the 880 fourth-order magic squares can be the basic square for a 4-by-4 grid, and the corresponding derived square can be inserted into the grid in 8 ways, $(880 \cdot 8)^9$ or 4 24770 09370 18688 57788 98944 x 10^9 twelfth-order squares (exclusive of rotations and reflections) can be constructed in this way from the first 144 positive integers.

1	15	14	4	2	5	11	16	3	2	14	15
12	6	7	9	14	9	7	4	13	16	4	1
8	10	11	5	15	8	10	1	12	9	5	8
13	3	2	16	3	12	6	13	6	7	11	10
4	6	9	15	5	2	15	12	6	3	15	10
13	11	8	2	10	16	1	7	4	9	5	16
7	1	14	12	11	13	4	6	13	8	12	1
10	16	3	5	8	3	14	9	11	14	2	7
7	4	14	9	8	5	11	10	9	1	8	16
16	13	3	2	9	12	6	7	14	12	5	3
1	12	6	15	2	3	13	16	4	6	11	13
10	5	11	8	15	14	4	1	7	15	10	2

FIGURE 3. Nine Basic 4-by-4 Magic Squares

(1)				*(2)*				*(3)*			
1	15	14	4	18	21	27	32	35	34	46	47
12	6	7	9	30	25	23	20	45	48	36	33
8	10	11	5	31	24	26	17	44	41	37	40
13	3	2	16	19	28	22	29	38	39	43	42
(4)				*(5)*				*(6)*			
52	54	57	63	69	66	79	76	86	83	95	90
61	59	56	50	74	80	65	71	84	89	85	96
55	49	62	60	75	77	68	70	93	88	92	81
58	64	51	53	72	67	78	73	91	94	82	87
(7)				*(8)*				*(9)*			
103	100	110	105	120	117	123	122	137	129	136	144
112	109	99	98	121	124	118	119	142	140	133	131
97	108	102	111	114	115	125	128	132	134	139	141
106	101	107	104	127	126	116	113	135	143	138	130

FIGURE 4. Nine Derived 4-by-4 Magic Squares

120	117	123	122	13	8	12	1	87	82	94	91
121	124	118	119	3	10	6	15	81	92	88	93
114	115	125	128	2	11	7	14	96	85	89	84
127	126	116	113	16	5	9	4	90	95	83	86
47	46	34	35	69	66	79	76	105	98	111	104
33	36	48	45	74	80	65	71	110	99	102	107
40	37	41	44	75	77	68	70	100	109	108	101
42	43	39	38	72	67	78	73	103	112	97	106
52	61	55	58	135	143	138	130	29	17	20	32
54	59	49	64	132	134	139	141	22	26	23	27
57	56	62	51	142	140	133	131	28	24	25	21
63	50	60	53	137	129	136	144	19	31	30	18

FIGURE 5. A 12-by-12 Magic Square

Together the two families contain $8^{17}(110)(8 \cdot 110^8 + 1)$ distinct twelfth-order magic squares.

This technique can be employed to produce two families of knth order magic squares from magic squares of the kth and nth orders. If $k = n$, there is one family. Such is the family of 134,217,728 ninth-order magic squares [2].

REFERENCES

1. William H. Benson & Oswald Jacoby. *New Recreations with Magic Squares.* New York: Dover, 1976. Pp. 187-198.
2. Charles W. Trigg. "A Family of Ninth-Order Magic Squares." *Mathematics Magazine* (accepted for publication).

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COIN TOSSING AND THE r-BONACCI NUMBERS

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In this paper we find the probability that a fair coin tossed n times will contain a run of at least r consecutive heads.

Let $X_n = \{x_1x_2 \ldots x_n / x_i \in \{h,t\},\ i = 1, 2, \ldots, n\}$ be the set of 2^n equi-probable outcomes and Y_n^r be the subset of X_n each of whose elements contains a run of at least r consecutive heads. Also, let $a(r,n)$ be the cardinality of Y_n^r. We can construct Y_n^r by noting that each of its elements must fall into one of the following two categories:

(1) $$HA_{n-r}$$

(2) $$W_j tHA_{n-j-1-r}$$

where H is the first run of r consecutive heads to appear when reading from left to right,

A_i is an i-string of any combination of heads and tails,

W_j is a j-string of heads and tails not containing H, and

t is a singleton tail.

Since there are $2^j - a(r,j)$ ways in which W_j can occur, the total number of elements of type (2) is

$$\sum_{j=0}^{n-1-r} [2^j - a(r,j)]2^{n-j-1-r}.$$

Summing over all possibilities for (1) and (2) we obtain

$$a(r,n) = 2^{n-r} + \sum_{j=0}^{n-1-r} [2^j - a(r,j)]2^{n-j-1-r}$$

$$(3) \qquad = 2^{n-r}\left[1 + (n - r)/2 - \sum_{j=r}^{n-1-r} a(r,j)/2^{j+1}\right].$$

The next three lemmas exhibit some relationships among the $a(r,n)$.

Lemma 1: If $n \geq r$, then

$$a(r,n) = 2^{n-r} + \sum_{j=n-r}^{n-1} a(r,j).$$

Proof: Clearly $a(k,k) = 1$ for all $k > 0$. If we rewrite (3) and assume the lemma true for $n - 1$, we have

$$a(r,n) = 2\left\{2^{n-r-1}\left[1 + (n - 1 - r)/2 - \sum_{j=r}^{n-2-r} a(r,j)/2^{j+1}\right] + 2^{n-2-r} - a(r,n - 1 - r)/2\right\}$$

$$= 2a(r,n - 1) + 2^{n-1-r} - a(r,n - 1 - r)$$

$$= a(r,n - 1) + \left\{2^{n-1-r} + \sum_{j=n-1-r}^{n-2} a(r,j)\right\} + 2^{n-1-r} - a(r,n - 1 - r)$$

$$= 2^{n-r} + \sum_{j=n-r}^{n-1} a(r,j);$$

thus, the lemma holds for n and the proof follows by induction.

The next lemma relates $a(r,n)$ to the r-Bonacci numbers $F_m^{(r)}$ where $F_1^{(r)} = 1$, $F_m^{(r)} = 2^{m-2}$ for $m = 2, \ldots, r + 1$, and $F_m^{(r)} = F_{m-r}^{(r)} + \cdots + F_{m-1}^{(r)}$ for $m = r + 2, r + 3, \ldots$.

Lemma 2: If $r \leq n$, then

$$(4) \qquad a(r,n) = \sum_{j=1}^{n-r+1} F_j^{(r)} 2^{n-r+1-j}.$$

Proof: We know that $a(r,r) = 1 = F_1^{(r)}$, which proves the case $n = r$. For $n > r$, assume that for $i = r, r + 1, \ldots, n - 1$,

$$a(r,i) = \sum_{j=1}^{i-r+1} F_j^{(r)} 2^{i-r+1-j}$$

then

$$a(r,n) = 2^{n-r} + \sum_{i=n-r}^{n-1} a(r,i)$$

$$= 2^{n-r} + \sum_{i=n-r}^{n-1}\left\{\sum_{j=1}^{i-r+1} F_j^{(r)} 2^{i-r+1-j}\right\}$$

$$= 2^{n-r} + \sum_{j=1}^{n-r}\left\{\sum_{i=j-r+1}^{j} F_i^{(r)}\right\}2^{n-r-j}$$

$$= 2^{n-r} + \sum_{j=1}^{n-r}\left\{\sum_{i=1}^{r} F_{j-r+1}^{(r)}\right\}2^{n-r-j},$$

which reduces to (4) and the proof is completed by induction.

The last lemma, proved by Swamy [4], is a generalization of a problem posed by Carlitz [3].

Lemma 3: $\sum_{j=1}^{m} F_j^{(r)} 2^{m-j} = 2^{m+r+1} - F_{m+r+1}^{(r)}$.

We are now in a position to calculate the desired probability.

Theorem: The probability that n tosses of a fair coin will contain a run of at least r consecutive heads, $r \leq n$, is given by $1 - F_{n+2}^{(r)}/2^n$.

Proof: Apply Lemma 3 to (4) with $m = n - r + 1$.

REFERENCES

1. S. Bezuszka and L. D'Angelo. "An Application of Tribonacci Numbers." *The Fibonacci Quarterly* 15 (1977):140-144.
2. M. Bicknell and V. E. Hoggatt, Jr. "Generalized Fibonacci Polynomials." *The Fibonacci Quarterly* 11 (1973):457-465.
3. L. Carlitz. Problem B-135. *The Fibonacci Quarterly* 6 (1968):90.
4. M. N. S. Swamy. "A Formula for $\sum_{k=1}^{n} F_k(x)y^{n-k}$ and Its Generalization to r-Bonacci Polynomials." *The Fibonacci Quarterly* 15 (1977):73-77.

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COMBINATORIAL IDENTITIES DERIVED FROM UNITS

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ABSTRACT

We shall derive two combinatorial identities by considering units in infinite classes of cubic fields. This is a comparatively new application of units.

0. INTRODUCTION

We shall begin by stating a result of Bernstein and Hasse [2] concerning systems of units in infinitely many fields.

Theorem: Let $P(x)$ be a polynomial of degree $n \geq 2$ with the form

$$P(x) = (x - D_0)(x - D_1) \ldots (x - D_{n-1}) - d,\ d \geq 1,\ D_i, d \in Z,\ D_0 \equiv D_i \pmod{d},$$
$$D_0 - D_i \geq 2d(n - 1),\ (i = 1, \ldots, n - 1),\ D_0 > D_1 > \ldots > D_{n-1}.$$

Then $P(x)$ has exactly n distinct real roots; $P(x)$ is irreducible over Q; and if w is the largest root of $P(x)$, then

$$e_i = \frac{(w - D_i)^n}{d} \qquad (i = 0, \ldots, n - 1)$$

are different units of $Q(w)$. Furthermore, any $n - 1$ of these units form a system of independent units.

1. COMBINATORIAL IDENTITIES FROM UNITS

Consider the cubic polynomials $P(x) = (x - D_0)(x - D_1)(x - D_2) - 1$; D_i as above. First we work with the case $D_2 = 0$; later we will eliminate this condition. Now it is clear that itself is a unit in $Q(w)$ with $N(w) = 1$. We proceed by expressing the integral powers of w. For any integer $n \geq 0$, let

$$w^n = x_n + y_n w + z_n w^2 \qquad (x_n, y_n, z_n \in Z). \tag{1.1}$$

Calculating directly and taking into account that $w^3 = 1 - Bw + Aw^2$ where $A = D_0 + D_1$ and $B = D_0 D_1$, we have

$$\begin{aligned} w^{n+1} &= z_n + (x_n - Bz_n)w + (y_n + Az_n)w^2 \\ w^{n+2} &= (y_n + Ax_n) + (z_n - By_n - ABz_n)w + (x_n - Bz_n + Ay_n + A^2 z_n)w^2; \end{aligned} \tag{1.2}$$

so that

$$x_{n+1} = z_n;\ y_{n+1} = x_n - Bx_{n+1};\ z_{n+1} = x_{n-1} - Bx_n + Ax_{n+1}. \tag{1.3}$$

From (1.1) and (1.3), we obtain

$$w^n = x_n + (x_{n-1} - Bx_n)w + (x_{n-2} - Bx_{n-1} + Ax_n)w^2 \tag{1.4}$$

along with the recursion formula

(1.5) $$x_{n+3} = x_n - Bx_{n+1} + Ax_{n+2};\ n \geq 0.$$

Now in order to write x_n explicitly, we shall make use of generating functions together with (1.5) to obtain

(1.6) $$\sum_{n=0}^{\infty} x_n u^n = (1 - Au + Bu^2) \sum_{n=0}^{\infty} (A - Bu + u^2)^n u^n.$$

It should be noted that for the sake of convergence, u can be chosen such that $|Au - Bu^2 + u^3| < 1$. Equating coefficients in (1.6),

(1.7) $$x_n = \sum_{i=2}^{n-2} \sum_{\ell=1} (-1)^i \begin{pmatrix} & n - i - 1 & \\ n - 2i + \ell & i - 2\ell & \ell - 1 \end{pmatrix} A^{n-2i+\ell} B^{i-2\ell}.$$

In the same manner we calculate the negative powers of w.

(1.8) $$w^{-n} = r_n + s_n w + t_n w^2, \quad (n \in N;\ r_n, s_n, t_n \in Z)$$

(1.9) $$w^{-n} = r_n + (r_{n-2} - Ar_{n-1})w + r_{n-1}w^2 \text{ where } r_{n+3} = r_n - Ar_{n+1} + Br_{n+2}.$$

As before, we apply generating functions to obtain

$$\sum_{n=0}^{\infty} r_n u^n = \sum_{n=0}^{\infty} (B - Au + u^2)^n u^n.$$

Comparing coefficients of equal powers of u gives us the relation

(1.10) $$r_n = \sum_{i=0}^{n-1} \sum_{\ell=0} (-1)^i \begin{pmatrix} & n - i & \\ n - 2i + \ell & i - 2\ell & \ell \end{pmatrix} A^{i-2\ell} B^{n-2i+\ell}.$$

We return to formulas (1.1) and (1.8) and multiply right and left sides together to obtain, after some rearrangements, the following three equations with r_n, s_n, t_n as unknowns:

$$1 = x_n r_n + z_n s_n + (y_n + Az_n)t_n;$$
$$0 = y_n r_n + (x_n - Bz_n)s_n + (-By_n + z_n - ABz_n)t_n;$$
$$0 = z_n r_n + (y_n + Az_n)s_n + (x_n + Ay_n - Bz_n + A_n^2 z_n)t_n.$$

The determinant of the system is equal to the norm of w^n as can be seen from (1.1) and (1.2). But $N(w) = 1$. Therefore, we have

(1.11) $$r_n = \begin{vmatrix} x_n - Bz_n & -By_n + z_n - ABz_n \\ y_n + Az_n & x_n + Ay_n - Bz_n + A_n^2 z_n \end{vmatrix}.$$

From (1.2), (1.5), and (1.11),

$$r_n = \begin{vmatrix} x_{n+3} - Ax_{n+2} & x_{n+4} - Ax_{n+3} \\ x_{n+2} & x_{n+3} \end{vmatrix}.$$

(1.12) $$r_n = x_{n+3}^2 - x_{n+2}x_{n+4}.$$

We have at last reached our first combinatorial identity by considering (1.12) in conjunction with (1.7) and (1.10), which express the x_n and the r_n as combinatorial functions.

If we now consider x_n as an unknown and solve the original system of equations, the determinant of the system becomes $-N(w^{-n}) = -1$; so that

$$x_n = -\begin{vmatrix} r_n - Bt_n & -Bs_n + t_n - ABt_n \\ s_n + At_n & r_n + As_n - Bt_n + A_n^2 t_n \end{vmatrix}.$$

Substituting for s_n and t_n in terms of r_n from (1.9) and recalling that $r_{n+1} = r_{n-2} - Ar_{n-1} + Br_n$, we obtain our second combinatorial identity

(1.13) $$x_n = r_{n-3}^2 - r_{n-2}r_{n-4} \quad (n \geq 5).$$

Note that no generality was lost by assuming $D_2 = 0$, since by setting $w - D_2 = \overline{w}$ and working with the equation

$$\overline{w}^3 + (2D_2 - D_0 - D_1)\overline{w}^2 + (D_2 - D_0)(D_2 - D_1)\overline{w} - 1 = 0,$$

we would obtain the same identities with A and B replaced by $\bar{A} = -2D_2 + D_0 + D_1$ and $\bar{B} = (D_2 - D_0)(D_2 - D_1)$, respectively.

REFERENCES

1. L. Bernstein. "Zeros of Combinatorial Functions and Combinatorial Identities." *Houston J. Math.* 2, No. 1 (1976):9-15.
2. L. Bernstein and H. Hasse. "An Explicit Formula for Units of an Algebraic Number Field of Degree $n \geq 2$." *Pacific J.* 30 (1969):293-365.

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A STOLARSKY ARRAY OF WYTHOFF PAIRS

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A *positive Fibonacci sequence* is a sequence $\{s_k\}$ such that $s_{k+1} = s_k + s_{k-1}$ and $s_k > 0$ for k sufficiently large. A *Stolarsky array* is an array $A = \{A_{m,n}:m,n \quad 0\}$ of natural numbers such that:

(a) the rows $\{A_{m,1}, A_{m,2}, \ldots\}$ are positive Fibonacci sequences;

(b) every natural number occurs exactly once in the array;

(c) every positive Fibonacci sequence is a row of the array, after a shift of indices. That is, given a positive Fibonacci sequence $\{s_j\}$, there exist m and k such that

$$A_{m,n} = s_{n+k}.$$

The first such array[1] was constructed by Stolarsky [8]. In this note, we will construct a new Stolarsky array using Wythoff pairs. By inspecting the tables of these two arrays, it is easy to obtain more Stolarsky arrays. (For example, in either table, the 4 may be shifted from the second to the third row.) It would be interesting to have a classification of the Stolarsky arrays.

Let $\alpha = \frac{1}{2}(1 + \sqrt{5})$ be the golden ratio, and let [ ] denote the greatest integer function. The *Wythoff pairs* are the pairs of numbers $([n\alpha], [n\alpha^2])$ which give the winning positions in Wythoff's game (see [5], for example). These pairs have two remarkable properties:

1. *Beatty complementarity* [2]—Every natural number m is either of the form $[n\alpha]$ or of the form $[n\alpha^2]$, but not both.

2. *Connell's formula* [4]—

$$[n\alpha] + [n\alpha^2] = [[n\alpha^2]\alpha].$$

<u>*Lemma 1*</u>: Let $s_1 = [k\alpha]$, $s_2 = [k\alpha^2]$ generate a positive Fibonacci sequence. Then (s_{2j-1}, s_{2j}) is a Wythoff pair for every $j > 0$.

<u>*Proof*</u>: Since $\alpha^2 = \alpha + 1$, we have

$$(*) \qquad n + [n\alpha] = [n\alpha^2].$$

Suppose $(s_{2j-3}, s_{2j-2}) = ([m\alpha], [m\alpha^2])$ is a Wythoff pair. Then by Connell's formula,

$$s_{2j-1} = [m\alpha] + [m\alpha^2] = [[m\alpha^2]\alpha],$$

while by formula (*),

$$s_{2j} = [m\alpha^2] + [[m\alpha^2]\alpha] = [[m\alpha^2]\alpha^2].$$

Thus, (s_{2j-1}, s_{2j}) is a Wythoff pair, and the lemma follows by induction.

We define the *Wythoff array* to be an array $W = \{W_{m,n}\}$ which is Fibonacci in its rows, and is generated by:

$$W_{m,1} = [[m\alpha]\alpha], \; W_{m,2} = [[m\alpha]\alpha^2].$$

The first 100 terms of the Wythoff array are listed in Table 1.

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[1]That (c) holds for Stolarsky's array does not seem to have been noticed. We will verify it as Corollary 2, below.

TABLE 1

1	2	3	5	8	13	21	34	55	89
4	7	11	18	29	47	76	123	199	322
6	10	16	26	42	68	110	178	288	466
9	15	24	39	63	102	165	267	432	699
12	20	32	52	84	136	220	356	576	932
14	23	37	60	97	157	254	411	665	1076
17	28	45	73	118	191	309	500	809	1309
19	31	50	81	131	212	343	555	898	1453
22	36	58	94	152	246	398	644	1042	1686
25	41	66	107	173	280	453	733	1186	1919

According to Lemma 1, $(W_{m,2k-1}, W_{m,2k})$ is a Wythoff pair for every m, k.

Lemma 2: The Wythoff array contains all the Wythoff pairs.

Proof: First note that $(W_{1,1}, W_{1,2}) = ([\alpha], [\alpha^2])$. Suppose the $\{[n\alpha], [n\alpha^2]\} \subset W$ for all $n < N$. By Beatty complementarity, $N = [m\alpha]$ or $N = [m\alpha^2]$. If $N = [m\alpha^2]$, then by Lemma 1, one of the rows of W contains

$$\ldots, [m\alpha], [m\alpha^2], [N\alpha], [N\alpha^2], \ldots .$$

On the other hand, if $N = [m\alpha]$, then $(W_{m,1}, W_{m,2}) = ([N\alpha], [N\alpha^2])$. In either case, $\{[N\alpha], [N\alpha^2]\} \subset W$, and the lemma is proved by induction.

Corollary: The Wythoff array contains each natural number exactly once.

After a shift of indices, any positive Fibonacci sequence $\{s_j\}$ will satisfy $0 \leq 2s_0 < s_1$ (see [1]). We say that such a sequence is in *standard form*.

Theorem: Let $\{s_j\}$ be a Fibonacci sequence in standard form. For any $\omega > 0$, and for all sufficiently large k,

$$s_{2k} < s_{2k-1}\alpha < s_{2k} + \omega.$$

Proof: A simple induction shows that

$$s_j = s_0 F_{j+2} + (s_1 - 2s_0)F_j ,$$

where $\{F_j\}$ is Fibonacci's sequence $F_0 = F_1 = 1$. Using the continued fraction expansion for α (cf. [7]) and the formula

$$F_{2k+1}F_{2k-1} = F_{2k}^2 + 1,$$

we see that

$$\frac{F_{2k}}{F_{2k+1}} < \alpha < \frac{F_{2k-1}}{F_{2k}} = \frac{F_{2k}}{F_{2k-1}} + \frac{1}{F_{2k}F_{2k-1}},$$

so that

$$F_{2k} < F_{2k-1}\alpha < F_{2k} + \frac{1}{F_{2k}}.$$

Let k be large enough so that

$$\frac{s_0}{F_{2k+2}} + \frac{s_1 - 2s_0}{F_{2k}} < \omega.$$

Then

$$\begin{aligned} s_{2k} &= s_0 F_{2k+2} + (s_1 - 2s_0)F_{2k} \\ &< s_0 F_{2k+1}\alpha + (s_1 - 2s_0)F_{2k-1}\alpha \\ &= s_{2k-1}\alpha \\ &< s_0 F_{2k+2} + (s_1 - 2s_0)F_{2k} + \frac{s_0}{F_{2k+2}} + \frac{s_1 - 2s_0}{F_{2k}} \\ &< s_{2k} + \omega, \end{aligned}$$

which proves the theorem.

Corollary 1: The Wythoff array is a Stolarsky array.

Proof: Let $\{s_j\}$ be a Fibonacci sequence in standard form. We must show that $\{s_j\}$ is a row of W, after a shift of indices. For k large, we have

$$s_{2k} < s_{2k-1}\alpha < s_{2k} + 1,$$

so that $[s_{2k-1}\alpha] = s_{2k}$. Thus, by formula (*), (s_{2k}, s_{2k+1}) is a Wythoff pair. The corollary now follows from Lemma 2.

Corollary 2: Stolarsky's array is a Stolarsky array.

Proof: We recall the definition of Stolarsky's array. Let $g(x) = [x\alpha + \frac{1}{2}]$. It is easy to check that for any natural number k, $\{k, g(k), g^2(k), \ldots\}$ forms a positive Fibonacci sequence. Stolarsky's array $S = \{S_{m,n}\}$ is defined by

$$S_{1,1} = 1;$$
$$S_{m,1} = \text{the smallest natural number not in } \{S_{k,n} : k < m\};$$
$$S_{m,n} = g^{n-1}(S_{m,1}).$$

The first 100 terms of Stolarsky's array are listed in Table 2.

TABLE 2

1	2	3	5	8	13	21	34	56	89
4	6	10	16	26	42	68	110	178	288
7	11	18	29	47	76	123	199	322	521
9	15	24	39	63	102	165	267	432	699
12	19	31	50	81	131	212	343	555	898
14	23	37	60	97	157	254	411	665	1076
17	28	45	73	118	191	309	500	809	1309
20	32	52	84	136	220	356	576	932	1508
22	36	58	94	152	246	398	644	1042	1686
25	40	65	105	170	275	445	720	1165	1885

By construction, S satisfies condition (a). Condition (b) was proved by Hendy [6; Theorem 1]. To check condition (c), we apply the above theorem with $\omega = \frac{1}{2}$. For k large enough,

$$s_{2k} < s_{2k-1}\alpha < s_{2k-1}\alpha + \tfrac{1}{2} < s_{2k} + 1,$$

so that $g(s_{2k-1}) = [s_{2k-1}\alpha + \frac{1}{2}] = s_{2k}$. By Hendy's theorem, $s_{2k-1} = S_{m,n}$ for some m, n. But then, $s_{2k} = g(S_{m,n}) = S_{m,n+1}$, so that $\{s_j\}$ is the mth row of S, after a shift in indices.

Stolarsky's conjecture, proved by Butcher [3] and by Hendy [6], says that

$$\{S_{m,2} - S_{m,1}\} = \{S_{m,1}\} \cup \{S_{m,2}\}.$$

There is an analogous statement for the Wythoff array:

Proposition: $\{W_{m,2} - W_{m,1}\} = \bigcup_{k \geq 0} \{W_{m,2k+1}\}$.

Proof: $W_{m,2} - W_{m,1} = [[m\alpha]\alpha^2] - [[m\alpha]\alpha] = [m\alpha]$ by formula (*). Since $(W_{m,2k+1}, W_{m,2k+2})$ is always a Wythoff pair,

$$\{[m\alpha]\} = \bigcup_{k \geq 0} \{W_{m,2k+1}\}.$$

REFERENCES

1. Brother U. Alfred. "On the Ordering of Fibonacci Sequences." *The Fibonacci Quarterly* 1. No. 4 (1963):43-46.
2. S. Beatty. Problem 3173. *American Math. Monthly* 33 (1926):159; 34 (1927):159-160.
3. J. C. Butcher. "On a Conjecture Concerning a Set of Sequences Satisfying the Fibonacci Difference Equation." *The Fibonacci Quarterly* 16 (1978):81-83.
4. I. G. Connell. "Some Properties of Beatty Sequences, I." *Canadian Math. Bulletin* 2 (1959):190-197.
5. H. S. M. Coxeter. "The Golden Section, Phyllotaxis, and Wythoff's Game." *Scripta Mathematica* 19 (1953):135-143.
6. M. D. Hendy. "Stolarsky's Distribution of the Positive Integers." *The Fibonacci Quarterly* 16 (1978):70-80.
7. C. D. Olds. *Continued Fractions*. New York: Random House, Inc., 1963.
8. K. B. Stolarsky. "A Set of Generalized Fibonacci Sequences Such That Each Natural Number Belongs to Exactly One." *The Fibonacci Quarterly* 15 (1977):224.

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AN APPLICATION OF THE FIBONACCI SEARCH TECHNIQUE TO DETERMINE OPTIMAL SAMPLE SIZE IN A BAYESIAN DECISION PROBLEM

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ABSTRACT

In cases where computational difficulties or lack of knowledge about the functional form of a curve preclude the use of analytical methods for determining a maximum, various search techniques can be employed. In Bayesian decision problems, an optimal sample size is based on a maximum expected net gain from sampling. When ENGS is plotted against a range of admissible values of n it is often computationally difficult to determine the maximum. This paper demonstrates how a sequential search technique based on the Fibonacci numbers can be used to determine that value with a minimum number of computations.

AN APPLICATION OF THE FIBONACCI SEARCH TECHNIQUE TO DETERMINE OPTIMAL SAMPLE SIZE IN A BAYESIAN DECISION PROBLEM

One of the most common problems in applied mathematics is the determination of an optimal (max, min) point on the curve of some functional relationship. Sometimes, either because of computational difficulty or lack of knowledge concerning the functional form of the curve itself, it is not feasible to find this optimal point analytically. In such cases, a search technique is a powerful tool. This paper deals with the application of the Fibonacci search technique to the problem of determining the optimal sample size for obtaining additional information in a two-action decision situation with a linear cost function.

THE OPTIMAL SAMPLE SIZE PROBLEM

In any decision problem the question of purchasing additional information is generally approached by comparing the expected value of perfect information, EVPI, with the cost of sampling. EVPI is also equivalent to the cost of uncertainty. Since perfect information can never be obtained from a sample and since it is uneconomical to pay more for information than it could be worth, an amount greater than EVPI should never be spent on sampling. Therefore, the only size samples that would even be considered are those for which the cost of sampling is not greater than EVPI.

For most samples, the cost of a sample of size n can generally be expressed as:

(1) $$C(n) = C_f + nC_v$$

where C_f is the fixed cost of sampling and C_v is the variable cost under the assumption that the incremental cost of each additional sampled unit is the same. The maximum sample size is therefore:

(2) $$n_{\max} \leq \frac{\text{EVPI} - C_f}{C_v} .$$

Any sample size such that $0 \leq n \leq n_{\max}$ is therefore feasible. The problem is to determine the value of n in this range which is optimal. We will designate the optimal value of n as n^*.

The expected value of the information obtained from any sample of size n can be determined from the expected reduction in the cost of uncertainty that could be achieved with the sample. That is the difference between the EVPI prior to taking the sample and the EVPI after or posterior to the sample. This is computed by means of an extensive form analysis or pre-posterior analysis as described by Sasaki [3], Schlaifer [4], and others. This expected value of sample information is abbreviated EVSI. The expected net gain from sampling, ENGS, is simply $\text{EVSI}(n) - C(n)$, that is, the expected value of information from the sample less the cost of obtaining that sample.

The optimal sample size in a decision problem is that value of n, n^*, in the range 0 to n_{max}, for which ENGS is a maximum. Whenever the cost of sampling is high relative to EVPI, n_{max} will be reasonably small and n^* can be determined by simply computing ENGS for every admissible n. However, when the cost of uncertainty is great and sampling costs are not high, this procedure requires a large amount of tedious computations even when performed by computer. Therefore, shortcuts for obtaining n^* are desirable. One such shortcut method would be a Fibonacci search technique.

For the Fibonacci search technique to be effective, it is necessary that ENGS have a single maximum value in the range 0 to n_{max}. Raiffa and Schlaifer [2] have shown that, for two-action problems with linear cost functions, if ENGS has any positive values at all in this range, it will have a single maximum. Consequently, the Fibonacci search technique can be used to find the maximum value of ENGS that corresponds to the optimal sample size in a decision problem of this type.

THE FIBONACCI SEARCH TECHNIQUE

Assume that we are looking for the maximum of a particular curve in the interval (a,b). Then by experimentation we gather information about the curve and reduce the length of our interval of uncertainty. In search techniques, all points of experimentation may be known in advance (preplanned) or information gathered from previous experiments may be used to select the next experimental point (sequential search). The Fibonacci technique is a sequential search technique.

In searching for the maximum of $f(x)$ on the interval (a,b) of length L we perform two experiments as shown in Figure 1. The experiments are performed at points c and d such that the length of $(a,d) = l_1$, and the length of $(c,b) = l_2$. In order to obtain equal intervals, we should let $l_1 = l_2$. If x^* is the true maximum in the interval (a,b), then it follows that:

1. if $f(c) > f(c)$, $x^* \in (a,d)$,
2. if $f(d) > f(c)$, $x^* \in (c,b)$,
3. if $f(c) = f(d)$, $x^* \in (c,d)$.

Case 3 would be extremely rare, and in general either 1 or 2 would occur.

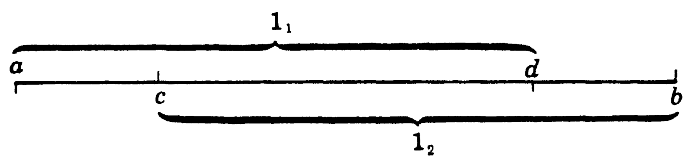

FIGURE 1. Points of Initial Experiments on Interval L

In any case, the new interval of uncertainty would be no greater than l_1. Utilizing the Fibonacci technique after the initial two experiments, it is necessary to perform at most one experiment to determine the next interval.

Now, let us look at the Fibonacci numbers. For any $n \geq 2$, the Fibonacci number, $F_n = F_{n-1} + F_{n-2}$. The Fibonacci series for the first few values of n is:

n:	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	...
F_n:	1	1	2	3	5	8	13	21	34	55	89	144	233	377	610	987	...

If we let k be the number of experiments to be performed and L be the length of the initial interval of uncertainty, then after k experiments the interval of uncertainty will be reduced to

$$L_k = \frac{1}{F_k} L.$$

The Fibonacci technique is employed to select sequentially the specific points of experimentation.

AN ILLUSTRATIVE EXAMPLE

As an illustration consider the regret table for a decision problem provided in Table 1 where θ is the unknown decision parameter representing the states of nature and $P_0(\theta)$ is the prior distribution on θ. The EVPI that corresponds to the expected regret of the better act is \$1,050. Assume that $C_f = 0$ and $C_v = \$50$ per unit sampled. From equation (2), n_{max} must be 21 and the initial interval of uncertainty within which n^* must lie is (0,21).

Since we want our search technique to reduce this interval of uncertainty and we know that

$$L_{final} = \frac{1}{F_k} L$$

where k is the number of experimental points that will be taken, we must find the smallest Fibonacci number such that $F_n \geq 21$. From the Fibonacci series above, we can see that $F_7 = 21$ and therefore $k = 7$. This is the maximum number of experiments that it will be necessary to perform.

TABLE 1. Regret Table

		Regret	
$\tilde{\theta}$	$P_0(\tilde{\theta})$	a_1	a_2
0.30	0.70	\$1500	\$ 0
0.10	0.30	\$ 0	\$6000
	Expected Regret	\$1050	\$1800
EVPI = \$1050			

The procedure is as follows (the calculation of ENGS for each experiment is provided in the Appendix):

$$L_1 = (0,21), \qquad k = 7;$$

$$L_2 = \frac{F_{k-1}}{F_k} \cdot L_1 = \frac{F_6}{F_7} \cdot L_1 = \frac{13}{21} \cdot 21 = 13.$$

Our initial two points for experimentation will be the two points that are exactly 13 units from the endpoints of the initial interval, L_1. Therefore, L_2 is either (0,13) or (8,21). To determine which of these possible intervals contains n^*, an experiment is conducted at the points $n = 8$ and $n = 13$. That is, ENGS is computed for these two sample sizes resulting in $f(8) = \$34.77$ and $f(13) = -\$50.65$. Since $f(8) > f(13)$, the new interval of uncertainty must be (0,13), that is, $L_2 = (0,13)$.

The next interval,

$$L_3 = \frac{F_5}{F_6} \cdot L_2 = \frac{8}{13} \cdot 13 = 8.$$

By the same procedure employed above, the new interval will be determined by points which are 8 units from tne endpoints of the previous interval. The two new points are $n = 8$ and $n = 5$, and L_3 is either (0,8) or (5,13). It is necessary to compare $f(8)$ with $f(5)$ to determine which of these two possible intervals contains n^*. Since we have already computed $f(8)$, it is now only necessary to determine $f(5)$, which is \$97.88, ENGS for a sample of size 5. This property of the Fibonacci search technique which, after the initial two experiments, makes it necessary to conduct only one additional experiment for each additional paired comparison, is one of its great advantages.

Since $f(5) = \$97.88 > f(8) = \$34.77$, the new interval of uncertainty is $L_3 = (0,8)$.

Proceeding,

$$L_4 = \frac{F_4}{F_5} \cdot L_3 = \frac{5}{8} \cdot 8 = 5,$$

and the new interval is either (0,5) or (3,8). Since $f(5) = \$97.88 > f(3) = \$59.22$, the new interval is $L_4 = (3,8)$.

$$L_5 = \frac{F_3}{F_4} \cdot L_4 = \frac{3}{5} \cdot 5 = 3.$$

The new interval is either (3,6) or (5,8).

Since $f(5) = \$97.88 > f(6) = \$80.04$, the new interval is $L_5 = (3,6)$.

$$L_6 = \frac{F_2}{F_3} \cdot L_5 = \frac{2}{3} \cdot 3 = 2.$$

The new interval is either (3,5) or (4,6).

Since $f(5) = \$97.88 > f(4) = \$70.93$, the new interval is (4,6).

$$L_7 = \frac{F_1}{F_2} \cdot L_6 = \frac{1}{2} \cdot 2 = 1.$$

The new interval is one unit (5,5), which is optimal.

At the second-to-last stage, where L was determined to be the interval (4,6), we had already computed $f(4)$, $f(5)$, and $f(6)$, and by comparing the three values could easily see that $n^* = 5$.

Figure 2 shows graphically how the original interval of uncertainty (0,21) was reduced to the interval (4,6). In order to arrive at n^*, ENGS had to be computed for only six values of n, $n = 8$, $n = 13$, $n = 5$, $n = 3$, $n = 6$, and $n = 4$. This is in contrast to having to compute ENGS for 21 integer values in the original interval.

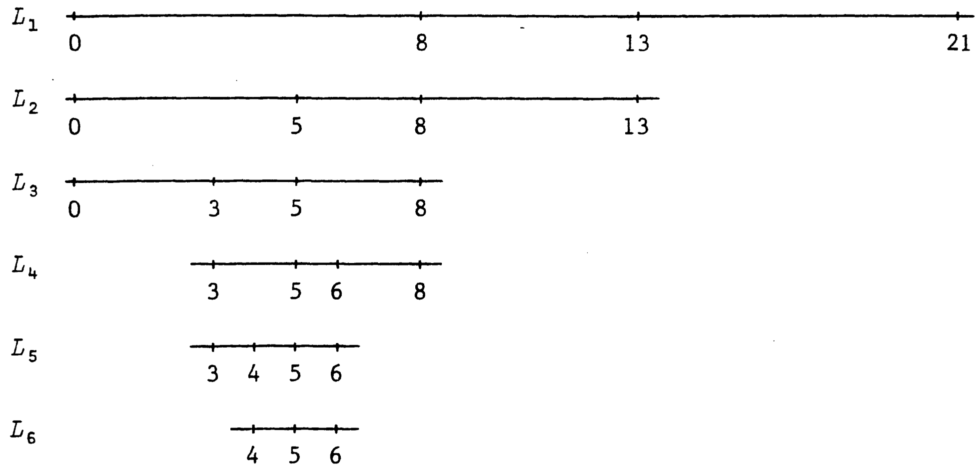

FIGURE 2. Reduction of the Interval of Uncertainty

The computational savings increase even more dramatically as the length of L increases. This is easily seen by looking back to the Fibonacci series and observing, for example, that if $L = (0,987)$, at most 15 experiments would be required, since $F_{15} = 987$.

APPENDIX

Calculation of ENGS

Experiment #1: n = 8

	$\tilde{\theta}$	$P_0(\tilde{\theta})$	$P(x\|\tilde{\theta})$	$P(x \cap \tilde{\theta})$	$P_1(\tilde{\theta})$	Regret a_1	Regret a_2
$x = 0$:	0.3	0.7	0.06	0.042	0.25	1500	0
	0.1	0.3	0.43	0.129	0.75	0	6000
				0.171		375*	4500
$x = 1$:	0.3	0.7	0.20	0.140	0.55	1500	0
	0.1	0.3	0.38	0.114	0.45	0	6000
				0.254		825*	2700
$x = 2$:	0.3	0.7	0.30	0.210	0.82	1500	0
	0.1	0.3	0.15	0.045	0.18	0	6000
				0.255		1230	1080*
$x = 3$:	0.3	0.7	0.25	0.175	0.95	1500	0
	0.1	0.3	0.03	0.009	0.05	0	6000
				0.184		1426	300*
$x = 4$:	0.3	0.7	0.14	0.098	0.97	1500	0
	0.1	0.3	0.01	0.003	0.03	0	6000
				0.101		1500	180*
$x = 5$:	0.3	0.7	0.05	0.035	1	1500	0
	0.1	0.3	0.00	0	0	0	6000
				0.035		1500	0*

(continued)

	$\tilde{\theta}$	$P_0(\tilde{\theta})$	$P(x\|\tilde{\theta})$	$P(x\tilde{\theta})$	$P_1(\tilde{\theta})$	Regret a_1	Regret a_2
$x = 6$:	0.3	0.7	0.01	0.007	1	1500	0
	0.1	0.3	0.00	0	0	0	6000
				0.007		1500	0*
$x = 7$:	0.3	0.7	0.0	0			
	0.1	0.3	0.0	0			
$x = 8$:	0.3	0.7	0.0	0			
	0.1	0.3	0.0	0			

Summary of Posterior Expected Regret for n = 8, X

X	Decision	Marginal Probability	Regret
0	a_1	0.171	375
1	a_1	0.254	825
2	a_2	0.255	1056
3	a_2	0.184	294
4	a_2	0.101	180
5	a_2	0.035	0
6	a_2	0.007	0
7	a_2	0	0
8	a_2	0	0
		Posterior Expected Regret:	615.23

Prior EVPI	1050.00
Post EVPI	-615.23
EVSI (8)	434.77
$C(n = 8)$	-400.00
ENGS (8)	34.77

Experiment #2: n = 13

	$\tilde{\theta}$	$P_0(\tilde{\theta})$	$P(x\|\tilde{\theta})$	$P(x\tilde{\theta})$	$P_1(\tilde{\theta})$	Regret a_1	Regret a_2
$x = 0$:	0.3	0.7	0.10	0.007	0.09	1500	0
	0.1	0.3	0.254	0.076	0.91	0	6000
				0.083		135*	5460
$x = 1$:	0.3	0.7	0.054	0.038	0.26	1500	0
	0.1	0.3	0.367	0.110	0.74	0	6000
				0.148		390*	4440
$x = 2$:	0.3	0.7	0.140	0.098	0.57	1500	0
	0.1	0.3	0.245	0.074	0.43	0	6000
				0.172		855*	2580
$x = 3$:	0.3	0.7	0.22	0.154	0.84	1500	0
	0.1	0.3	0.10	0.030	0.16	0	6000
				0.184		1260	960*
$x = 4$:	0.3	0.7	0.230	0.161	0.85	1500	0
	0.1	0.3	0.028	0.008	0.05	0	6000
				0.169		1425	300*
$x = 5$:	0.3	0.7	0.180	0.126	0.99	1500	0
	0.1	0.3	0.006	0.002	0.01	0	6000
				0.128		1485	60*

(continued)

	$\tilde{\theta}$	$P_0(\tilde{\theta})$	$P(x\|\tilde{\theta})$	$P(x\tilde{\theta})$	$P_1(\tilde{\theta})$	Regret	
						a_1	a_2
x = 6:	0.3	0.7	0.103	0.072	1	1500	0
	0.1	0.3	0	0	0	0	6000
				0.072		1500	0*

For x = 7 through x = 13, $E(R_{a_1})$ = 1500$E(R_{a_2})$ = 0

Summary of Posterior Expected Regret for n = 13, X

X	Decision	Marginal Probability	Regret
0	a_1	0.083	135
1	a_1	0.148	390
2	a_1	0.172	855
3	a_2	0.184	960
4	a_2	0.169	300
5	a_2	0.128	60
6	a_2	0.072	0
7	a_2	0.031	0
8	a_2	0.0	0
9	a_2	0.0	0
10	a_2	0.0	0
11	a_2	0.0	0
12	a_2	0.0	0
13	a_2	0.0	0
		Posterior Expected Regret:	450.65

Prior EVPI	1050.00
Post EVPI	-450.65
EVSI (13)	599.35
$C(n = 13)$	-650.00
ENGS (13)	- 50.65

Experiment #3: n = 5

	$\tilde{\theta}$	$P_0(\tilde{\theta})$	$P(x\|\tilde{\theta})$	$P(x\tilde{\theta})$	$P_1(\tilde{\theta})$	Regret	
						a_1	a_2
x = 0:	0.3	0.7	0.17	0.119	0.40	1500	0
	0.1	0.3	0.59	0.177	0.60	0	6000
				0.296		600*	3600
x = 1:	0.3	0.7	0.36	0.252	0.72	1500	0
	0.1	0.3	0.33	0.099	0.28	0	6000
				0.351		1080*	1680
x = 2:	0.3	0.7	0.31	0.217	0.91	1500	0
	0.1	0.3	0.07	0.021	0.09	0	6000
				0.238		1365	540*
x = 3:	0.3	0.7	0.13	0.091	0.97	1500	0
	0.1	0.3	0.01	0.003	0.03	0	6000
				0.094		1455	180*
x = 4:	0.3	0.7	0.03	0.04	1	1500	0
	0.1	0.3	0	0	0	0	6000
				0.021		1500	0*

(continued)

	$\tilde{\theta}$	$P_0(\tilde{\theta})$	$P(x\|\tilde{\theta})$	$P(x\tilde{\theta})$	$P_1(\tilde{\theta})$	Regret a_1	Regret a_2
$x = 5$:	0.3	0.7	0	0		1500	0
	0.1	0.3	0	0		0	6000
							0*

Summary of Posterior Expected Regret for $n = 5$, X

X	Decision	Marginal Probability	Regret
0	a_1	0.296	600
1	a_1	0.351	1080
2	a_2	0.238	540
3	a_2	0.094	180
4	a_2	0.021	0
5	a_2	0	0
		Posterior Expected Regret:	702.12

Prior EVPI	1050.00
Post EVPI	-702.12
EVSI (5)	347.88
$C(n = 5)$	-250.00
ENGS (5)	97.88

Experiment #4: $n = 3$

	$\tilde{\theta}$	$P_0(\tilde{\theta})$	$P(x\|\tilde{\theta})$	$P(x\tilde{\theta})$	$P_1(\tilde{\theta})$	Regret a_1	Regret a_2
$x = 0$:	0.3	0.7	0.34	0.238	0.52	1500	0
	0.1	0.3	0.73	0.219	0.48	0	6000
				0.457		780*	2880
$x = 1$:	0.3	0.7	0.44	0.308	0.81	1500	0
	0.1	0.3	0.24	0.072	0.19	0	6000
				0.380		1215	1140*
$x = 2$:	0.3	0.7	0.19	0.133	0.94	1500	0
	0.1	0.3	0.03	0.009	0.06	0	6000
				0.142		1410	360*
$x = 3$:	0.3	0.7	0.03	0.021	1	1500	0
	0.1	0.3	0	0	0	0	6000
				0.021		1500	0*

Summary of Posterior Expected Regret for $n = 3$, X

X	Decision	Marginal Probability	Regret
0	a	0.457	780
1	a	0.380	1140
2	a	0.142	360
3	a	0.021	0
		Posterior Expected Regret:	840.78

Prior EVPI	1050.00
Post EVPI	-840.78
EVSI (3)	209.22
$C(n = 3)$	-150.00
ENGS (3)	59.22

Experiment #5: n = 6

	$\tilde{\theta}$	$P_0(\tilde{\theta})$	$P(x\mid\tilde{\theta})$	$P(x\tilde{\theta})$	$P_1(\tilde{\theta})$	Regret	
						a_1	a_2
x = 0:	0.3	0.7	0.12	0.084	0.35	1500	0
	0.1	0.3	0.53	0.159	0.65	0	6000
				0.243		525*	3910
x = 1:	0.3	0.7	0.31	0.217	0.67	1500	0
	0.1	0.3	0.35	0.105	0.33	0	6000
				0.322		1005*	1980
x = 2:	0.3	0.7	0.32	0.224	0.88	1500	0
	0.1	0.3	0.10	0.030	0.12	0	6000
				0.254		1320	720*
x = 3:	0.3	0.7	0.19	0.133	0.96	1500	0
	0.1	0.3	0.02	0.006	0.04	0	6000
				0.139		1440	240*
x = 4:	0.3	0.7	0.060	0.0420	0.99	1500	0
	0.1	0.3	0.001	0.0003	0.01	0	6000
				0.0423		1485	60*
x = 5:	0.3	0.7	0.01	0.007	1	1500	0
	0.1	0.3	0	0	0	0	6000
				0.007		1500	0*
x = 6:	0.3	0.7	0	0		1500	0
	0.1	0.3	0	0		0	6000
							0*

Summary of Posterior Expected Regret for n = 6, X

X	Decision	Marginal Probability	Regret
0	a_1	0.243	525
1	a_1	0.322	1005
2	a_2	0.254	720
3	a_2	0.139	240
4	a_2	0.042	60
5	a_2	0.007	0
6	a_2	0	0
		Posterior Expected Regret:	669.96

Prior EVPI	1050.00
Post EVPI	-669.96
EVSI (6)	380.04
$C(n = 6)$	-300.00
ENGS (6)	80.04

Experiment #6: n = 4

	$\tilde{\theta}$	$P_0(\tilde{\theta})$	$P(x\mid\tilde{\theta})$	$P(x\tilde{\theta})$	$P_1(\tilde{\theta})$	Regret	
						a_1	a_2
x = 0:	0.3	0.7	0.24	0.168	0.46	1500	0
	0.1	0.3	0.66	0.198	0.54	0	6000
				0.366		690*	3240
x = 1:	0.3	0.7	0.41	0.287	0.77	1500	0
	0.1	0.3	0.29	0.087	0.23	0	6000
				0.374		1155*	1380

(continued)

	$\tilde{\theta}$	$P_0(\tilde{\theta})$	$P(x\|\tilde{\theta})$	$P(x\tilde{\theta})$	$P_1(\tilde{\theta})$	Regret a_1	Regret a_2
$x = 2$:	0.3	0.7	0.26	0.182	0.92	1500	0
	0.1	0.3	0.29	0.015	0.23	0	6000
				0.197		1380	480*
$x = 3$:	0.3	0.7	0.08	0.056	1	1500	0
	0.1	0.3	0.0	0	0	0	6000
				0.056		1500	0*
$x = 4$:	0.3	0.7	0.01	0.007	1	1500	0
	0.1	0.3	0.0	0	0	0	6000
				0.007		1500	0*

Summary of Posterior Expected Regret for n = 4, X

X	Decision	Marginal Probability	Regret
0	a_1	0.366	690
1	a_1	0.374	1155
2	a_2	0.197	480
3	a_2	0.056	0
4	a_2	0.007	0
		Posterior Expected Regret:	779.07

Prior EVPI	1050.00
Post EVPI	-779.07
EVSI (4)	270.93
$C(n = 4)$	-200.00
ENGS (4)	70.93

REFERENCES

1. G. Beveridge and R. Schecter. *Optimization: Theory and Practice.* New York: McGraw-Hill, 1970.
2. H. Raiffa and R. Schlaifer. *Applied Statistical Decision Theory.* Cambridge, Mass.: The MIT Press, 1961.
3. K. Sasaki. *Statistics for Modern Business Decision Making.* Belmont, Calif.: Wadsworth, 1968.
4. R. Schlaifer. *Probability and Statistics for Business Decisions.* New York: McGraw-Hill, 1959.

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SIMULTANEOUS TRIBONACCI REPRESENTATIONS

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1. INTRODUCTION AND DEFINITIONS

The two-sided sequence $\{t_n\}_{-\infty}^{\infty}$ of Tribonacci numbers is defined by $t_{-1} = 0$, $t_0 = 0$, $t_1 = 1$ and the recursion $t_{n+3} = t_{n+2} + t_{n+1} + t_n$. A Tribonacci representation of the integer a is an expression $a = \Sigma K_i t_i$ where $\{K_n\}_{-\infty}^{\infty}$ is a finitely nonzero sequence of integers.

This paper attempts to generalze to Tribonacci representations some of the results of Robert Silber's and my joint paper [7], "The Ring of Fibonacci Representations." I advise reading that paper before this one because, among other reasons, there one can see how much can be done in the order 2 case.

It is a pleasure to acknowledge here the extensive and essential assistance that Professor Silber gave me in working on the present paper.

Although I had originally planned to attempt to generalize all of [7], for a variety of reasons only parts of Section 3 of [7] were attempted. Some terminology must be introduced to explain these generalizations.

A finitely nonzero sequence of integers $\{K_i\}_{-\infty}^{\infty}$ will be called canonical (of order *three*) iff

A. Either (a) all the nonzero K_n are +1 or (b) all the nonzero K_n are -1;

B. No *three* consecutive k's are nonzero.

If (a) holds, we call the sequence positive canonical; if (b) holds, it is negative canonical.

Theorem 3.6 of [7] generalizes straightforwardly to: Every triple of integers (a,b,c) can be written $(\Sigma K_i t_i,\ \Sigma K_i t_{i+1},\ \Sigma K_i t_{i+2})$ for a unique canonical sequence $\{K_i\}$. These are the "simultaneous Tribonacci representations" of the title. The resolution algorithm, which (among other things) enables one to find the sequence given the triple, was altered from that of [7] not in an essential way.

For a finitely nonzero sequence $\{K_n\}$ define the upper (lower) degree to be the largest (smallest) integer p (r) such that $K_p \neq 0$ ($K_r \neq 0$). By definition, the identically zero sequence has lower degree $+\infty$ and upper degree $-\infty$. By a straightforward generalization of the order 2 case, those triples for which the associated canonical sequence has given upper degree are found.

Theorem 3.4 in [7], "Every integer r has a unique positive canonical Fibonacci representation with negative upper degree" has the not so obvious generalization "Every integer pair (a,b) can be written $(\Sigma K_i t_i,\ \Sigma K_i t_{i+1})$ for a unique positive canonical sequence $\{K_n\}$ of upper degree ≤ -2."

The above results which comprise Sections 2 and 3 of this paper, are mostly obvious enough generalizations of [7] that they are included here only because they are needed in parts of Sections 4 and 5, which attempt to answer the question of which triples have canonical sequences with given lower degree. The answer to the analogous question for the Fibonacci case is fairly easy to state (see Theorem 3.10, [7]).

Carlitz, Scoville, and Hoggatt [2] show that the solution to a problem intimately related to the Tribonacci lower degree problem is not the obvious generalization of the order 2 answer.

I have not solved the lower-degree problem. However, a computer-drawn region in the complex plane is shown to have the property that a certain algebraic expression in a, b, c, and r lies in this region if the associated canonical sequence has lower degree r.

For the above problem of Carlitz-Scoville-Hoggatt, a computer draws a diagram which divides the unit square into regions, and it is shown that this diagram is a solution to the problem in the sense that explicit formulas for this diagram would solve the problem.

In practice, however, accuracy is guaranteed only for a (probabilistic) propostion of integers. One must make some calculations with irrational numbers and plot a point on the unit square. If this point is far enough away from the curves of the diagram, one is assured of accuracy. The probability of accuracy can be increased by improving the accuracy of the calculations, of plotting points on the square, and of the diagram itself by increase computer time and improving the accuracy of the computer's sketching ability, finer tipped pens, etc.

Since there is no practical need at present for more accurate approximate solutions of this problem, I have made fiarly rough diagrams, and paid more attention to the variety of theoretical questions which appear.

Certain questions can be answered completely even with the rough sketches, though.

I had hopes before the first very rough diagram was drawn that it would turn out to be some familiar shape which would indicate the correct analytic solution. However, the complicated and unfamiliar shape that appeared indicates that any analytic solution is likely to be very complicated.

2. THE RESOLUTION ALGORITHM

Approximations to the three roots of $x^3 - x^2 - x - 1 = 0$ are:

$$\alpha = 1.839286754$$
$$\beta = -.419643377 + .606290729i$$
$$\gamma = -.419643377 - .606290729i$$

$Z[\alpha]$ forms a ring and also is a free module of dimension 3 over Z; $\{1, \alpha, \alpha^2\}$ is one basis. α is invertible in $Z[\alpha]$. In fact, $0 = \alpha^3 - \alpha^2 - \alpha - 1$ implies $\alpha^{-1} = -1 - \alpha + \alpha^2$. We take $\{\alpha^{-2}, \alpha^{-1}, 1\}$ as the standard basis of $Z[\alpha]$ over Z.

Let A be the linear transformation of IR^3 defined by $A(d,e,f) = (f,\ d+f,\ e+f)$.

Lemma 1: Given any three integers d_0, e_0, and f_0 not all zero, let $(d_n, e_n, f_n) = A\ (d_0, e_0, f_0)$. Then, for sufficiently large n, the three integers d_n, e_n, and f_n are of the same sign.

Proof: The characteristic polynomial of A is $x^3 - x^2 - x - 1$. An eigenvector associated with maximum eigenvalue α is $(1, \alpha^{-1} + 1, \alpha)$. Thus, as $n \to \infty$, either

(a) $(1, e_n/d_n, f_n/d_n) \to (1, \alpha^{-1} + 1, \alpha)$

or

(b) $(d_n, e_n, f_n) \to (0, 0, 0)$ (since $|\beta| = |\gamma| < 1$).

(This is an application of the "power method," see [5, Section 9.6].) Since d_n, e_n, and f_n are integers not all zero, (b) cannot hold. Then (a) implies that d_n, e_n, and f_n must all eventually have the same sign.

The next theorem is found generalized in [3, Theorem A] except for a slightly different version of uniqueness. Also the alternate proof of existence here is by means of a practical algorithm.

We shall call two finitely nonzero sequences of integers $\{K_n\}_{-\infty}^{\infty}$ and $\{K_n'\}_{-\infty}^{\infty}$ equivalent if $\Sigma K_n X_n = \Sigma K_n' X_n$ for every complex sequence $\{X_n\}_{-\infty}^{\infty}$ which satisfies $X_{n+3} = X_{n+2} + X_{n+1} + X_n$ for all n.

Theorem 2: For any finitely nonzero sequence of integers $\{K_n\}_{-\infty}^{\infty}$ there is a unique equivalent canonical sequence $\{K_n'\}_{-\infty}^{\infty}$. $\{K_n'\}_{-\infty}^{\infty}$ is in fact the unique canonical sequence satisfying $\Sigma K_n'\alpha^n = \Sigma K_n \alpha^n$.

Proof: *Uniqueness*.—First note that the sequence is positive or negative canonical according as $\Sigma K_n'\alpha^n$ is nonnegative or nonpositive. By factoring out a minus sign if necessary, we may now assume without loss of generality that $\{K_n'\}$ is positive.

Claim.—p is the upper degree of the sequence iff $\alpha^p \le \Sigma K_n'\alpha^n < \alpha^{p+1}$. Since $K_p' = 1$, the left-hand inequality is clear. For the right-hand inequality, note that $\{K_n'\}$ has zeros at least in one of every three consecutive terms and thus $\Sigma K_n'\alpha^n$ would be increased if the one's in the series were moved upward in position (if necessary) and then more one's added (when necessary) to form a new series $\Sigma K_n''\alpha^n$ with $K_n'' = 0$ for $n > p$, $K_p'' = 1$, $K_{p-1}'' = 1$, $K_{p-2}'' = 0$, and successive decreasing terms 1, 1, 0, 1, 1, 0, ..., ad infinitum. We obtain

$$\begin{aligned}\Sigma K_n'\alpha^n < \sum_{n=0}^{\infty} \alpha^{p-3i} + \alpha^{p-1-3i} &= (\alpha^p + \alpha^{p-1})/(1 - \alpha^{-3}) \\ &= \alpha^{p+1}(\alpha^{-1} + \alpha^{-2})/(\alpha^{-1} + \alpha^{-2}) \\ &= \alpha^{p+1}.\end{aligned}$$

Thus p is determined by the value of $\Sigma K_n'\alpha^n$. To find the next lower "one" in the sequence, merely examine the powers of a that $\Sigma K_n\alpha^n - K_p\alpha^p$ lies between. Successively subtracting off suitable powers of a will determine the positions of each of the other "ones" in the sequence.

Existence.—It is clear that the following operation replaces sequences by equivalent ones:

Choose integers n and K

Replace K_n by $K_n + K$

K_{n-1} by $K_{n-1} - K$

K_{n-2} by $K_{n-2} - K$

and K_{n-3} by $K_{n-3} - K$

Since the following *resolution algorithm* involves only repeated applications of the proper choice of this operation, existence will be proven if the algorithm is proven to terminate with a canonical sequence.

Step I: To replace $\{K_n\}_{-\infty}^{\infty}$ by an equivalent sequence in which all nonzero terms are of the same sign. If the upper degree of the sequence is p, replace K_p by 0 and add K_p to K_{p-1}, K_{p-2}, and K_{p-3}. Repeat this procedure until all nonzero terms are of like sign.

For analyzing Step I, let r denote the lower degree. If $p - r \ge 3$, p is reduced by at least 1 but r is unaltered. Thus, eventually, no more than three terms are nonzero. If $K_{p-2} = d$, $K_{p-1} = e$, $K_p = f$ and all other terms are zero, application of the procedure yields a new sequence with consecutive terms $(f, d + f, e + f)$ and all other terms zero. It now follows from Lemma 1 that eventually all nonzero terms are of the same sign.

Step II: If all nonzero terms are negative, factor out a minus sign, and treat the sequence as if all terms were nonnegative. Thus, without loss of generality, assume henceforth that the sequence is nonnegative.

Step III: (i) If any three consecutive terms are nonzero, choose three such terms K_{n-3}, K_{n-2}, and K_{n-1}, pick a positive integer $K \leq \min\{K_{n-3}, K_{n-2}, K_{n-1}\}$, subtract K from each of K_{n-3}, K_{n-2}, and K_{n-1} and add K to K_n.

(ii) If no three consecutive terms are nonzero, either all nonzero terms are 1, in which case the sequence is canonical and Step III terminates, or else (a) choose any $K_n > 1$, choose positive $J < K_n$, replace K_n by $K_n - J$, and replace K_{n-1}, K_{n-2}, and K_{n-3} by $K_{n-1} + J$, $K_{n-2} + J$, and $K_{n-3} + J$, respectively; then (b) [actually applying (i) in a specific way] choose positive $K \leq \min\{K_n - J, K_{n-1} + J, K_{n-2} + J\}$ and replace $K_n - J$, $K_{n-1} + J$, and $K_{n-2} + J$ by $K_n - J - K$, $K_{n-1} + J - K$, and $K_{n-2} + J - K$, respectively, and replace K_{n+1} by $K_{n+1} + K$. Repeat Step III until the sequence obtained is canonical.

In order to show that Step III terminates in a finite number of repetitions, first introduce the parameter $N = \Sigma K_n$. Note that (i) reduces N by $2K > 0$. Thus (i) cannot be repeated consecutively indefinitely. Any infinite repetition of Step III would have an infinite number of times (ii) is applied. The next thing to show is that from the position before one use of (ii) to the position before the next use of (ii) N is not increased. (ii) itself adds $2J - 2K$ to N so (ii) increases N only if $J > K$. In this case, the new consecutive nonzero terms $K_n + J - K$, $K_{n-2} + J - K$, and $K_{n-3} + J$ have minimum $\geq J - K$. Thus (i) must next be applied, and must be repeated until at least one of these three terms is reduced to zero. But if (i) reduces an individual term by K', then that application of (i) reduces N by $2K'$. Thus (i) must be repeated at least until N is brought back down to its value before the most recent use of (ii).

There still remains the possibility of an infinite sequence of Step III's, each with $N = N_0$ just before each application of (ii). To show this is impossible, order the set of all finitely nonzero nonnegative integer sequences lexicographically. Note that (i) and (ii) both strictly increase the lexicographic order. Consider only those $\{K_n\}_{-\infty}^{\infty}$ produced with $N = N_0$. These form a sequence of nonnegative sequences of given entry-sum $N = N_0$ which is increasing in lexocographic order. Such a sequence of sequences must be finite it if is bounded above. The following is a proof of this.

Consider the highest-position nonzero term in each of the sequences. This single term will be nondecreasing (in lexicographic order) and because of the existence of the upper bound and the requirement $N = N_0$ can only move through a finite number of values. Thus, the highest-position term becomes fixed after a certain point. Beyond this point, consider also the next highest-position term. This term must now be nondecreasing and so also eventually becomes fixed. Continuing on, one by one each of the successive nonzero terms becomes fixed and since there are at most N_0 nonzero terms, eventually all the terms become fixed.

It now only remains to show the existence of an upper bound for the sequences under consideration. Pick m such that $\alpha^m > \Sigma K_n \alpha^n$. If $\{K_n\}$ and $\{K_n''\}$ are equivalent nonnegative sequences and p'' is the upper degree of $\{K_n''\}$, then $p'' < m$. Reason: if $p'' \geq m$, then $\alpha^m \leq K''_{p''} \alpha^{p''} \leq \Sigma K_n'' \alpha^n = \Sigma K_n \alpha^n$. Thus, one can choose the sequence with 1 in the mth place and zeros elsewhere as a lexicographic upper bound to all nonnegative sequences equivalent to $\{K_n\}$. The proof is complete.

The above resolution algorithm is most conveniently done by first writing the sequence $\{K_n\}$ in usual *positional* notation (the reverse of usual sequential order) with a dot setting off position zero from position -1. Thus, the sequence with $K_{-2} = 2$, $K_{-1} = 1$, $K_0 = 3$, $K_1 = 0$, $K_2 = 4$, and all other terms 0, would be written 403.12 in this notation.

Applying the algorithm,

$$403.12 \xrightarrow{(i)} 412.01 \xrightarrow{(i)} 1301.01 \xrightarrow{(iia)} 1212.11 \xrightarrow{(iib)} 2101.11 \xrightarrow{(i)} 2110. \xrightarrow{(i)} 11000.$$

This shows that the sequence $\{K_n'\}$ with $K_3' = 1$, $K_4' = 1$, and all other terms 0, is the canonical sequence equivalent to $\{K_n\}$.

3. SIMULTANEOUS TRIBONACCI REPRESENTATIONS

The following is found greatly generalized in [8, Theorem 2.4].

Lemma 3: For all $n \in Z$,

$$\alpha_i = t_i\alpha^{-2} + (t_{i+2} - t_{i+1})\alpha^{-1} + t_{i+1}.$$

Proof: First check this formula explicitly for $i = 0, -1, -2$ (making suitable use of the relations $\alpha^{n+3} = \alpha^{n+2} + \alpha^{n+1} + \alpha^n$). The formula then follows for all i, since both left and right sides satisfy the recursion $X_{i+3} = X_{i+2} + X_{i+1} + X_i$ for all i.

Proposition 4: There are unique integers a, b, and c such that $\Sigma K_i \alpha^i = a\alpha^{-2} + (c - b)a^{-1} + b$. These integers are given by

$$a = \Sigma K_i t_i, \; b = \Sigma K_i t_{i+1}, \text{ and } c = \Sigma K_i t_{i+2}.$$

Proof: Applying Lemma 3,

$$\Sigma K_i\alpha^i = (\Sigma K_i t_i)\alpha^{-2} + (\Sigma K_i t_{i+2} - \Sigma K_i t_{i+1})\alpha^{-1} + (\Sigma K_i t_{i+1}).$$

Now use the fact that α^{-2}, α^{-1}, and 1 are a basis of $Z[\alpha]$ over Z.

Theorem 5: Existence, uniqueness, and construction of simultaneous Tribonacci representations.

(a) For every integer triple (a, b, c) there is a unique canonical sequence $\{K_i\}$ such that

$$a = \Sigma K_i t_i,\ b = \Sigma K_i t_{i+1},\ \text{and}\ c = \Sigma K_i t_{i+2}.$$

(b) The sequence $\{K_i\}$ can be found by resolving the sequence with a in the -2 position, $c - b$ in the -1 position, b in the 0 position, and zeros elsewhere, that is, $(b).(c - b)(a)$ in positional notation.

Proof: By Theorem 2, there is a unique canonical sequence $\{K_i\}$ such that

$$a\alpha^{-2} + (c - b)\alpha^{-1} + b = \Sigma K_i\alpha^i.$$

Now apply Propostion 4.

Comment: Theorem 5(a) was stated first in [4]. I believe that the use of the resolution algorithm to find the canonical sequence $\{K_n\}$ is in the majority of cases the most efficient method now available.

Example: Find the canonical sequence $\{K_i\}$ such that $a = \Sigma K_i t_i = -1$, $b = \Sigma K_i t_{i+1} = 4$, and $c = \Sigma K_i t_{i+2} = 3$.

$$(b).(c - b)(a) = 4.(-1)(-1) \xrightarrow{I} .334 \xrightarrow{(i)} 2.112 \xrightarrow{(i)} 11.002 \xrightarrow{(iia)} 11.001111 \xrightarrow{(iib)} 11.010001$$

Verification:

$$t_{-6} + t_{-2} + t_0 + t_1 = -3 + 1 + 0 + 1 = -1$$

$$t_{-5} + t_{-1} + t_1 + t_2 = 2 + 0 + 1 + 1 = 4$$

$$t_{-4} + t_0 + t_2 + t_3 = 0 + 0 + 1 + 2 = 3$$

Theorem 6(a)—First proven in [4, (5.2)]: The triple (a, b, c) has its simultaneous Tribonacci representation using a positive canonical sequence $\{K_n\}$ iff $a\alpha^{-2} + b(1 - \alpha^{-1}) + c\alpha^{-1} \geq 0$.

Theorem 6(b): In addition, the sequence $\{K_n\}$ will have upper degree p iff

$$\alpha^p \leq a\alpha^{-2} + b(1 - \alpha^{-1}) + c\alpha^{-1} < \alpha^{p+1}.$$

Proof: As was noted in the proof of uniqueness in Theorem 2, $\{K_i\}$ is positive canonical iff $\Sigma K_i\alpha^i \geq 0$, and in addition has upper degree p iff $\alpha^p \leq \Sigma K_i\alpha^i < \alpha^{p+1}$. By Propositions 4 and 5, $\Sigma K_i\alpha^i = a\alpha^{-2} + (c - b)\alpha^{-1} + b$. Substituting this into the above inequalities yields the conclusion.

Theorem 7: For each pair (a,b) of integers, there is a unique positive canonical sequence of upper degree ≤ -2 such that $a = \Sigma K_i t_i$ and $b = \Sigma K_i t_{i+1}$. The triple (a,b,c) for the given pair (a,b) is found by the formula

$$c = -[a\alpha^{-1} + b(\alpha - 1)].$$

Proof: By Theorem 6, $\{K_i\}$ is positive canonical and of upper degree ≤ -2 iff

$$0 \leq a\alpha^{-2} + b(1 - \alpha^{-1}) + c\alpha^{-1} < \alpha^{-1},$$

or, equivalently,

$$-a\alpha^{-1} - b(\alpha - 1) \leq c < 1 - a\alpha^{-1} - b(\alpha - 1).$$

This formula gives a unique integer c for each pair (a,b) and the existence of uniqueness of $\{K_i\}$ then follows by Theorem 5.

4. THE LOWER DEGREE I

In this section and in Section 5, all sequences $\{K_i\}_{-\infty}^{\infty}$ not otherwise described will be assumed canonical. This property will not be explicitly stated again.

Define $S = \{\Sigma K_i\beta^i\colon\ \{K_i\}$ is positive and of lower degree $\geq 0\}$. (β is defined at the beginning of Section 2.) Also define $S^0 = \{\Sigma K_i\beta^i\colon\ \{K_i\}$ is positive and of lower degree $0\}$. We wish to describe S and S^0, at least approximately, as subsets of the complex plane.

Let $S_{11} = \{\Sigma K_i\beta^i\colon\ \{K_i\}$ is positive of lower degree ≥ 0 and upper degree $\leq 11\}$. S_{11} is a finite set of complex numbers.

$$\text{Let } S_{11}^0 = S^0 \cap S_{11}.$$

$$\text{Let } \mathcal{T} = \{\Sigma K_i\beta^i\colon\ \{K_i\} \text{ is positive}\}.$$

Proposition 8: If $\Sigma K_i\beta \in \mathcal{T}$, then $|\Sigma K_i\beta^i - \sum^{11} K_i\beta^i| < .075$. In particular, every point of $\mathcal{S}$ is within .075 of a point in $\mathcal{S}_{11}$ and every point of $\mathcal{S}^0$ is within .075 of a point in $\mathcal{S}^0_{11}$.

Proof: $|\Sigma K_i\beta^i - \sum^{11} K_i\beta^i| \leq \sum_{i=12}^{\infty} K_i|\beta^i| \leq \sum_{i=1}^{\infty} |\beta|^{12+3i} + |\beta|^{13+3i} = (|\beta|^{12} + |\beta|^{13})/(1 - 1\beta1^3) < .075$

where the second inequality uses an argument similar to that in the uniqueness part of Theorem 2.

The above proposition indicates that in a certain sense $\mathcal{S}_{11}$ is a good approximation to $\mathcal{S}$ and that $\mathcal{S}^0_{11}$ is a good approximation to $\mathcal{S}^0$. This, together with the following propositions, will help explain the correctness of our "sketch" of $\mathcal{S}$ and $\mathcal{S}^0$ (Figure 1, which appears at the end of this section).

Proposition 9: If $\Sigma K_i\beta^i \in \mathcal{S}^0$, then $|\Sigma K_i\beta^i| > .425$.

Proof: We calculated the minimum of the moduli of the hundreds of points of $\mathcal{S}^0_{11}$ obtaining .50088 (attained at $1 + \beta^2 + \beta^4 + \beta^7 + \beta^{10}$) and, applying Proposition 8, subtracted .075, thus obtaining the lower bound of the theorem. Clearly, a listing of the details of this proof would be unprofitable. The skeptical reader with access to a computer can easily reproduce them for himself.

Proposition 10: If $\{K_i\}$ is positive and of lower degree r, then $|\Sigma K_i\beta^i| > .425|\beta^r|$.

Proof: The lower degree of $\{K_{i+r}\}_{i=-\infty}^{\infty}$ is zero. Hence $|\Sigma K_{i+r}\beta^i| > .425$.

$$|\Sigma K_i\beta^i| = |\Sigma K_{i+r}\beta^{i+r}| = |\beta^r||\Sigma K_{i+r}\beta^i| > .425|\beta^r|.$$

Proposition 11: If $\Sigma K_i\beta^i \in \mathcal{S}$, then $|\Sigma K_i\beta^i| < 1.69$.

Proof: The proof is analogous to that of Proposition 9. The point of $\mathcal{S}_{11}$ of maximum modulus is $1 + \beta^3 + \beta^6 + \beta^8 + \beta^9 + \beta^{11}$. Its modulus is 1.6055.

Proposition 12: If $\Sigma K_i\beta^i \in \mathcal{T}$ and $|\Sigma K_i\beta^i| < 1.69$, then lower degree $\{K_i\}$ is ≥ -4.

Proof: Note that $.425|\beta^{-5}| > 1.69$ and apply Proposition 10.

To sketch $\mathcal{S}$ and $\mathcal{S}^0$, plot $\mathcal{S}_{11}$, identifying $\mathcal{S}^{11}_0$, and on the same graph plot all points $\Sigma K_i\beta^i$ where $|\Sigma K_i\beta^i| < 1.69$, lower degree of $\{K_i\}$ is between -4 and -1 inclusive, and upper degree of $\{K_i\}$ is ≤ 11. Then sketch a simple closed curve which separates $\mathcal{S}_{11}$ from the other points plotted, and add a simple curve to separate $\mathcal{S}^0_{11}$ from the rest of $\mathcal{S}_{11}$.

By Proposition 8, every point in $\mathcal{S}$ is (approximately) inside the curve drawn, and by Propositions 8 and 10, every point in $\mathcal{T}$ but not in $\mathcal{S}$ is (approximately) outside the curve, where the approximation includes the value .075 of Proposition 10 and sketching errors which will probably be smaller than .075.

The next few propositions and theorems help justify the drawing of a simple closed curve, since such a curve indeed encloses a simply-connected domain.

The polynomial $X^3 - X^2 - X - 1$ is irreducible and hence its Galois group is transitive on the roots α, β, γ [6, Chapter 3, Section 5].

Hence,

$$a\alpha^{-2} + b\alpha^{-1} + c = \Sigma K_i\alpha^i \quad \text{iff}$$
$$a\beta^{-2} + b\beta^{-1} + c = \Sigma K_i\beta^i \quad \text{iff}$$
$$a\gamma^{-2} + b\gamma^{-1} + c = \Sigma K_i\gamma^i.$$

Thus $\mathcal{T}$ = those elements of $Z[\beta]$ which become positive when Z is held fixed and α is substituted for β.

Proposition 13: $\mathcal{T}$ is dense in the complex plane.

Proof: Consider the complex set (with polar coordinates) $0 = \{(r,\theta)\colon\ r_1 < r < r_2,\ \theta_1 < \theta < \theta_2\}$. Such sets are a base for the complex topology. Because $|\beta| < 1$ and argument β is not a rational multiple of 2π (see the proof of Theorem 18 in [2]), therefore, for certain sufficiently large integers m, $|\beta^m| < r_2 - r_1$ and $\theta_1 <$ argument $\beta^m < \theta_2$. Thus, $n\beta^m \in 0$ for a correctly chosen positive integer r. Now $n\alpha^m > 0$, so $n\beta^m \in \mathcal{T}$.

Proposition 14: If $\Sigma K_i\beta^i \in \mathcal{T}$ and $|\Sigma K_i\beta^i| < .425|\beta^{r-1}|$, then $\{K_i\}$ has lower degree $\geq r$. In particular, if $\Sigma K_i\beta^i \in \mathcal{T}$ and $|\Sigma K_i\beta^i| < .425|\beta^{-1}|$, then $\Sigma K_i\beta^i \in \mathcal{S}$.

Proof: This is an immediate corollary of Proposition 10.

Theorem 15: The lower degree is locally constant at points of $\mathcal{T}$. If $\Sigma K_i\beta^i \in \mathcal{T}$, $\{K_i\}$ has lower degree r and upper degree p, $\Sigma K_i'\beta^i \in \mathcal{T}$ and $|\Sigma K_i'\beta^i - \Sigma K_i\beta^i| < .425|\beta^{p+1}|$, then $\{K_i\}$ has lower degree r.

Proof: Let $\Sigma K_i'\beta^i - \Sigma K_i\beta^i = \Sigma K_i''\beta^i$.

Claim.—$\Sigma K_i''\beta^i \in \mathcal{T}$. If not, $\Sigma - K_i''\beta^i \in \mathcal{T}$, and by Proposition 14, $\{-K_i''\}$ would have lower degree $\geq p + 2$ and thus also would have upper degree $\geq p + 2$. Consequently, by Theorem 6, $\Sigma K_i\alpha^i - \Sigma K_i'\alpha^i = \Sigma - K_i''\alpha^i \geq \alpha^{p+2}$. But $\Sigma K_i\alpha^i < \alpha^{p+1}$ since p is the upper degree of $\{K_i\}$. This yields $\Sigma K_i'\alpha^i = \Sigma K_i\alpha^i - (\Sigma K_i\alpha^i - \Sigma K_i'\alpha^i) < \alpha^{p+1} - \alpha^{p+2} < 0$ contradicting the fact that $\Sigma K_i'\beta^i \in \mathcal{T}$.

Thus $\Sigma K_i''\beta^i \in \mathcal{T}$. As above, the lower degree of $\{K_i''\}$ is $\geq p + 2$, while the upper degree of $\{K_i\}$ is p. Hence $\{K_i + K_i''\}$ is canonical, and since $\Sigma(K_i + K_i'')\beta^i = \Sigma K_i'\beta^i$, it follows that $\{K_i'\} = \{K_i + K_i''\}$. Thus, lower degree of $\{K_i'\}$ = lower degree of $\{K_i\}$ and the proof is finished.

Let $\mathcal{U}$ be the interior of the closure of $\mathcal{S}$.

Let $\mathcal{U}^0$ be the interior of the closure of $\mathcal{S}^0$.

Theorem 16: $\mathcal{S} = \mathcal{T} \cap \mathcal{U}$. $\mathcal{S}^0 = \mathcal{T} \cap \mathcal{U}^0$.

Proof: Here is the proof for $\mathcal{S}$. The proof for $\mathcal{S}^0$ is similar.

For each point $z = \Sigma K_i\beta^i \in \mathcal{T}$, consider the open disc $\mathcal{B}_z$ with center z and radius $.425|\beta^{p+1}|$ where p is the upper degree of $\{K_i\}$. By Theorem 15, if $z \in \mathcal{S}$, then $\mathcal{B}_z \cap \mathcal{T} \subseteq \mathcal{S}$. Hence $z \in \mathcal{B}_z \subseteq$ Closure $(\mathcal{B}_z \cap \mathcal{T}) \subseteq$ Closure $\mathcal{S}$. Thus, $\mathcal{S} \subseteq \mathcal{U}$.

On the other hand, if $z \in \mathcal{T}\backslash\mathcal{S}$ with lower degree $r < 0$, then $\mathcal{B}_z$ does not meet $\mathcal{S}$, hence $z \notin \mathcal{U}$.

Lemma 17: If $\Sigma K_i\beta^i \in \mathcal{T}$ has lower degree r and upper degree p, if $\Sigma K_i'\beta^i \in \mathcal{T}$ and $|\Sigma K_i'\beta^i - \Sigma K_i\beta^i| < .425|\beta^p|$, then $\Sigma K_i'\beta^i$ has lower degree $\geq r$.

Proof: The proof follows most of that of Theorem 15 word for word except that the lower degree of $\{K_i''\}$ is $\geq p + 1$, while the upper degree of $\{K_i\}$ is p, so that while $\{K_i'\}$ is equivalent to $\{K_i + K_i''\}$, the latter may not be canonical. However, every term of $\{K_i + K_i''\}$ is either 0 or 1 and thus this sequence may be resolved as follows: In choosing three consecutive 1's, choose such a triple with next higher term 0. Applying operation III(i) produces a new sequence with, again, all terms either 0 or 1. The parameter N is reduced by 2. Repeat until the equivalent canonical sequence $\{K_i\}$ is obtained. Since operation III(i) never lowers the lower degree, the conclusions is obtained.

Theorem 18: $\mathcal{U}$ is connected.

Proof: Let $z \in \mathcal{U}$. Then, since $\mathcal{U}$ is open and $\mathcal{S}$ is dense in $\mathcal{U}$, the connected component of $\mathcal{U}$ containing z also contains a point $\sum_0^\infty K_i\beta^i$ in $\mathcal{S}$.

It suffices to show that $\sum_0^{n-1} K_i\beta^i$ and $\sum_0^n K_i\beta^i$ lie in the same connected component for all $n \geq 0$ $\left(\text{where } \sum_0^{-1} K_i\beta^i = 0\right)$. This clearly holds if $K_n = 0$, so assume $K_n = 1$, $K_i = 0$ for $p < i < n$ and $K_p = 1$ $\left(\text{if } \sum_0^{n-1} K_i\beta^i = 0, \text{ set } p = -1 \text{ and succeeding statements will still hold true}\right)$.

By Lemma 17 $\left(\text{or Proposition 14 if } \sum_0^{n-1} K_i\beta^i = 0\right)$, the open disc with center $\sum_0^{n-1} K_i\beta^i$ and radius $.425|\beta^p|$ is contained in $\mathcal{U}$ as is the open disc with center $\sum_0^n K_i\beta^i$ and radius $.425|\beta^n|$. But since $.425 + .425|\beta^{-1}| > 1$, $\left|\sum_0^n K_i\beta^i - \sum_0^{n-1} K_i\beta^i\right| = |\beta^n| < .425|\beta^n| + .425|\beta^{n-1}| \leq .425|\beta^n| + .425|\beta^p|$ so the two discs overlap, and the proof is complete.

Theorem 19: $\mathcal{U}$ is simply connected.

Proof: Since $\mathcal{U}$ is the interior of a closure, if z is in a bounded component of the complement of $\mathcal{U}$ then z is in the closure of the (open) exterior of $\mathcal{U}$, thus $z \in$ Closure $(\mathcal{T}\backslash\mathcal{U})$. By Propositions 11 and 12, $z \in$ Closure $\mathcal{A}r$, where $\mathcal{A}r = \{\Sigma K_i\beta^i \mid \{K_i\}$ has lower degree $r\}$ where r is some number between -4 and -1 inclusive. Now $\mathcal{A}r = \mathcal{C}r \cup \mathcal{D}r$ where $\mathcal{C}r = \{\beta^r + \beta^{r+2}\Sigma K_i\beta^i\colon\ \Sigma K_i\beta^i \in \mathcal{S}\}$ and $\mathcal{D}r = \{\beta^r + \beta^{r+1} + \beta^{r+3}\Sigma K_i\beta^i\colon\ \Sigma K_i\beta^i \in \mathcal{S}\}$. Then just as it was shown in Theorem 18 that $\mathcal{S} \subseteq$ interior (Closure $\mathcal{S}$), similarly, $\mathcal{C}r \subseteq$ interior [Closure($\mathcal{C}r$)] $= \beta^r + \beta^{r+2}$ and $\mathcal{D}r \subseteq$ interior [Closure($\mathcal{D}r$)] $= \beta^r + \beta^{r+1} + \beta^{r+3}\mathcal{U}$, and these latter sets are open, connected, and contained in the exterior of $\mathcal{U}$, and z is in the closure of one of these sets. Hence, it suffices to show that each of these eight connected sets lies in the unbounded component of the complement of $\mathcal{U}$. Now $\mathcal{U}$ lies within the open disc of center zero and radius 1.69, but

$$\beta^{-1} + \beta^2 \in \mathcal{C}_{-1},\ \beta^{-1} + 1 + \beta^2 + \beta^5 \in \mathcal{D}_{-1},\ \beta^{-2} \in \mathcal{C}_{-2},\ \beta^{-2} + \beta^{-1} + \beta^1 \in \mathcal{D}_{-2},$$

$$\beta^{-3} \in \mathcal{C}_{-3},\ \beta^{-3} + \beta^{-2} \in \mathcal{D}_{-3},\ \beta^{-4} \in \mathcal{C}_{-4},\ \beta^{-4} + \beta^{-3} \in \mathcal{D}_{-4},$$

and each of these points has modulus > 1.69.

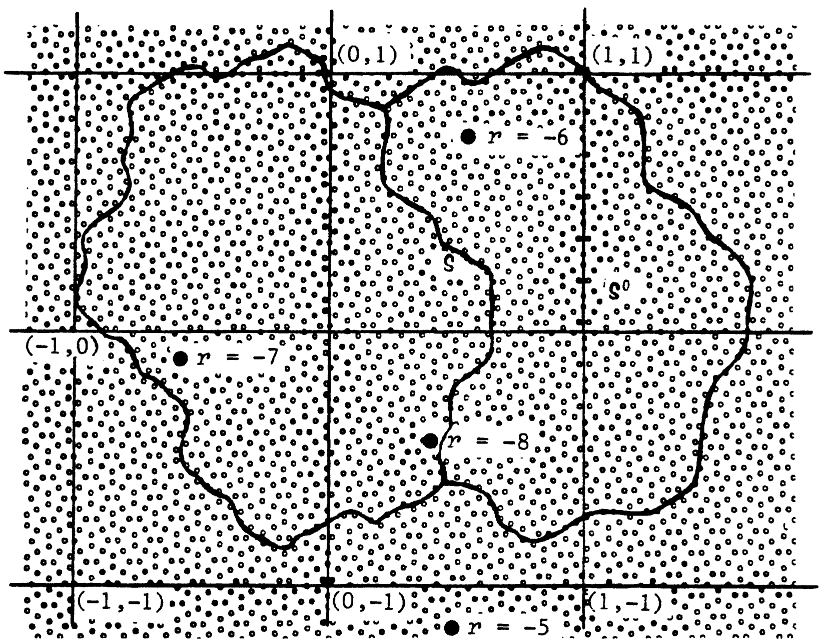


FIGURE 1. Sketches of $\mathcal{S}$ and $\mathcal{S}^0$. The dots were plotted by computer, using different colors in each of the three regions shown. Everything else was sketched by hand. The four plotted points illustrate Application 21. Only dots of modulus less than 1.69 were plotted. Thus, the right edge of the figure is a sketch of part of a circle of radius 1.69. This gives an idea of the accuracy of the rest of the sketch.

5. THE LOWER DEGREE II

If we assume that the set $\mathcal{S}$ is known, we can then solve the "lower degree problem" for simultaneous Tribonacci representations in terms of $\mathcal{S}^0$.

Theorem 20: $a = \Sigma K_i t_i$, $b = \Sigma K_i t_{i+1}$, and $c = \Sigma K_i t_{i+2}$, where $\{K_i\}$ is positive of lower degree r iff $\beta^{-r}[a\beta^{-2} + b(1 - \beta^{-1}) + c\beta^{-1}] \in \mathcal{S}^0$.

Proof: By Proposition 4 and Theorem 5, the first set of conditions is equivalent to $a\alpha^{-2} + (c - b)\alpha^{-1} + b = \Sigma K_i\alpha^i$, where lower degree of $\{K_i\} = r$. Substituting β for α and making other rearrangements yields $a\beta^{-2} + b(1 - \beta^{-1}) + c\beta^{-1} = \sum_i K_{i+r}\beta^{i+r} = \beta^r\Sigma K_{i+r}\beta^i$, which is equivalent to the second condition of the theorem.

Theorem 21: $a = \Sigma K_i t_i$, $b = \Sigma K_i t_{i+1}$, and $c = \Sigma K_i t_{i+2}$, where $\{K_i\}$ is positive of lower degree $\geq r$ iff $\beta^{-r}[a\beta^{-2} + b(1 - \beta^{-1}) + c\beta^{-1}] \in \mathcal{S}$.

Proof: The proof is similar to that of Theorem 19.

Application 22: Here is how Theorem 20 can be used in practice. Note that by Propositions 9 and 11, if $\Sigma K_i \beta^i \in \mathcal{S}^0$, then $.425 < |\Sigma K_i \beta^i| < 1.69$.

Compute $x = a\beta^{-2} + b(1 - \alpha^{-1}) + c\alpha^{-1}$. If $x \geq 0$, then $\{K_i\}$ is positive. If $x < 0$, replace (a, b, c) by $(-a, -b, -c)$.

Next compute $z = a\beta^{-2} + b(1 - \beta^{-1}) + c\beta^{-1}$. Taking absolute values, we require $.425 < |\beta|^{-r}|z| < 1.69$, or taking logs and rearranging,

$$\frac{\log|z| - \log(.425)}{\log|\beta|} < r < \frac{\log|z| - \log(1.69)}{\log|\beta|}. \tag{1}$$

This gives four or five possible values for r. Plot $\beta^{-r}z$ for these values and see which lies within $\mathcal{S}^0$.

Example: Find by this method the lower degree of the canonical sequence $\{K_n\}$ such that $a = \Sigma K_i t_i = -1$, $b = \Sigma K_i t_{i+1} = 4$, and $c = \Sigma K_i t_{i+2} = 3$.

$$-1\alpha^{-2} + 4(1 - \alpha^{-1}) + 3\alpha^{-1} = 3.16 > 0, \text{ so } \{K_n\} \text{ is positive,}$$

$$z = -1\beta^{-2} + 4(1 - \beta^{-1}) + 3\beta^{-1} = 5.42 + 3.39i, \ |z| = 6.3945131,$$

so (1) above gives

$$-8.90 < r < -4.37.$$

Plotting $\beta^{-r}z$ for $r = -8, -7, -6$, and -5 on Figure 1 shows the lower degree is -6, which agrees with the result of the Example following Theorem 5. The above method will be must more efficient than the resolution algorithm most of the time. However, accuracy will not be guaranteed if one (and therefore two) of the points plotted falls near the boundary of $\mathcal{S}^0$. If this problem comes up, r will still be known for sure to be one of two consecutive integers.

It was shown in [2, Theorem 1] that each positive integer a has a unique (Zeckendorf) representation $a = \Sigma K_i t_i$, where $\{K_i\}$ is positive and of lower degree ≥ 2.

Problem.—For a given a with this representation, find formulas for $b = \Sigma K_i t_{i+1}$ and $c = \Sigma K_i t_{i+2}$ in terms of a. Let us call such a triple (a, b, c) a Zeckendorf triple. We shall solve this problem, not with a precise formula, but rather, in terms of a picture.

Let $x + iy$ be an arbitrary point in $\mathcal{S}$. Rewrite the condition of Theorem 21 for the problem:

$$a\beta^{-4} + (c - b)\beta^{-3} + b\beta^{-2} = x + iy.$$

Equivalently,

$$a2\beta^{-2} + (c - b)2\beta^{-1} + 2b = 2\beta^2(x + iy). \tag{2}$$

We wish to break this into real and imaginary parts. To do this, we need to find the real and imaginary parts of $2\beta^n$ for various n. By the recursion relation, the values for all other n can be obtained from the values for $n = -1$, 0, and 1. Since α, β, and $\gamma = \overline{\beta}$ are the roots of $x^3 - x^2 - x - 1 = 0$, we have $\alpha + \beta + \gamma = 1$ and $\alpha\beta\gamma = 1$: Hence, $\mathrm{Re}(2\beta) = \beta + \gamma = 1 - \alpha$ and $\beta\gamma = \alpha^{-1}$. Thus, $\mathrm{Re}(2\beta^{-1}) = \beta^{-1} + \gamma^{-1} = (\beta + \gamma)/\beta\gamma = \alpha - \alpha^2 = -1 - \alpha^{-1}$. By the recursion $\mathrm{Re}(2\beta^2) = 3 - \alpha^2$ and $\mathrm{Re}(2\beta^{-2}) = -1 - \alpha^{-2}$. Let $\delta = \mathrm{Im}(2\beta) = (\beta - \gamma)/i \doteq 1.21258146$. Then $\mathrm{Im}(2\beta^{-1}) = (\beta^{-1} - \gamma^{-1})/i = -(\beta - \gamma)/\beta\gamma i = -\alpha\delta$. By the recursion $\mathrm{Im}(2\beta^2) = (1 - \alpha)\delta$ and $\mathrm{Im}(2\beta^{-2}) = (1 + \alpha)\delta$. Taking real and imaginary parts of (2) using the above data yields

$$(-1 - \alpha^{-2})a + (-1 - \alpha^{-1})(c - b) + 2b = (3 - \alpha^2)x - (1 - \alpha)\delta y,$$

and

$$(1 + \alpha)\delta a + (-\alpha\delta)(c - b) = (3 - \alpha^2)y + (1 - \alpha)\delta y. \tag{3}$$

Solving (3) for b and c yields,

$$b = a\alpha + u$$
$$c = a\alpha^2 + v,$$

where $u = (4 - \alpha^{-2} - \alpha^2)x/2 + [(\alpha - 3\alpha^{-1} + 1 - 3\alpha^{-2})\delta^{-1} - \delta(1 - \alpha)]y/2$ and $v = u + (1 - \alpha^{-1})x + (\alpha - 3\alpha^{-1})\delta^{-1}y$.

Thus, there is a well-defined real matrix $\boldsymbol{T}$ such that $(u,v) = \boldsymbol{T}(x,y)$.

Now define $\mathcal{W} = \{(u,v): \ (u,v) = \boldsymbol{T}(x,y), \ x + iy \in \mathcal{S}\}$. Let $\mathcal{X}$ = interior (Closure $\mathcal{W}$). Clearly, $\mathcal{X} = \{(u,v): \ (u,v) = (x,y), \ x + iy \in \mathcal{U}\}$. A sketch of $\mathcal{W}$ (or $\mathcal{X}$) appears in Figure 2(b).

We have proven

Theorem 23: (a, b, c) is a Zeckendorf triple if there is a point $(u,v) \in \mathcal{W}$ such that $b = a\alpha + u$ and $c = a\alpha^2 + v$. Every point in $\mathcal{W}$ corresponds in this way to a unique Zeckendorf triple.

Corollary 24: The pair (b,c) in a Zeckendorf triple (a, b, c) can and does take on only the values $([\![a\alpha]\!] + j, [\![a\alpha^2]\!] + k)$ where $(j,k) = (0,0), (0,1), (1,0), (1,1)$, or $(2,1)$.

Proof: $\mathcal{W}_{11}$, $\mathcal{W}$ computed exactly with $\mathcal{S}$ replaced by $\mathcal{S}_{11}$, gives an approximation to $\mathcal{W}$. The "error" of .075 in replacing $\mathcal{S}$ by $\mathcal{S}_{11}$ is propagated into an error of no more than .098 in replacing $\mathcal{W}$ by $\mathcal{W}_{11}$.

Note that (b,c) can assume the value $([\![a\alpha]\!] + j, [\![a\alpha^2]\!] + k)$ iff there is a $(u,v) \in \mathcal{W}$ with $j - 1 < u \le j$ and $k - 1 < v \le k$. An examination of the thousands of points in $\mathcal{W}_{11}$ yields the results of the corollary.

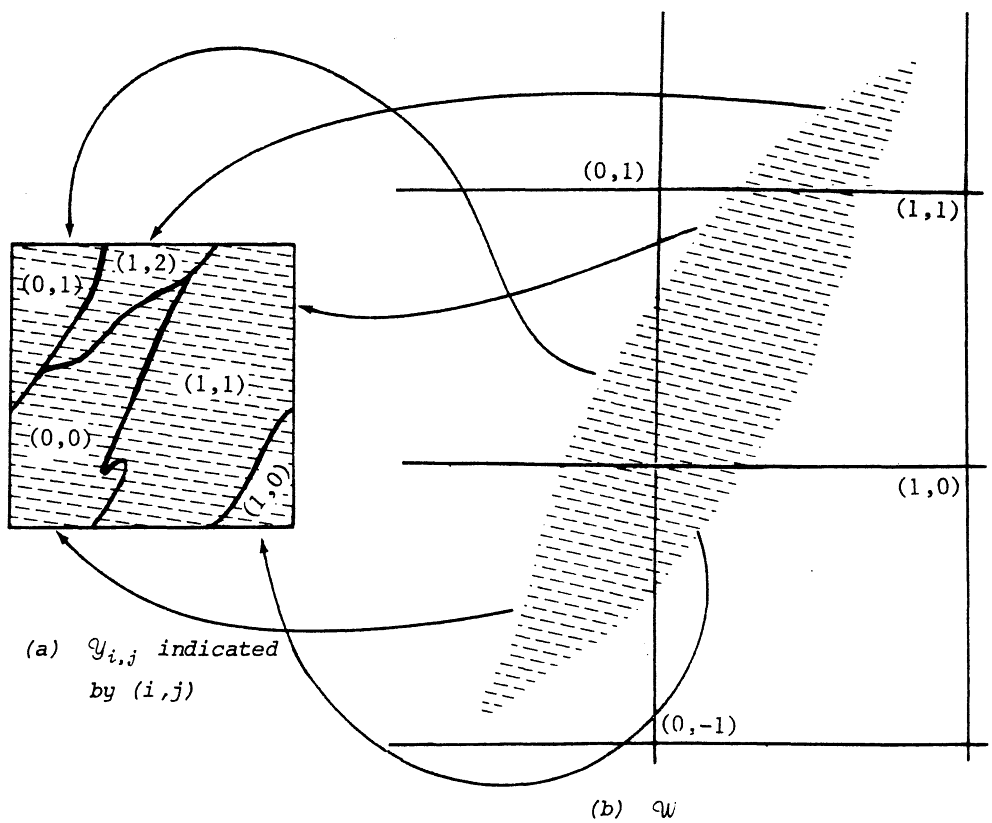
Let $\mathcal{Q}$ be the open unit square $\{(s,t): 0 < s < 1, 0 < t < 1\}$.

For each of the pairs (j,k) of Corollary 24, define $\mathcal{Y}_{j,k} = \{(s,t) \in \mathcal{Q}: (j - s, k - t) \in \mathcal{W}\}$.

Theorem 25: If (a, b, c) is a Zeckendorf triple, then $b = [\![a\alpha]\!] + j$ and $c = [\![a\alpha^2]\!] + k$ iff $(a\alpha - [\![a\alpha]\!], a\alpha^2 - [\![a\alpha^2]\!]) \in \mathcal{Y}_{j,k}$.

Proof: By Theorem 24, $b = [\![a\alpha]\!] + j$ and $c = [\![a\alpha^2]\!] + k$ iff $[\![a\alpha]\!] + j = a\alpha + u$ and $[\![a\alpha^2]\!] + k = a\alpha^2 + v$ with $(u,v) \in \mathcal{W}$ if $j - (a\alpha - [\![a\alpha]\!]), k - (a\alpha^2 - [\![a\alpha^2]\!]) \in \mathcal{W}$. Also $(a\alpha - [\![a\alpha]\!], a\alpha^2 - [\![a\alpha^2]\!])$ is always in $\mathcal{Q}$ for positive integer a.

The sets $\mathcal{Y}_{j,k}$ can be drawn approximately by rotating $\mathcal{W}$ through 180°, cutting $\mathcal{W}$ up along lines of integer value and translating the pieces by integer distances vertically + horizontally into $\mathcal{Q}$. The result is shown in Figure 2(a). It is seen that these sets appear to disjointly cover $\mathcal{Q}$. This is worth proving. Let $\mathcal{Z}_{j,k}$ = interior(Closure $\mathcal{Y}_{j,k}$). Clearly $\mathcal{Y}_{j,k} \subseteq \mathcal{Z}_{j,k} = [(j,k) - \mathcal{X}] \cap \mathcal{Q}$.



(a) $\mathcal{Y}_{i,j}$ *indicated by (i,j)*

(b) $\mathcal{W}$

FIGURE 2

Theorem 26(a): The union of the $\mathcal{Y}_{j,k}$ is dense in $\mathcal{Q}$.

Theorem 26(b): The sets $\mathcal{Z}_{j,k}$ are disjoint.

Proof (a): By Theorems 23 and 25, it is seen that the union of the $\mathcal{Y}_{j,k}$ consists of the set $\{(a\alpha - [\![a\alpha]\!], a\alpha^2 - [\![a\alpha^2]\!]):$ a positive integer$\}$. Since 1, α, and α^2 are linearly independent over $\mathcal{Z}$, this set is dense in the unit square (see [1, Chapter IV]).

Proof (b): It suffices to prove that if j', j'', k', and k'' are integers, if (u',v') and $(u'',v'') \in \mathcal{X}$ and if $(j' - u', k' - v') = (j'' - u'', k'' - v'')$, then $u' = u''$ and $v' = v''$. These hypotheses imply that $u'' = u' + j$ and $v'' = v' + k$ where j and k are integers.

Let $\{(u_n', v_n')\}$ be a sequence in $\mathcal{W}$ converging to (u',v'). Corresponding to each (u_n', v_n') there is a unique Zeckendorf triple (a_n, b_n, c_n) with

$$b_n = a_n\alpha + u_n' \tag{4}$$

$$c_n = a_n\alpha^2 + v_n'.$$

By deleting from the sequence a finite number of terms, we can assume without loss of generality that $a_n\alpha^2 + ([\![a_n\alpha]\!] + j)(1 - \alpha^{-1}) + ([\![a_n\alpha^2]\!] + k)\alpha^{-1} > 0$ for all n. Since $b_n \geq [\![a_n\alpha]\!]$ and $c_n \geq [\![a_n\alpha^2]\!]$, this guarantees by Theorem 6 that a positive canonical sequence is obtained for simultaneous Tribonacci representation of the triples $(a_n, b_n + j, c_n + k)$. If we suppose that $(j,k) \neq (0,0)$, these are not Zeckendorf triples and thus they are simultaneously represented by a positive sequence $\{K_i\}$ of lower degree ≤ 1. Going through precisely the same calculations as precede the proof of Theorem 24, it is seen that $b_n + j = a_n\alpha + u_n''$ and $c_n + k = a_n\alpha^2 + v_n''$ where $(u_n'', v_n'') = \boldsymbol{T}(x_n'', y_n'')$ where $x_n'' + iy_n'' \mathcal{T}\backslash\mathcal{S}$. Hence, $(u_n'', v_n'') \in$ exterior $\mathcal{X}$. In light of (4), we have $(u_n'', v_n'') = (u_n', v_n') + (j,k)$. Hence (u_n'', v_n'') converges to (u'', v''), which thus cannot lie in $\mathcal{X}$, contradicting our hypothesis. The theorem is proven.

Example: Use Figure 2(a) to find the Zeckendorf triple with $a = 650$.

First we compute $a\alpha = 1195.5364$ and $a\alpha^2 = 2198.9343$. Then observe from Figure 2 that $(.5364, .9343) \in \mathcal{Y}_{(1,2)}$. Hence, $b = 1195 + 1 = 1196$, and $c = 2198 + 2 = 2200$, which results can be verified by direct calculation.

It was shown in [2, Theorem 12] that each integer has a unique (2nd canonical) representation $a = \Sigma K_i t_i$ where $\{K_i\}$ is positive and of lower degree positive and congruent to 1 modulo 3. For such a representation, we call (a, b, c) a 2nd canonical triple if $b = \Sigma K_i t_{i+1}$ and $c = \Sigma K_i t_{i+2}$. The following facts about 2nd canonical triples are proved similarly to their analogues for Zeckendorf triples.

Corollary 27: The pair (b,c) in a 2nd canonical triple (a, b, c) can and does take on only the values $([\![a\alpha]\!] + j, [\![a\alpha^2]\!] + k)$ where $(j,k) = (-1,-1)$, $(-1,0)$, $(0,-1)$, $(0,0)$, $(0,1)$, $(0,2)$, $(1,0)$, $(1,1)$, or $(1,2)$. In [2] it was shown that j takes on the values -1, 0, and 1. Figure 3 is the analogue for 2nd canonical triples of Figure 2. A region marked (j,k) in Figure 3(a) denotes the region $\mathcal{Y}^2_{j,k}$.

Theorem 28: If (a, b, c) is a 2nd canonical triple when $b = [\![a\alpha]\!] + j$ and $c = [\![a\alpha^2]\!] + k$ iff $(a\alpha - [\![a\alpha]\!], b\alpha^2 - [\![b\alpha^2]\!]) \in \mathcal{Y}^2_{j,k}$.

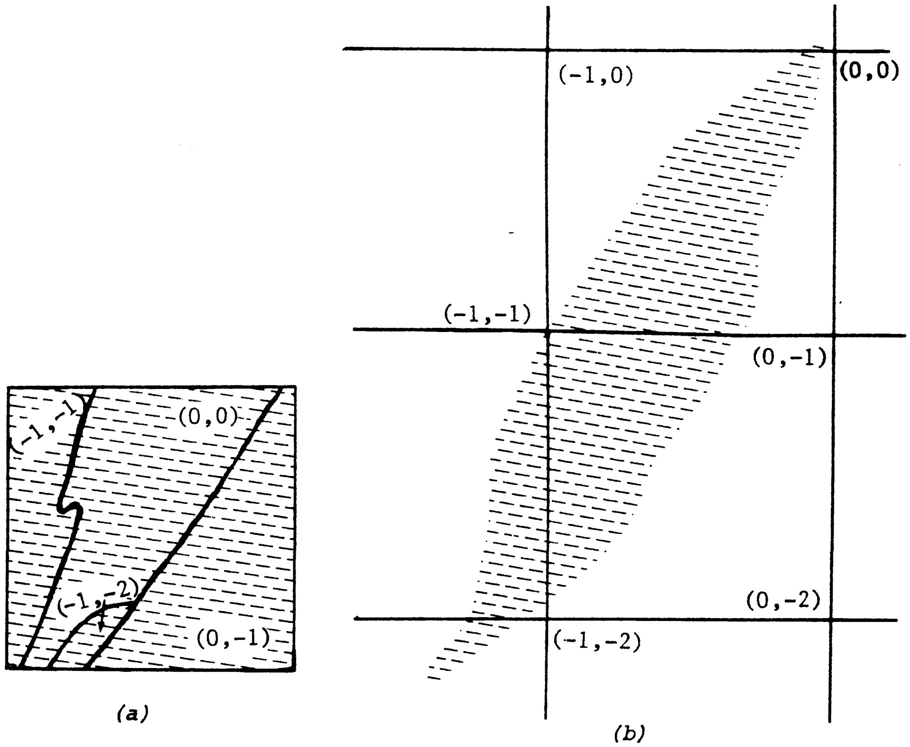


FIGURE 3. *(a) shows the unit square divided into the regions* $\mathcal{Y}^2_{i,j}$. *(b) shows the region* $\mathcal{W}^2$.

It is seen that with the given limits of accuracy, the computer sketch of $\mathcal{W}^2$ *does not indicate for sure whether* $\mathcal{Y}^2_{i,j}$ *is nonempty for (j,k) = (1,1), (1,0), (0,-2), (0,1) and (-1,0). However, accurate calculations of carefully chosen points of* $\mathcal{W}^2$ *corresponding to points of* $\mathcal{S}$ *of high upper degree show that these sets are indeed nonempty. Further theoretical considerations show that the areas near all four corners of* $\mathcal{Q}$ *are covered by infinitely many "strips" periodically alternating between those three sets* $\mathcal{Y}^2_{i,j}$ *which the sketch "allows" into the corner (for example, only* $\mathcal{Y}^2_{0,0}$, $\mathcal{Y}^2_{-1,-1}$, *and* $\mathcal{Y}^2_{-1,-2}$ *fit near (0,0) in* $\mathcal{Q}$*). Since all these strips are below accuracy level size, the corners of* $\mathcal{Q}$ *have been blacked out.*

REFERENCES

1. J. W. S. Cassels. *An Introduction to Diophantine Approximation.* Cambridge, Mass.: Cambridge University Press, 1957.
2. L. Carlitz, Richard Scoville, and V. E. Hoggatt, Jr. "Fibonacci Representations of Higher Order." *The Fibonacci Quarterly* 10 (1972):43-69, 94.
3. L. Carlitz, Richard Scoville, and V. E. Hoggatt, Jr. "Fibonacci Representations of Higher Order-II." *The Fibonacci Quarterly* 10 (1972):72-80.
4. Verner E. Hoggatt, Jr., and Marjorie Bicknell. "Generalized Fibonacci Polynomials and Zeckendorf's Theorem." *The Fibonacci Quarterly* 11 (1973):399-419.
5. Ben Noble. *Applied Linear Algebra.* Englewood Cliffs, N.J.: Prentice Hall, 1969.
6. M. M. Postnikov. *Fundamentals of Galois Theory.* Groningen, The Netherlands: P. Noordhoff, Ltd., 1962.
7. Robert Silber and Ralph Gellar. "The Ring of Fibonacci Representations." *The Fibonacci Quarterly* 14 (1976):289-326.
8. Robert Silber. "On the *N* Canonical Fibonacci Representations of Order *N*." *The Fibonacci Quarterly* 15 (1977):57-66.

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POLYNOMIAL FIBONACCI-LUCAS IDENTITIES OF THE FORM $\sum_{r=1}^{n} P(r)F_r$

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INTRODUCTION

$\sum_{r=1}^{n} P(r)$ can be evaluated by substitution in the mnemonic chain of formulas:

(1) $$\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$$

(2) $$\sum_{r=1}^{n} r(r+1) = \frac{n(n+1)(n+2)}{3}$$

- -

(K) $$\sum_{r=1}^{n} r(r+1)\ \ldots\ (r+k-1) = \frac{n(n+1)\ \cdots\ (n+k)}{k+1}.$$

The proof of (K) by mathematical induction also establishes the validity of (1), (2), ..., (K + 1).

Example 1: From (a) on the right,

(a)

	1	0	0	0
-1	1	-1	1	
-2	1	-3		

$$r^3 = r(r+1)(r+2) - 3r(r+1) + r$$

$$\sum_{r=1} r^3 = \sum_{r=1}^{n} r(r+1)(r+2) - 3\sum_{r=1}^{n} r(r+1) + \sum_{r=1}^{n} r$$

$$= \frac{n(n+1)(n+2)(n+3)}{4} - n(n+1)(n+2) + \frac{n(n+1)}{2}$$

$$= \frac{n^2(n+1)^2}{4}.$$

THE FIBONACCI-LUCAS CHAIN OF POLYNOMIAL IDENTITIES

I. Using the Fibonacci lists of identities, or otherwise, it is possible to write the following relations:

(1) $$\sum_{r=1}^{n} F_r = F_{n+2} - 1$$

(2) $$\sum_{r=1}^{n} rF_r = nF_{n+2} - F_{n+3} + 2$$

(3) $$\sum_{r=1}^{n} r(r+1)F_r = n(n+1)F_{n+2} - 2nF_{n+3} + 2F_{n+4} - 6$$

(4) $$\sum_{r=1}^{n} r(r+1)(r+2)F_r = n(n+1)(n+2)F_{n+2} - 3n(n+1)F_{n+3} + 6nF_{n+4} - 6F_{n+5} + 30$$

(5) $$\sum_{r=1}^{n} r(r+1)(r+2)(r+3)F_r = n(n+1)(n+2)(n+3)F_{n+2} - 4n(n+1)(n+2)F_{n+3} + 12n(n+1)F_{n+4} - 24nF_{n+5} + 24F_{n+6} - 192$$

- -

(K) $$\sum_{r=1}^{n} r(r+1)\ \cdots\ (r+k-1)F_r = n(n+1)\ \cdots\ (n+k-1)F_{n+2} - kn(n+1)\ \cdots\ (n+k-2)F_{n+3} + \cdots + (-1)^{k+1}k!F_{n+k+2} + (-1)^{k+1}k!F_{k+2}.$$

Using iterated integration by parts for finite differences:

$$\sum_{r=1}^{n} r(r+1)\cdots(r+k-1)F_r = n(n+1)\cdots(n+k+1)(F_{n+1}+F_n)$$

$$-\ kn(n+1)\cdots(n+k-2\ (F_{n+1}+2F_n+F_{n-1})$$

$$+\ k(k-1)n(n+1)\cdots(n+k-3)(F_{n+1}+3F_n+3F_{n-1}+F_{n-2})$$

$$-\ k(k-1)(k-2)n(n+1)\cdots(n+k-4)(F_{n+1}+4F_n+6F_{n-1}+4F_{n-2}+F_{n-3})$$

$$+\ \cdots + (-1)^{k+1}k!\left[F_{n+1}+\binom{k+1}{1}F_{n+2}+\binom{k+1}{2}F_{n+3}+\cdots\right]+C_k.$$

Catalan has an operator formula $U^{n+p} = U^{n-p}(u+1)^p$. After the algebraic operations are performed on the right, all powers become subscripts of U. U_k can be replaced by either F_k or L_k. Using this formula, or mathematical induction,

$$\sum_{r=1}^{n} r(r+1)\cdots(r+k-1)F_r = n(n+1)\cdots(n+k-1)F_{n+2}$$

$$-\ kn(n+1)\cdots(n+k-2)F_{n+3}$$

$$+\ \cdots + (-1)^{k+1}k!F_{n+k+2} + C_k.$$

Let $n = 1$ in (K) and (K - 1), C_k, an integer, must be a multiple of $k!$ Dividing out the common factor $(k-1)!$ in (K - 1) and $k!$ in (K), $|C| = F_{k+3} - F_{k+1} = F_{k+2}$.

<u>*Example 2:*</u> $$\sum_{r=1}^{n} r^3F_r = \sum_{r=1}^{n} r(r+1)(r+2)F_r - 3\sum_{r=1}^{n} r(r+1)F_r + \sum_{r=1}^{n} F_r$$

$$= n^3F_{n+2} - (3n^2 - 3n + 1)F_{n+3} + (6n-6)F_{n+4} - 6F_{n+5} + 50.$$

II. Using a Lucas list of identities, or otherwise, it is possible to establish (using proofs similar to those in (I) the following identities:

(1) $$\sum_{r=1}^{n} L_r = L_{n+2} - 3$$

(2) $$\sum_{r=1}^{n} rL_r = nL_{n+2} - L_{n+3} + 4$$

(3) $$\sum_{r=1}^{n} r(r+1)L_r = n(n+1)L_{n+2} - 2nL_{n+3} + 2L_{n+4} - 14$$

(4) $$\sum_{r=1}^{n} r(r+1)(r+2)L_r = n(n+1)(n+2)L_{n+2} - 3n(n+1)L_{n+3} + 6nL_{n+4} - 6L_{n+5} + 66$$

(5) $$\sum_{r=1}^{n} r(r+1)(r+2)(r+3)L_r = n(n+1)(n+2)(n+3)L_{n+2} - 4n(n+1)(n+2)L_{n+3} + 12n(n+1)L_{n+4} - 24nL_{n+5} + 24L_{n+6} - 432$$

- -

(K) $$\sum_{r=1}^{n} r(r+1)\cdots(r+k-1)L_r = n(n+1)\cdots(n+k-1)L_{n+2} - kn(n+1)\cdots(n+k-2)L_{n+3} + k(k-1)n(n+1)\cdots(n+k-3)L_{n+4} - \cdots + (-1)^{k+1}k!L_{n+k+2} + (-1)^{k+1}k!L'_{k+2}.$$

Example 3: $\sum_{r=1}^{n} r^3 L_r = \sum_{r=1}^{n} r(r+1)(r+2)L_r - 3\sum_{r=1}^{n} r(r+1)L_r + \sum_{r=1}^{n} rL_r$

$$= n^3 L_{n+2} - (3n^2 - 3n + 1)L_{n+3} + (6n - 6)L_{n+4} - 6L_{n+5} + 112.$$

REFERENCES

1. Leonard Dickson. *History of the Theory of Numbers*. Chapter 17.
2. V. C. Harris. "Fibonacci-Lucas Identities." *The Fibonacci Quarterly*.

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A GENERALIZATION OF SOME L. CARLITZ IDENTITIES

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Leonard Carlitz [1], by factoring $(x + y)^p - x^p - y^p$, developed the following identities:

1. $F^3_{n+1} - F^3_n - F^3_{n-1} = 3F_{n-1}F_nF_{n+1}$
   $L^3_{n+1} - L^3_n - L^3_{n-1} = 3L_{n-1}L_nL_{n+1}$
2. $F^4_{n+1} + F^4_n + F^4_{n-1} = 2[F^2_{n+1} - F_nF_{n-1}]^2$
   $L^4_{n+1} + L^4_n + L^4_{n-1} = 2[L^2_{n+1} - L_nL_{n-1}]^2$
3. $F^5_{n+1} - F^5_n - F^5_{n-1} = 5F_{n-1}F_nF_{n+1}(F^2_{n+1} - F_nF_{n-1})$
   $L^5_{n+1} - L^5_n - L^5_{n-1} = 5L_{n-1}L_nL_{n+1}(L^2_{n+1} - L_nL_{n-1})$
4. $F^7_{n+1} - F^7_n - F^7_{n-1} = 7F_{n-1}F_nF_{n+1}(F^2_{n+1} - F_nF_{n-1})^2$
   $L^7_{n+1} - L^7_n - L^7_{n-1} = 7L_{n-1}L_nL_{n+1}(L^2_{n+1} - L_nL_{n-1})^2$

The common subscript difference is 1. A generalization consists in forming identities with (a) a common subscript difference $2r + 1$; (b) a common subscript difference $2r$.

1. $F^3_{n+2r+1} - L^3_{2r+1}F^3_n - F^3_{n-2r-1} = 3L_{2r+1}F_{n-2r-1}F_nF_{n+2r+1}$
   $L^3_{n+2r+1} - L^3_{2r+1}L^3_n - L^3_{n-2r-1} = 3L_{2r+1}L_{n-2r-1}L_nL_{n+2r+1}$

   $L^3_{2r}F^3_n - F^3_{n+2r} - F^3_{n-2r} = 3L_{2r}F_{n-2r}F_nF_{n+2r}$
   $L^3_{2r}L^3_n - L^3_{n+2r} - L^3_{n-2r} = 3L_{2r}L_{n-2r}L_nL_{n+2r}$

2. $F^4_{n+2r+1} + L^4_{2r+1}F^4_n + F^4_{n-2r-1} = 2[F^2_{n+2r+1} - L_{2r+1}F_nF_{n-2r-1}]^2$
   $L^4_{n+2r+1} + L^4_{2r+1}L^4_n + L^4_{n-2r-1} = 2[L^2_{n+2r+1} - L_{2r+1}L_nL_{n-2r-1}]^2$

   $F^4_{n+2r} + L^4_{2r}F^4_n + F^4_{n-2r} = 2(F^2_{n+2r} + L_{2r}F_nF_{n-2r})^2$
   $L^4_{n+2r} + L^4_{2r}L^4_n + L^4_{n-2r} = 2(L^2_{n+2r} + L_{2r}L_nL_{n-2r})^2$

3. $F^5_{n+2r+1} - L^5_{2r+1}F^5_n - F^5_{n-2r-1} = 5L_{2r+1}F_{n-2r-1}F_nF_{n+2r+1}(F^2_{n+2r+1} - L_{2r+1}F_nF_{n-2r-1})$
   $L^5_{n+2r+1} - L^5_{2r+1}L^5_n - L^5_{n-2r-1} = 5L_{2r+1}L_{n-2r-1}L_nL_{n+2r+1}(L^2_{n+2r+1} - L_{2r+1}L_nL_{n-2r-1})$

   $L^5_{2r}F^5_n - F^5_{n+2r} - F^5_{n-2r} = 5L_{2r}F_{n-2}\ F_nF_{n+2r}(F^2_{n+2r} + L_{2r}F_nF_{n-2r})$
   $L^5_{2r}L^5_n - L^5_{n+2r} - L^5_{n-2r} = 5L_{2r}L_{n-2}\ L_nL_{n+2r}(L^2_{n+2r} + L_{2r}L_nL_{n-2r})$

4. (a) $F^7_{n+2r+1} - L^7_{2r+1}F^7_n - F^7_{n-2r-1} = 7L_{2r+1}F_{n-2r-1}F_nF_{n+2r+1}(F^2_{n+2r+1} - L_{2r+1}F_nF_{n-2r-1})^2$
   (b) $L^7_{n+2r+1} - L^7_{2r+1}L^7_n - L^7_{n-2r-1} = 7L_{2r+1}L_{n-2r-1}L_nL_{n+2r+1}(L^2_{n+2r+1} - L_{2r+1}L_nL_{n-2r-1})^2$

(c) $L_{2r}^7 F_n^7 - F_{n+2r}^7 - F_{n-2r}^7 = 7L_{2r}F_{n-2r}F_nF_{n+2r}(F_{n+2r}^2 + L_{2r}F_nF_{n-2r})^2$

(d) $L_{2r}^7 L_n^7 - L_{n+2r}^7 - L_{n-2r}^7 = 7L_{2r}L_{n-2r}L_nL_{n+2r}(L_{n+2r}^2 + L_{2r}L_nL_{n-2r})^2$

The proofs of 4(a) and 4(c) could serve as proof models for the remaining identities.

4(a): $F_{n+2r+1}^7 - L_{2r+1}F_n^7 - F_{n-2r-1}^7 = -(L_{2r+1}F_n)^7 + F_{n+2r+1}^7 - F_{n-2r-1}^7$

$$= -(F_{n+2r+1} - F_{n-2r-1})^7 + F_{n+2r+1}^7 - F_{n-2r-1}^7$$

$$= 7F_{n+2r+1}^6F_{n-2r-1} - 21F_{n+2r+1}^5F_{n-2r+1}^2 + 35F_{n+2r-1}^4F_{n-2r+1}^3 - 35F_{n+2r+1}^3F_{n-2r-1}^4 + 21F_{n+2r+1}^2F_{n-2r-1}^5 - 7F_{n+2r+1}F_{n-2r-1}^6$$

$$= 7F_{n+2r+1}F_{n-2r-1}(F_{n+2r+1}^5 - 3F_{n+2r+1}^4F_{n-2r-1} + 5F_{n+2r+1}^3F_{n-2r-1}^2 - 5F_{n+2r+1}^2F_{n-2r-1}^3 + 3F_{n+2r+1}F_{n-2r-1}^4 - F_{n-2r-1}^5)$$

$$= 7F_{n+2r+1}F_{n-2r-1}(F_{n+2r+1} - F_{n-2r-1})(F_{n+2r+1}^4 - 2F_{n+2r+1}^3F_{n-2r-1} + 3F_{n+2r+1}^2F_{n-2r-1}^2 - 2F_{n+2r+1}F_{n-2r-1}^3 + F_{n-2r-1}^4)$$

$$= 7F_{n+2r+1}F_{n-2r-1}L_{2r+1}F_n(F_{n+2r+1}^2 - F_{n+2r+1}F_{n-2r-1} + F_{n-2r-1}^2)^2$$

$$= 7L_{2r+1}F_{n-2r-1}F_nF_{n+2r+1}(F_{n+2r+1}^2 - L_{2r+1}F_nF_{n-2r-1})^2$$

4(c): $F_n^7L_{2r}^7 - F_{n+2r}^7 - F_{n-2r}^7 = (F_nL_{2r})^7 - F_{n+2r}^7 - F_{n-2r}^7 = (F_{n+2r} + F_{n-2r})^7 - F_{n+2r}^7 - F_{n-2r}^7$

$$= 7F_{n-2r}F_{n+2r}(F_{n+2r}^5 + 3F_{n+2r}^4F_{n-2r} + 5F_{n+2r}^3F_{n-2r}^2 + 5F_{n+2r}^2F_{n-2r}^3 + 3F_{n+2r}F_{n-2r}^4 + F_{n-2r}^5)$$

$$= 7F_{n-2r}F_{n+2r}(F_{n+2r} + F_{n-2r})(F_{n+2r}^4 + 2F_{n+2r}^3F_{n-2r} + 3F_{n+2r}^2F_{n-2r}^2 + 2F_{n+2r}F_{n-2r}^3 + F_{n-2r}^4)$$

$$= 7F_{n-2r}F_{n+2r}L_{2r}F_n(F_{n+2r}^2 + F_{n+2r}F_{n-2r} + F_{n-2r}^2)^2$$

$$= 7L_{2r}F_{n-2r}F_nF_{n+2r}(F_{n+2r}^2 + L_{2r}F_nF_{n-2r})^2$$

NOTE: On the assumption that Type I primitive units are given by

$$\left(\frac{a + b\sqrt{D}}{2}\right)^n = \frac{L_n + F_n\sqrt{D}}{2},$$

these sixteen generalized F-L identities are valid Type I identities.

REFERENCE

1. Problem H-112 (and its solution), proposed by Leonard Carlitz. *The Fibonacci Quarterly* 7 (1969).

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A CHARACTERIZATION OF THE PYTHAGOREAN TRIPLES

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The Pythagorean triples are all the systems of positive integers x, y, z which satisfy the "Pythagorean equation"

(1) $$x^2 + y^2 = z^2.$$

It is well known (see Uspensky and Heaslet [2]) that the Pythagorean triples can be characterized by the formulas

(2) $$x = M(r^2 - s^2),\ y = M2rs,\ z = M(r^2 + s^2),$$

where r and s are any two relatively prime numbers of different parity with $r > s$ and M is an arbitrary positive integer.

In this note we characterize the Pythagorean triples that satisfy (1) in terms of the integer k, where

(3) $$z = y + k$$

for some $k \geq 1$.

The case where $k = 1$ and thus $z = y + 1$ is also well known and a proof appears in Ore [1]. The solutions are characterized by the formulas

(4) $$x = 2n + 1,\ y = 2n(n + 1),\ z = 2n(n + 1) + 1$$

where n is any integer ≥ 1.

In order to generalize the result for all positive integers k, we observe that any positive integer k can be written in the form

(5) $$k = p^2q$$

where p and q are positive integers and $q = P_1P_2 \ldots P_m$ for distinct primes $P_1, P_2, \ldots, P_m$. Consequently, we have the following characterization.

Theorem: Let (x, y, z) be a Pythagorean triple where $z = y + k$ for $k \geq 1$. Then

(i) if k is odd and $k = p^2q$, then for $n \geq 1$,

$$x = pq(2n + p)$$
$$y = 2nq(n + p)$$
$$z = 2nq(n + p) + k,$$

(ii) if k is even and $k = 2p^2q$, then for $n \geq 1$,

$$x = 2pq(n + p)$$
$$y = nq(n + 2p)$$
$$z = nq(n + 2p) + k.$$

Proof: (i) Suppose k is odd, $k = p^2q$ and $q = P_1P_2 \ldots P_m$ where $P_1, P_2, \ldots, P_m$ are distinct odd primes. Then

$$x^2 + y^2 = (y + k)^2$$

implies

$$x^2 = 2yk + k^2$$

or

$$x^2 = p^2(2yq + p^2q^2).$$

Hence,

(6) $$x = p\sqrt{2yq + p^2q^2}.$$

Since x is an integer, $2yq + p^2q^2 = t^2$ for some integer t. Solving for y,

(7) $$y = \frac{t^2 - p^2q^2}{2q}.$$

But y is positive, hence, $t = s + pq$ for some integer $s \geq 1$. Substituting t into (7) yields

(8) $$y = \frac{s(s + 2pq)}{2q}.$$

Hence, s must be even, say $s = 2w$ for some integer $w \geq 1$, and substituting into (8) we have

(9) $$y = \frac{2w(w + pq)}{q}.$$

Since q is odd and a product of distinct primes, q must divide w, i.e., $w = nq$ for some integer $n \geq 1$. Substituting w into (9) yields the desired formula for

(10) $$y = 2nq(n + p),$$

and substituting (10) for y in (6) yields

$$x = pq(2n + p).$$

(ii) Suppose k is even, $k = 2p^2q$ and q is a product of distinct primes. Then

$$x^2 = 4p^2(yq + p^2q^2),$$

and

(11) $$x = 2p\sqrt{yq + p^2q^2}.$$

Again, $yq + p^2q^2 = t^2$ for some integer t. Solving for y,

(12) $$y = \frac{t^2 - p^2q^2}{q}.$$

But y is positive, hence $t = s + pq$ for some integer $s \geq 1$. Substituting t into (12) yields

(13) $$y = \frac{s(s + 2pq)}{q}.$$

Since q is a product of distinct primes, q must divide s, i.e., $s = nq$ for some integer $n \geq 1$. Substituting s into (13) yields the desired formula for y,

(14) $$y = nq(n + 2p),$$

and substituting (14) for y in (11) yields

$$x = 2pq(n + p).$$

REFERENCES

1. O. Ore. *Number Theory and Its History*. New York: McGraw-Hill, 1948.
2. J. V. Uspensky and M. A. Heaslet. *Elementary Number Theory*. New York: McGraw-Hill, 1939.

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ON PRIMITIVE WEIRD NUMBERS

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1. INTRODUCTION

Let n be a positive integer. Denote by $\sigma(n)$ the sum of divisors of n. It is called n perfect if $\sigma(n) = 2n$, abundant if $\sigma(n) \geq 2n$, and deficient if $\sigma(n) < 2n$. Further, n is defined to be pseudoperfect if it is the sum of some of its proper divisors that all are distinct (d is a proper divisor of n, if d/n and $d < n$).

An integer n is called weird if n is abundant but not pseudoperfect. It is primitive abundant if it is abundant but all its proper divisors are deficient. If n is primitive abundant but not pseudoperfect, it is called primitive weird.

It is not known [1] if there are infinitely many primitive weird numbers or any odd weird numbers. A list of weird and primitive weird numbers not exceeding 10^6 is given in [1]. However, there is a misprint in [1] on page 618: instead of 539774 one should read 539744.

In this note we let n specially be of the form

(1) $$n = 2^\alpha pq \quad (\alpha \geq 1,\ p < q,\ p \text{ and } q \text{ odd primes}),$$

and give necessary and sufficient conditions under which n is primitive weird. As far as we know this cannot be found in the literature. As an application, we list some primitive weird numbers exceeding 10^6.

Throughout this note, let p and q be odd primes and $p < q$.

We use the following notations:

$$S = \sum_{\nu=0}^{\alpha} 2^\nu = 2^{\alpha+1} - 1, \quad S' = \sum_{(\nu)} 2^\nu$$

(the sum being taken over some of the indices ν);

$$S_p = \sum_{\nu=0}^{\alpha} 2^\nu p = (2^{\alpha+1} - 1)p, \quad S_p^m = S_p - mp \quad (0 \leq m \leq 2^{\alpha+1} - 1);$$

$$S_q = \sum_{\nu=0}^{\alpha} 2^\nu q = (2^{\alpha+1} - 1)q, \quad S_q^n = S_q - nq \quad (0 \leq n \leq 2^{\alpha+1} - 1);$$

$$S_{pq} = \sum_{\nu=0}^{\alpha-1} 2^\nu pq = (2^\alpha - 1)pq, \quad S_{pq}^k = S_{pq} - kpq \quad (0 \leq k \leq 2^\alpha - 1).$$

Theorem 1: The integer n in (1) is primitive weird iff

(2) $$2^{\alpha+1} + 1 \le p \le q < \frac{(2^{\alpha+1} - 1)(p + 1)}{p - (2^{\alpha+1} - 1)}$$

is true and

(3) $$pq = S_p^m + S_q^n + S' \quad \text{for some } m,n$$

is false.

Theorem 2: Assume that the primes p and q are of the forms

$$p = 2^{\alpha+1} + x, \quad 1 \le x \le 2^{\alpha+1} - 3$$

$$q = \frac{\tau p - i}{x + 1}, \quad 1 \le i \le x \text{ and } \tau \text{ an integer}$$

such that

$$\frac{2^{\alpha+1}p - i}{x + 2} \le q \le \frac{2^{\alpha+1}p + x - 1}{x + 1}.$$

Then the integer n in (1) is primitive weird.

2. APPLICATIONS

Theorem 2 gives, e.g., the following primitive weird numbers $n = 2^\alpha pq$.

2^α	p	q	n
2	5	7	70
4	11	19	836
8	17	127	17272
	19	71	10792
		61	9272
	23	43	7912
	29	31	7192
16	37	191	113072
	41	127	83312
	43	107	73616
32	67	1021	2189024
		971	2081824
		887	1901728
	71	541	1229152
		523	1188256
	79	311	786208
	83	257	682592
	97	179	555616
	101	167	539744
	109	149	519712
64	131	4159	34869056
		4093	34315712
		3733	31297472
		3373	28279232
	137	1657	14528576
	139	1471	13086016
		1469	12979264
		1447	12872512
	149	853	8134208
		839	8000704
	151	773	7470272
	157	659	6621632
	167	521	5568448
	179	433	4960448
	191	379	4632896
	239	271	4145216
	251	257	4128448
128	257	30197	993360512
		29683	976451968

(continued)

2^α	p	q	n
		25057	824275072
		24029	790457984
	263	8317	279983488
		8087	272240768
		7561	254533504
128	269	4861	167373952
		4649	160074368
	271	4217	146279296
	277	3109	110232704
	283	2557	92624768
	307	1499	58904704
		1493	58668928
		1487	58433152
	311	1399	55691392
	317	1303	52870528
	337	1039	44818304
	409	677	35442304
	499	521	33277312
256	521	25997	3467375872
		25841	3446569216
		25633	3418827008
		24851	3314526976
		24799	3307591424
	523	22271	2981819648
		21617	2894256896
		20963	2806694144
		20789	2783397632
	547	7703	1078666496
		7673	1074465536
	557	6163	878794496
		6151	877083392
256	563	5521	795730688
	569	5003	728756992
		4993	727300352
		4973	724387072
	577	4441	655988992
		4423	653330176
	587	3931	590719232
		3923	589517056
	593	3673	557590784
		3659	555465472
	599	3457	530110208
	619	2917	462239488
	631	2687	434047232
		2671	431462656
	661	2251	380905216
	683	2029	354766592
	769	1523	299823872
	811	1381	286717696
	839	1307	280722688
	911	1163	271230208
	919	1151	270788864
	937	1123	269376256
	947	1109	268857088
	1013	1031	267367168

3. PROOF OF THEOREM 1

The divisors of n in (1) are:

$$2^\nu,\ 2^\nu p,\ 2^\nu q,\ 2^\nu pq \quad (\nu = 0, 1, \ldots, \alpha).$$

We note that divisors 2^ν are always deficient. All the divisors $2^\nu p$ and $2^\nu q$ are deficient iff

(4) $$p \geq 2^{\alpha+1} + 1.$$

For such p, n is abundant iff

(5) $$q \le \frac{(2^{\alpha+1} - 1)(p + 1)}{p - (2^{\alpha+1} - 1)}.$$

Last we see that all the divisors $2^{\nu}pq$, where $\nu < \alpha$ and p satisfies (4), are deficient. This shows that n is primitive abundant iff (2) holds.

It is clear that [if the condition (4) holds]

$$S' \le S < p < q - 1,$$

so

$$2pq - S' > 2pq - p = (2q - 1)p > (p + q)p > S_p + S_q \ge S_p^m + S_q^n$$

or

(6) $$S_p^m + S_q^n + S' < 2pq.$$

Since

$$S_{pq}^k \le S_{pq} = n - pq,$$

we see from (6) that n is pseudoperfect iff (3) holds.

4. PROOF OF THEOREM 2

On the basis of our choice of p and q, the condition (2) is satisfied. Write (3) in the form

(7) $$mp + nq + (2^{\alpha+1} - 1)q + (x + 1)q = (2^{\alpha+1} - 1)(p + q) + S'.$$

This implies

(8) $$nq = (2^{\alpha+1} - 1 - \tau - m)p + i + S'.$$

Write, for brevity,

$$M = 2^{\alpha+1} - 1 - \tau.$$

If $m \ge M + 1$, the right side of (8) is $\le -p + i + S' \le -1 < 0$, while the left side is always ≥ 0.

In the case $m = M$, (8) is equivalent to $nq = S' + i$, which cannot hold for any n, because $0 < S' + i < q$.

Finally we have the case $m < M$. Equation (8) trivially fails for $n = 0$. If $n \ge 1$, we see that

$$nq > (M - m)p + i + S'$$

if $q \ge (M + 1)p$, and this is true for

$$q \ge \frac{2^{\alpha+1}p - i}{x + 2}.$$

5. REMARK

An integer

(10) $$n = 2^{\alpha}\prod_{i=1}^{t} p_i \quad (2^{\alpha+1} < p_1 < \cdots < p_t)$$

is abundant, and all its proper divisors are deficient, if

(11) $$\frac{2^{\alpha+1}}{2^{\alpha+1} - 1}\left(1 + \frac{1}{p_1}\right) > \prod_{i=1}^{t}\left(1 + \frac{1}{p_i}\right) \ge \frac{2^{\alpha+1}}{2^{\alpha+1} - 1}$$

or

(12) $$\frac{(2^{\alpha+1} - 1)\prod_{i=1}^{t-1}(p_i + 1)}{2^{\alpha+1}\left(1 + \frac{1}{p_1}\right)\prod_{i=1}^{t-1}p_i - (2^{\alpha+1} - 1)\prod_{i=1}^{t-1}(p_i + 1)} < p_t \le \frac{(2^{\alpha+1} - 1)\prod_{i=1}^{t-1}(p_i + 1)}{2^{\alpha+1}\prod_{i=1}^{t-1}p_i - (2^{\alpha+1} - 1)\prod_{i=1}^{t-1}(p_i + 1)}.$$

We see that n is not pseudoperfect if

(13) $$\sigma(n) - 2n = 2^{\alpha+1},$$

because

$$\sigma(n) - n - \sum_{\nu=0}^{\alpha} 2^{\nu} = n + 1 > n$$

and

$$\sigma(n) - n - p_1 = n - (p_1 - 2^{\alpha+1}) < n.$$

Write (13) into the form

$$p_t = \frac{(2^{\alpha+1} - 1)\prod_{i=1}^{t-1}(p_i + 1) - 2^{\alpha+1}}{2^{\alpha+1}\prod_{i=1}^{t-1}p_i - (2^{\alpha+1} - 1)\prod_{i=1}^{t-1}(p_i + 1)}. \tag{14}$$

We see that p_t from (14) also satisfies (12) and this remains valid if we replace $2^{\alpha+1}$ in (13) and (14), e.g., by any constant $A \geq 2^{\alpha+1}$ provided that $p_1 > A$.

We can now present an algorithm for computing arbitrary long (great) primitive weird numbers n satisfying (10) and (14) if they exist.

For given α choose first the prime $p_1 > (A \geq) 2^{\alpha+1}$ and then p_2 from (14). If this is not a prime, choose p_2 an arbitrary prime $> p_1$ and calculate p_3 from (14). If this is not a prime, choose p_3 an arbitrary prime $> p_2$, and so on. The algorithm ends when we obtain a prime p_t from (14).

REFERENCE

1. S. J. Benkoski and P. Erdös. "On Weird and Pseudoperfect Numbers." *Math. of Comp.* 126 (1974):617-623.

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FIBONACCI CONCEPT: EXTENSION TO REAL ROOTS OF POLYNOMIAL EQUATIONS

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It was in November 1973 when Professor T. A. Davis was conducting a biocensus that he introduced me to the well-known Fibonacci numbers. He told me that certain limbs of a normal human body are in the Golden Ratio, viz. 1.618... . I observed that the reciprocal of the Golden Ratio (0.618...) is nothing but a root of the quadratic equation

$$x^2 + x = 1 \tag{1}$$

or

$$x^2 + x - 1 = 0 \tag{2}$$

which is formed by equating the three ratios of human limbs (each ratio, in fact, is equal to the Golden Ratio).

As is well known, this root 0.618 of (1) is the fixed ratio of the successive terms (ignoring some of the initial terms) of the Fibonacci sequence. I considered the sequence $\{U_r\}$ defined as follows:

$$U_r = 1,\ \forall_r = 1, 2, 3;\ U_r = U_{r-1} + U_{r-2} + U_{r-3},\ \forall_r \geq 4. \tag{3}$$

Using a computer program, I found that after 21 terms of the sequence, the ratios $\left\{\frac{U_{r-1}}{U_r}\right\}$ become constant up to the 9th decimal place and is 0.543689013, which is found to be a root of the polynomial equation (cubic),

$$x^3 + x^2 + x = 1. \tag{4}$$

Now, consider the sequences defined, analogously, as follows:

Sequence (Definitions):

(i) $U_r = 1,\ \forall_r = 1, 2, 3, 4;\ U_r = U_{r-1} + U_{r-2} + U_{r-3} + U_{r-4},\ \forall_r \geq 5;$

(ii) $U_r = 1,\ \forall_r = 1, 2, 3, 4, 5;\ U_r = U_{r-1} + U_{r-2} + U_{r-3} + U_{r-4} + U_{r-5},\ \forall_r \geq 6;$

⋮

(vii) $U_r = 1,\ \forall_r = 1, 2, 3, \ldots, 10;\ U_r = U_{r-1} + U_{r-2} + \cdots + U_{r-10},\ \forall_r \geq 11.$

The approximate limit points (which do exist) of sequences of ratios $\left\{\frac{U_{r-1}}{U_r}\right\}$ as obtained by computer, are 0.518790064, 0.508660392, 0.504138258, 0.502017055, 0.500994178, 0.500493118, and

0.500245462, respectively. We see that the nature of these sequences of ratios is also similar. These fixed ratios are the roots of the following polynomials (q) where q can be any one of the symbols i, ii, iii, ..., vii.

(i) $x^4 + x^3 + x^2 + x = 1$

(ii) $x^5 + x^4 + x^3 + x^2 + x = 1$

$\vdots$

(vii) $x^{10} + x^9 + x^8 + \cdots + x^2 + x = 1$

[I also observed that these ratios are tending to 0.5 ($=\frac{1}{2}$) as n becomes larger and larger, where n is the number of prefixed terms, each equal to unity, and also is the number of terms on R.H.S. of recurrence relations used in the definitions of the sequences $\{U_r\}$.] This can be explained mathematically on the ground that

$$\left(\frac{1}{2}\right)^n + \left(\frac{1}{2}\right)^{n-1} + \cdots + \left(\frac{1}{2}\right)^n + \frac{1}{2} \to 1 \text{ as } n \to \infty.$$

I observed this link only up to $n = 10$. For $n = 1$, it is obvious; also, for $n = 2$, it is easily seen to be valid. Intuitively, it can be stated that this fact is valid for all infinite n.

Then I considered the sequence:

(5) $$U_r = 1, \forall_r = 1, 2; U_r = 2U_{r-1} + 3U_{r-2}, \forall_r \geq 3.$$

I studied the ratios of the consecutive terms of this sequence and I found that the ratio tends to a root (= 0.3333...) of the equation

(6) $$3x^2 + 2x = 1.$$

Now consider the cubic equation

(7) $$2x^3 + x^2 + x = 1,$$

and form the sequence $\{U_r\}$ as follows:

(8) $$U_r = 1, \forall_r = 1, 2, 3; U_r = U_{r-1} + U_{r-2} + 2U_{r-3}, \forall_r \geq 4.$$

Take the ratios of the consecutive terms of this sequence. The sequence of these ratios comes out to be tending to 0.5, which is a root of the cubic equation.

Let us now slightly generalize the concept. Consider the polynomial equation

(9) $$a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_2 x^2 + a_1 x = 1$$

where all a_i's are positive real constants and n is any positive integer. Construct a corresponding sequence $\{U_r\}$ as follows:

(10) $$U_r = 1, \forall_r = 1, 2, 3, \ldots, n; U_r = a_1 U_{r-1} + a_2 U_{r-2} + \cdots + a_n U_{r-n}, \forall_r \geq n + 1.$$

Take the consecutive ratios of the terms of this sequence. The sequence $\left\{\frac{U_{r-1}}{U_r}\right\}$ of ratios comes out to be tending to a root of the polynomial equation (9).

Now consider the following polynomial

(11) $$x^2 - 2x = 1$$

involving negative coefficients also. Construct the sequence as follows:

(12) $$U_r = 1, \forall_r = 1, 2; U_r = -2U_{r-1} + U_{r-2}.$$

Find the ratios of the consecutive terms of this sequence. These again tend to -0.414213..., a root of this polynomial equation.

In case the roots of a quadratic equation are not real, the said sequence of ratios does not converge. In some cases it fluctuates in a manner that is readily observed. In other cases it is quite difficult to know the fluctuation pattern. I believe that there is some mathematical relation between this fluctuation pattern of the sequence of the ratios and the discrimanant of the quadratic equation when the constant term is made -1, by suitably disposing the coefficients. As an example, one can observe the quadratic equation

(13) $$-x^2 - x = 1.$$

Sometimes it happens that the sequence of the said ratios behaves in such a manner that it is quite difficult to assess even whether it is converging to some constant or fluctuating in some pattern. In such cases, with the help of a computer, one can assess the nature of the sequence of the ratios observing fairly large numbers of terms of this sequence (say 200 and 300, etc.). For example, I could not see any pattern easily in the sequence of the said ratios for the polynomial equation

(14) $$2x^3 + x^2 - x = 1.$$

However, when the fluctuations pattern is readily observable, it is my belief that there is a relation between the oscillating ratios and an imaginary root of the considered polynomial.

I could not get any case where the sequence of the said ratios converged to some constant, say x_0, when x_0 was not a root of the considered polynomial equation. This led me to state the following:

"Given a polynomial equation of the type

(15) $$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x = 1$$

(all a_n's are real and n any positive integer)."

We observe the sequence of the ratios of the successive terms of the sequence $\{U_r\}$ defined as follows:

(16) $$U_r = 1, \forall_r = 1, 2, \ldots, n; \; U_r = a_1 U_{r-1} + a_2 U_{r-2} + \cdots + a_n U_{r-n}, \; \forall_r \geq n + 1.$$

If the sequence $\left\{\frac{U_{r-1}}{U_r}\right\}$ converges to some fixed number x_0 (and I believe if there is a real root it converges more often than not), then x_0 satisfies this polynomial equation.

This fact can be utilized to attempt to find out the roots of a polynomial equation

(α) $$A_n x^n + A_{n-1}x^{n-1} + \cdots + A_1 x + A_0 = 0,$$

where the A_i's are all real and n is a positive integer. The method is summarized as follows: "If $A_0 = 0$, clearly $x = 0$ satisfies (α). So zero is one root of (α). Divide, then, (α) by x to get an equation of $(n - 1)$st degree, and again treat this new polynomial equation of degree $(n - 1)$ as (α). If $A_0 \neq 0$, we can write (α) in the following form:

(17) $$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x = 1, \text{ where } a_i = -\frac{A_i}{A_0}.$$

Now form a sequence $\{U_r\}$ as in (16). A fixed quantity x_0, to which the sequence $\left\{\frac{U_{r-1}}{U_r}\right\}$ tends, is one of the roots of (α). I am sure that it does tend, at least when all a_i's are positive. Divide (α) by $(x - x_0)$ to obtain a polynomial equation of degree $(n - 1)$. Again treat this new polynomial equation as (17) and (if possible) obtain another root, and so on."

I believe that the whole phenomenon is not merely a magic of numbers; instead, there is some mathematics behind this, though I could not get hold of it. Also, I could not find the system by which the sequence of the ratios chooses one of the roots to converge to.

Lastly, I quote an interesting example. Call the set of unities used in defining the sequence $\{U_r\}$ as "generators." In fact, this example will show the importance of generators.

"Consider the quadratic equation

(18) $$2x^2 - x - 1 = 0 \quad \text{or} \quad 2x^2 - x = 1.$$

Form $\{U_r\}$ as usual (i.e., taking 1,1 as generator) so as to get 1, 1, 1, 1, 1, 1, ..., which gives us 1 as one of the roots of (18). Now form another sequence $\{U_r\}$ taking the coefficients 2 and -1 as generators to get 2, -1, 5, -7, 17, -31, 65, ..., which converges to $-\frac{1}{2}$ and, interestingly, $-\frac{1}{2}$ is another root of (18)."

I shall conclude this article by posing a problem regarding the Fibonacci sequence. Up to the 36th term of the sequence (F_{36} = 24, 157, 817 ...) none is a perfect number. It remains to be solved whether any Fibonacci number is a perfect one. If not, then what is the mathematical logic behind it?

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TRIANGULAR ARRAYS ASSOCIATED WITH SOME PARTITIONS

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A partition of a positive integer n by a set of integers S is set $S' = \{s_i\}$ $(1 \le i \le r)$ of integers s_i drawn from S such that

$$n = \sum_{i=1}^{r} s_i . \tag{1}$$

The general problem for pattitions is to discuss the number of partitions of n by S, a number that depends, in each particular problem, on the restrictions that are placed on the representation (1); for example, r may not be fixed, S' may or may not contain repetitions, or the order of the terms in (1) may or may not matter according to the question at issue. A typical, but difficult, problem is that of unrestricted partitions: what is the number $p(n)$ of partitions of n by S when S is the set of all positive integers, repetitions are allowed, and neither the number of terms nor their order in (1) matters? (See [2, pp. 273-296] for an introduction to the theory of partitions as well as an account of some results concerning $p(n)$.)

A much simpler problem for which the answer is known is: what is the number $b(n,k)$ of partitions of $(n + 1)$ by S when S is the set of all positive integers, $r = k + 1$, repetitions are allowed, and the order of the terms in (1) matters? The number in question, as may easily be seen, is just a binomial coefficient

$$b(n,k) = \binom{n}{k}, \quad n \ge k \ge 0. \tag{2}$$

The associated triangular array $\{b(n,k)\}$ $(n \ge k \ge 0)$ is the familiar Pascal triangle and, among many identities for the $b(n,k)$, we have

$$b(n,k) = \sum_{m=1}^{n-k+1} b(n - m,\ k - 1), \quad n \ge k \ge 1. \tag{3}$$

Introducing the generating functions

$$B_k(x) = \sum_{n=k}^{\infty} b(n,k)x^{n-k}; \; B(x) = B_0(x) = \sum_{n=0}^{\infty} x^n,$$

we have, at least formally,

$$B_k(x) = [B(x)]^{k+1}, \quad k \ge 1 \tag{4}$$

$$(1 - x)B(x) = 1. \tag{5}$$

Hale [1] has recently enquired about partitions using k ones rather than partitions into k parts: what is the number $f(n,k)$ of partitions of n by S when S is the set of all integers, r is arbitrary, repetitions are allowed, the order of terms matters, and $s_i = 1$ for exactly k values of i, $1 \le i \le r$? Carson and Oates, in reply [3], have given a formula analogous to (3), namely,

$$f(n,k) = f(n - 1,\ k - 1) + \sum_{m=2}^{n-k} f(n - m,\ k), \quad n \ge k + 2 \ge 3, \tag{6}$$

noting also that, for $n \ge 2$, the $f(n,\ 0)$ are the Fibonacci numbers, since

$$f(n,\ 0) = f(n - 1,\ 0) + f(n - 2,\ 0), \quad n \ge 3, \tag{7}$$

with

$$f(1,\ 0) = 0;\ f(2,\ 0) = 1,$$

Since, as Hale [1] noted,

$$f(n,n) = 1;\ f(n,\ n - 1) = 0, \quad n \ge 1,$$

the associated triangular array $\{f(n,k)\}$ $(n \ge k \ge 0;\ n \ge 1)$ is completely determined by (6, 7).

To prove (7), note that either the first (and only) term s_1 in (1) is n or form some m, $2 \le m < n$, $s_1 = m$, and then the remaining terms $\{s_i\}$ $(2 \le i \le r)$ form a partition of $(n - m)$ by S. Hence

(8) $$f(n,\ 0) = 1 + \sum_{m=2}^{n-1} f(n - m,\ 0), \qquad n \geq 3$$

$$= 1 + \sum_{m=2}^{n-2} f(n - 1 - m,\ 0) + f(n - 2,\ 0), \qquad n \geq 4$$

$$= f(n - 1,\ 0) + f(n - 2,\ 0), \qquad n \geq 4$$

where the last equation also holds for $n = 3$. The proof of (6) is similar to that of (8), except that now $s_1 = 1$ is possible, giving the additional term $f(n - 1,\ k - 1)$. Notice that taking

(9) $$f(0,\ 0) = 1$$

both gives an apex to the triangular array $\{f(n,k)\}$ and eases the above proofs, allowing (7) to be extended to $n = 2$. Moreover, taking $f(n,k) = 0$ for $k < 0$ or $k > n$, allows (6) to be extended to $n \geq k + 2 \geq 2$ with (8) as a special case.

An alternative approach, complementing this additive theory, is by way of convolutive or multiplicative identities analogous to (4). By way of illustration, consider $f(n,\ 1)$, so that exactly one of the s_i in (1) is equal to 1. Now either $s_1 = 1$ and $\{s_i\}$ $(2 \leq i \leq r)$ is a partition of $(n - 1)$ by S, or for some i, $1 < i < r$, $s_i = 1$, and then for some m, $2 \leq m \leq n - 1$, $\{s_j\}$ $(1 \leq j < i)$ is a partition of m by S, while $\{s_j\}$ $(1 < i \leq r)$ is a partition of $(n - m - 1)$ by S; or $s_r = 1$ and $\{s_i\}$ $(1 \leq i < r)$ is a partition of $(n - 1)$ by S. Since these cases are exclusive and exhaustive,

$$f(n,\ 1) = f(n - 1,\ 0) + \sum_{m=2}^{n-1} f(m,\ 0)\ (n - m - 1,\ 0) + f(n - 1,\ 0), \quad n \geq 3$$

or, making use of (9),

(10) $$f(n,\ 1) = \sum_{m=1}^{n} f(n,\ 0) f(n - m - 1,\ 0), \quad n \geq 2.$$

Similarly, by considering the least i $(1 \leq i \leq r)$ for which $s_i = 1$ in (1), we have for $k \geq 1$

$$f(n,k) = f(n - 1,\ k - 1) + f(1,\ 0) f(n - 2,\ k - 1) + \cdots + f(n - k,\ 0) f(k - 1,\ k - 1), \qquad n \geq k + 1$$

(11) $$= \sum_{m=0}^{n-k} f(m,\ 0) f(n - m - 1,\ k - 1), \qquad n \geq k \geq 1$$

On introducing the generating functions

$$F_k(x) = \sum_{n=k}^{\infty} f(n,k) x^{n-k};\ F(x) = F_0(x),$$

we have, from (10, 11),

$$F_1(x) = F_0(x) F_0(x) = [F(x)]^2$$

$$F_k(x) = F_{k-1}(x) F_0(x) = F_{k-1}(x) F(x), \quad k \geq 1,$$

and it follows that

(12) $$F_k(x) = [F(x)]^{k+1}, \qquad k \geq 1.$$

From (12), which is the analogue of (4), further identities may be obtained in turn, for example,

$$F_k(x) = F_s(x) F_{k-s-1}(x), \quad 0 \leq s < k.$$

Moreover, by (7, 9), $F(x)$ satisfies the functional equation [cf. (5)],

(13) $$(1 - x - x^2) F(x) = 1 - x.$$

Similar results hold if we now take S to be the set of the first ℓ positive integers $(\ell \geq 2)$ rather than the set of all integers. Thus, let $b_\ell(n,k)$ be the number of partitions of $n + 1$ by S_ℓ where $S_\ell = \{i\}$ $(1 \leq i \leq \ell;\ \ell \geq 2)$, $r = k + 1$, repetitions are allowed in (1) and the order of the terms in (1) matters; and let $f_\ell(n,k)$ be the number of partitions of n by S_ℓ, r is arbitrary, repetitions are allowed, order matters, and k of the s_i are equal to 1. We further make the conventions that

$$b_\ell(n,k) = 0 = f\ (n,k), \quad k < 0 \text{ or } k > n,$$

$$f_\ell(0,0) = 1,$$

then the results for the triangular arrays $\{b_\ell(n,k)\}$, $\{f_\ell(n,k)\}$ $(n \geq k \geq 0)$ are truncated versions of those for the case of unrestricted S and may be summarized as follows, the proofs also being similar to those above.

First, we have the additive recurrence relations [cf. (3, 6)],

(14a) $$b_\ell(n,k) = \sum_{m=1}^{\ell} b_\ell(n - m,\ k - 1), \qquad n \geq k \geq 1,$$

(14b) $$b_\ell(n,0) = 1, \qquad 0 \leq n < \ell; \ = 0, \qquad n \geq \ell,$$

and

(15a) $$f_\ell(n,k) = f_\ell(n - 1,\ k - 1) + \sum_{m=2}^{\ell} f_\ell(n - m,\ \ell), \qquad n \geq k \geq 0,$$

(15b) $$f_\ell(0,0) = 1; \ f_\ell(1,0) = 0.$$

Secondly, writing

$$B_{k,\ell}(x) = \sum_{n=k}^{\infty} b_\ell(n,k)x^{n-k}; \ F_{k,\ell}(x) = \sum_{n=k}^{\infty} f_\ell(n,k)x^{n-k},$$

$$B_\ell(x) = B_{0,\ell}(x); \ F_\ell(x) = F_{0,\ell}(x),$$

we have [cf. (4, 12)]

(16) $$B_{k,\ell}(x) = [B_\ell(x)]^{k+1}; \ F_{k,\ell}(x) = [F_\ell(x)]^{k+1}, \quad k \geq 1,$$

with, from (14b, 15, $k = 0$)

(17) $$(1 - x)B_\ell(x) = 1 + x^\ell; \ \left(1 - \sum_{2}^{\ell} x^m\right)F_\ell(x) = 1.$$

Moreover, since

$$b_\ell(n,k) = b(n,k), \qquad n - \ell < k \leq n,$$
$$f_\ell(n,k) = f(n,k), \qquad n - \ell \leq k \leq n,$$

we have, in a natural way (see [2, p. 275]) the limiting results

(18) $$\lim_{\ell \to \infty} B_{k,\ell}(x) = B_k(x); \ \lim_{\ell \to \infty} F_{k,\ell}(x) = F_k(x).$$

Not all partition problems have the multiplicative structure exhibited in (4, 12, 16). For example, returning to the problem of unrestricted partitions mentioned at the beginning, let $p(n,k)$ be the number of partitions of n by S when S is the set of all positive integers, repetitions are allowed, neither the number of terms nor their order in (1) matters, and k of the s_i in (1) are equal to 1, and let

$$P_k(x) = \sum_{n=k}^{\infty} p(n,k)x^{n-k}; \ P(x) = P_0(x).$$

Then, without determining $P(x)$, it is at least straightforward to show that

(19) $$P_k(x) = P(x).$$

Shapiro [4] has asked whether there is an arithmetic of triangular arrays where a simple function of the generating function of the first column yields the generating function of the other columns as in (4, 12, 16) and indeed also (19). For example, given a sequence $\{a_n\}$ $(n \geq 0)$, we may define a triangular array $\{t_{n,k}\}$ by

(20a) $$t_{n,k} = \sum_{m=0}^{n-k} a_n t_{n-1,\,k-1+m},$$

(20b) $$t_{0,0} = a_0,$$

and

(20c) $$t_{n,k} = 0, \quad k < 0 \text{ or } k > n.$$

and then if

(21a) $$T_k(x) = \sum_{n=k}^{\infty} t_{n,k} x^{n-k}; \; T(x) = T_0(x),$$

(21b) $$T_k(x) = [T(x)]^{k+1}.$$

Conversely, given a triangular array satisfying (21), we may recover a sequence $\{a_n\}$ $(n \geq 0)$ via (20). What are the sequences arising in this way in the partition problems considered above [see (4, 12, 16)]?

REFERENCES

1. D. S. Hale. "Peter's Number Triangle." *Mathematics Teaching* 73 (1975):32.
2. G. H. Hardy and E. M. Wright. *Introduction to the Theory of Numbers*, 4th ed. O.U.P., 1960.
3. I. F. Carson and F. H. C. Oates. "Letters." *Mathematics Teaching* 74 (1976):4-5.
4. L. W. Shapiro. "A Catalan Triangle." *Discrete Math.* 14 (1976):83-90.

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BREAK-UP OF INTEGERS AND BRACKET FUNCTIONS IN TERMS OF BRACKET FUNCTIONS

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ABSTRACT

We have presented a general formula for the break-up of integers into bracket functins, and some formulas for the break-up of bracket functions into other bracket functions.

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It is interesting to find break-ups of variable integers into a sum of bracket functions involving the integer we want to break up and other integers. Two well-known examples of this are

(1) $$x = \sum_{i=0}^{m-1}\left[\frac{x+i}{m}\right] \quad \text{integers } m > 0;$$

(2) $$x = \left[\frac{(p+1)x}{2p+1}\right] + \sum_{i=1}^{p}\left[\frac{x+2i}{2p+1}\right] \quad \text{integers } p > 0.$$

Here we shall find a general break-up of the variable integer into bracket functions involving two other integers (equation 12). The above-mentioned break-ups are special cases of this more general formula.

To derive the general formula, we shall need to use the h-function (defined in [1]) defined by

(3) $$\begin{cases} h(x, m) = 1 & \text{if } m/x \\ \qquad\quad\;\; = 0 & \text{if } m\nmid x \end{cases}$$

It is easily seen that it satisfies the following properties (which we shall use later);

(4) $$\{h(x, m)\}^j = h(x, m) \quad \text{integers } j > 0;$$

(5) $$\sum_{j=1}^{m} h(x+j, m) = 1;$$

(6) $$h(x, m_1)h(x, m_2) = h(x, m) \text{ where } m = (m_1, m_2);$$

(7) $$h(x+mk, m) = h(x, m) \quad \text{integers } k;$$

(8) $$h(nx, m) = h(x, m) \text{ if } \langle n, m\rangle = 1.$$

Now, considering the difference operator, Δ, acting on the bracket function $\left[\frac{x-1}{m}\right]$:

$$\Delta\left[\frac{x-1}{m}\right] = \left[\frac{x}{m}\right] - \left[\frac{x-1}{m}\right] = \begin{cases} 1 & \text{if } m/x \\ 0 & \text{if } m\nmid x \end{cases}$$

we see that we can put

$$\Delta\left[\frac{x-1}{m}\right] = h(x, m)$$

(9)
$$\Delta^{-1}h(x, m) = \left[\frac{x-1}{m}\right] + c_1$$

where c_1 is an arbitrary constant. Applying the inverse difference operator to equation (5), we obtain

$$x = \sum_{j=1}^{m} \Delta^{-1}h(x+j, m) + c_2 = \sum_{j=1}^{m}\left[\frac{x+j-1}{m}\right] + c_3.$$

To evaluate the constant here, take $x = 1$. Clearly the lefthand side is equal to the bracket function. Thus, c_3 is zero.

$$\therefore\ x = \sum_{j=1}^{m}\left[\frac{x+j-1}{m}\right],$$

which is the same as equation (1).

To derive the general formula, consider

$$\sum_{r=1}^{n} h(nx+y+r, m) = \left|\Delta^{-1}h(nx+y+r, m)\right|_{r=1}^{n+1}$$

$$= \left[\frac{nx+y+r-1}{m}\right] - \left[\frac{nx+y}{m}\right]$$

$$= \Delta\left[\frac{nx+y}{m}\right]$$

(10)
$$\therefore\ \left[\frac{nx+y}{m}\right] = \sum_{r=1}^{n}\Delta^{-1}h(nx+y+r, m) + c.$$

We restrict our attention to relatively prime integers n and m. There must, then, exist two integers a and b such that

$$an + bm = 1$$

$$\therefore\ \left[\frac{nx+y}{m}\right] = \sum_{r=1}^{n}\Delta^{-1}h(nx+(an+bm)(y+r)m) + c.$$

Using equation (7), we now get

$$\left[\frac{nx+y}{m}\right] = \sum_{r=1}^{n}\Delta^{-1}h(nx+na(y+r), m) + c.$$

As $\langle n, m\rangle = 1$, using equation (8) gives

$$\left[\frac{nx+y}{m}\right] = \sum_{r=1}^{n}\Delta^{-1}h(x+a(y+r), m) + c = \sum_{r=1}^{n}\left[\frac{x+a(y+r)-1}{m}\right] + c.$$

Putting $x = 0$ in the above equation, we obtain

$$c = \left[\frac{y}{m}\right] - \sum_{r=1}^{n}\left[\frac{a(y+r)-1}{m}\right]$$

(11)
$$\therefore\ \left[\frac{nx+y}{m}\right] = \sum_{r=1}^{n}\left[\frac{x+a(y+r)-1}{m}\right] - \sum_{r=1}^{n}\left[\frac{a(y+r)-1}{m}\right] + \left[\frac{y}{m}\right].$$

We now further restrict our attention to the case $n < m$. We can then write

$$n = pq + 1$$

$$m = pq + p + 1$$

as these numbers are relatively prime (as can be easily checked). Then, taking $y = 0$, we obtain

$$\left[\frac{(pq+1)x}{pq+p+1}\right] = \sum_{r=1}^{pq+1}\left[\frac{x+ra-1}{pq+p+1}\right] - \sum_{r=1}^{pq+1}\left[\frac{ra-1}{pq+p+1}\right].$$

Now a solution to the constraint on a and b with the above values of m and n is

$$a = q+1,\ b = -q.$$

Thus we get

$$\left[\frac{(pq+1)x}{pq+p+1}\right] = \sum_{r=1}^{pq+1}\left[\frac{x+r(q+1)-1}{pq+p+1}\right] - \sum_{r=1}^{pq+1}\left[\frac{r(q+1)-1}{pq+p+1}\right].$$

To obtain the required formula, we shall break up the summation into the ranges $r = 1, \ldots, p$; $r = p+1, \ldots, 2p$; $r = p(q-1)+1, \ldots, pq$, and the last term $r = pq+1$. This may be written as a double summation over i and j by writing $r = pj + i + 1$ where j goes from 0 to $q-1$ and i from 0 to $p-1$, apart from the last term. Thus we have

$$\left[\frac{(pq+1)x}{pq+p+1}\right] = \sum_{j=0}^{q-1}\sum_{i=0}^{p-1}\left\{\left[\frac{x+(pj+i+1)(q+1)-1}{pq+p+1}\right] - \left[\frac{(pj+i+1)(q+1)-1}{pq+p+1}\right]\right\} + \left[\frac{x}{pq+p+1}\right]$$

as the last term $(r = pq+1)$ is just

$$\left[\frac{x+q(pq+p+1)}{pq+p+1}\right] - \left[\frac{q(pq+p+1)}{pq+p+1}\right].$$

Now we have

$$(pj+i+1)(q+1) - 1 = j(pq+p+1) + i(q+1) + q - j.$$

Cancelling the multiples of $pq+p+1$ in both bracket functions, we obtain

$$\left[\frac{(pq+1)x}{pq+p+1}\right] = \sum_{j=0}^{q-1}\sum_{i=0}^{p-1}\left\{\left[\frac{x+i(q+1)+q-j}{pq+p+1}\right] - \left[\frac{i(q+1)+q-j}{pq+p+1}\right]\right\} + \left[\frac{x}{pq+p+1}\right].$$

Inverting the order of summation of j, we can replace $q - j$ by $j + 1$.

$$\therefore\ \left[\frac{(pq+1)x}{pq+p+1}\right] = \sum_{j=0}^{q-1}\sum_{i=0}^{p-1}\left\{\left[\frac{x+i(q+1)+j+1}{pq+p+1}\right] - \left[\frac{i(q+1)+j+1}{pq+p+1}\right]\right\} - \left[\frac{x}{pq+p+1}\right].$$

Now the second bracket function on the righthand side is zero, as the maximum value of the numerator is $pq+p-1$. Changing the range of summation of j from 0 to $q-1$ to 1 to q and replacing j in the bracket function by $j-1$, we obtain

$$\left[\frac{(pq+1)x}{pq+p+1}\right] = \sum_{j=1}^{q}\sum_{i=0}^{p-1}\left[\frac{x+(q+1)i+j}{pq+p+1}\right] - \left[\frac{x}{pq+p+1}\right].$$

Adding and subtracting the term for $j = 0$,

$$\left[\frac{(pq+1)x}{pq+p+1}\right] = \sum_{j=0}^{q}\sum_{i=0}^{p-1}\left[\frac{x+(q+1)i+j}{pq+p+1}\right] - \sum_{i=0}^{p-1}\left[\frac{x+(q+1)i}{pq+p+1}\right] - \left[\frac{x}{pq+p+1}\right].$$

Now the $i = 0$ term in the second bracket on the righthand side cancels the last term. We can now again replace the double summation over i and j by a summation over t from 0 to $pq+p-1$. Adding and subtracting the term for $pq+p$, we obtain

$$\left[\frac{(pq+1)x}{pq+p+1}\right] = \sum_{t=0}^{pq+p}\left[\frac{x+t}{pq+p+1}\right] - \sum_{i=1}^{p-1}\left[\frac{x+(q+1)i}{pq+p+1}\right] - \left[\frac{x+pq+1}{pq+p+1}\right].$$

Using equation (1) for the first bracket function on the righthand side and transposing, we finally obtain

$$x = \left[\frac{(pq+1)x}{pq+p+1}\right] + \sum_{i=1}^{p-1}\left[\frac{x+(q+1)i}{pq+p+1}\right] + \left[\frac{x+(q+1)p}{pq+p+1}\right]$$

(12) $$\therefore \quad x = \left[\frac{(pq+1)x}{pq+p+1}\right] + \sum_{i=1}^{p}\left[\frac{x+(q+1)i}{pq+p+1}\right].$$

This is the general formula which we were searching for.

The special case $q = 0$ in equation (12) gives equation (1). The case $q = 1$ in equation (12) gives equation (2). Similarly, $q = 2$ gives us

(13) $$x = \left[\frac{(2p+1)x}{3p+1}\right] + \sum_{i=1}^{p}\left[\frac{x+3i}{3p+1}\right].$$

which is a new break-up of the type in equation (2). We can generate any number of such series. Separately, by choosing the special values of p we generate other break-ups. Thus, for $p = 1$

(14) $$x = \left[\frac{rx}{r+1}\right] + \left[\frac{x+r}{r+1}\right]$$

(where r is $q + 1$). We can in fact take $r \geq 0$. The next break-up in the series is, for $= 2$,

(15) $$x = \left[\frac{(2q+1)}{2q+3}\right] + \left[\frac{x+q+1}{2q+3}\right] + \left[\frac{x+2q+2}{2q+3}\right].$$

Again we can generate any number of such break-ups. It is obvious that equation (12) provides a considerable generalization of equations (1) and (2).

We are able to obtain an identity involving bracket functions by using euqation (11). It is clearly going to be equivalent to take $y = x$ and to take $y = 0$ and replace n by $n + 1$. Thus,

$$\sum_{r=1}^{n}\left[\frac{x+a(x+r)-1}{m}\right] - \sum_{r=1}^{n}\left[\frac{a(x+r)-1}{m}\right] + \left[\frac{x}{m}\right] = \sum_{r=1}^{n+1}\left[\frac{x+ar-1}{m}\right] - \sum_{r=1}^{n+1}\left[\frac{ar-1}{m}\right]$$

(16) $$\therefore \quad \left[\frac{x}{m}\right] = \sum_{r=1}^{n+1}\left[\frac{x+ar-1}{m}\right] - \sum_{r=1}^{n}\left[\frac{x+a(x+r)-1}{m}\right] + \sum_{r=1}^{n}\left[\frac{ax+ar-1}{m}\right] - \sum_{r=1}^{n+1}\left[\frac{ar-1}{m}\right].$$

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NOTE

We can also derive the result using Euler's ϕ-function, by using

$$\left[\frac{nx+y}{m}\right] = \sum_{r=1}^{n}\left[\frac{x+P_r}{m}\right] - \sum_{r=1}^{n}\left[\frac{P_r}{m}\right], \text{ where } P_r = \frac{(n-y-r)(m^{\phi(r)}-1)}{n}.$$

REFERENCE

1. H. N. Malik and A. Qadir. "Solution of Pseudo Periodic Difference Equations." This work, pp. 179-186, below.

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PSEUDO-PERIODIC DIFFERENCE EQUATIONS

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ABSTRACT

Periodic difference equations are generalized to pseudo-periodic (ψ_p) difference equations, and Minkowski's method extended to solve them. This is seen to lead to an identity involving Fibonacci and Lucas sequences.

1. INTRODUCTION

There are no general methods for the solution of all difference equations. However, periodic difference equations having the form

(1.1) $$P(E)f(x) = (a_1,\ \ldots,\ a_n)_n$$

where $P(E)$ is some polynomial of the shift operator, $f(x)$ is the unknown function in the discrete variable x and $a_1, \ldots, a_n$ are n constants, can be solved using Minkowski's operational calculus [1]. It would, clearly, be of interest to find a method for solving a more general class of equations.

In this paper we define a wider class of equations,

(1.2) $$P(E)f(x) = (a_1(x),\ \ldots,\ a_n(x))_n$$

which are not, strictly speaking, periodic, and call them *pseudo-periodic* (ψ_p) *equations*. We shall be extending Minkowski's method to solve these equations. This will be done using a discrete function, $h(x, m)$, defined by

(1.3) $$h(x, m) = \begin{cases} 1 & \text{when } m/x \\ 0 & \text{when } m\nmid x \end{cases}$$

which has the following properties:

(P1) $[h(x, m)]^j = h(x, m)\ \forall$ integers $j > 0$;

(P2) $\sum_{j=0}^{m-1} h(x + j, m) = 1$;

(P3) $h(x, m_1)h(x, m_2) = h(x, m)$ where $m = (m_1, m_2)$;

(P4) $h(x + mk, m) = h(x, m)\ \forall$ integers k;

(P5) $h(nx, m) = h(x, m)$ where $\langle n, m\rangle = 1$.

We shall then be able to evaluate the expression

(1.4) $$f(x) = \frac{1}{(E^{m_1 m_2} - a^{m_1 m_2})}\left[\frac{nx}{m_1}\right] h(x, m_2)$$

where $\left[\frac{p}{q}\right]$ is the usual bracket function. This will enable us to solve almost all ψ_p difference equations. Clearly we could always use it to solve periodic difference equations.

2. SOLUTION OF ψ_p DIFFERENCE EQUATIONS

We will start by finding the particular solution of the difference equation (Δ being the difference operator $E - 1$):

(2.1) $$\Delta f(x) = \left(\frac{x}{m}\right)^k h(x, m).$$

Using different values of m and k, any polynomial can be constructed by appropriate combinations of terms on the righthand side (apart from functions like $\sqrt{x}$, etc.). Thus, we can construct almost any linear, first-order, ψ_p difference euqation from equation (2.1). This clearly leads to more general ψ_p difference equations.

Consider the difference of the $(k + 1)$st Bernoulli polynomial,

(2.2) $$\Delta B_{k+1}\left(\left[\frac{x-1}{m}\right]+1\right) = \sum_{i=0}^{k}\binom{k+1}{1}(1+B)^{k-i+1}\left[\frac{x-1}{m}\right]^i.$$

$B_k \equiv B^k$ being the kth Bernoulli number, using (P2) we can write

$$\Delta B_{k+1}\left(\left[\frac{x-1}{m}\right]+1\right) = \sum_{i=1}^{k+1}(-1)^{k-i+1}\binom{k+1}{1}B_{k-i+1}\sum_{j=1}^{i}(-1)^{j+1}\binom{i}{j}\left(\frac{x}{m}\right)^{i-j}h(x, m).$$

Putting $j = i - r$, changing the order of summation of j and then putting $i = s + r$,

$$\Delta B_{k+1}\left(\left[\frac{x-1}{m}\right]+1\right) = \sum_{r=0}^{k}\sum_{s=1}^{k-r+1}(-1)^{k-r}\binom{k+1}{s+r}\binom{s+r}{r}B_{k-s-r+1}\left(\frac{x}{m}\right)^r h(x, m),$$

Now $$\binom{k+1}{s+r}\binom{s+r}{r} = \binom{k+1}{r}\binom{k-r+1}{s}$$

and $$\sum_{s=1}^{k-r+1}B_{k-s-r+1}\binom{k-s+1}{s} = (1+B)^{k-r+1} - B^{k-r+1}.$$

$$\therefore \quad \Delta B_{k+1}\left(\left[\frac{x-1}{m}\right]+1\right) = \sum_{r=0}^{k}(-1)^{k-r}\binom{k+1}{r}\left(\frac{x}{m}\right)^r[(1-B)^{k-r+1} - B^{k-r+1}]h(x, m)$$

$$= (k+1)\left(\frac{x}{m}\right)^k h(x, m).$$

Thus the solution of equation (2.1) is

(2.3) $$f(x) = \frac{1}{k+1}B_{k+1}\left(\left[\frac{x-1}{m}\right]+1\right) + c.$$

Now it can be seen that if

$$\Delta^{-1}f(x) = F(x) + c$$

$$\Delta^{-1}f\left(\left[\frac{x}{m}\right]\right) = \sum_{i=0}^{m-1}F\left(\left[\frac{x+i}{m}\right]\right) + c.$$

Thus, equation (2.3) gives us

$$\Delta^{-1}\left[\frac{x}{m}\right]^k = \frac{1}{k+1}\sum_{r=0}^{m-1}B_{k+1}\left[\frac{x+r}{m}\right]$$

(2.4) $$= \frac{m}{k+1}B_{k+1}\left(\left[\frac{x}{m}\right]\right) - \frac{1}{k+1}\left(x - m\left[\frac{x}{m}\right]\right)\left[\frac{x}{m}\right]^k.$$

To solve equation (1.4), we write $f(x)$ in the form

(2.5) $$f(x) = \sum_{i=0}^{k-j}\sum_{j=0}^{k}a_{ij}\left[\frac{nx}{m_1}\right]^{k-i}h(x+j, m),$$

where a_{ij} are coefficients chosen to fit the given $f(x)$. Operating on both sides of equation (2.5) with $(E^{m_1m_2} - a^{m_1m_2})$ and comparing with equation (1.3) we see that

$$a_{00} = \frac{1}{1-a^{m_1m_2}}$$

(2.6) $$a_{i0} = -a_{00}\sum_{s=0}^{i-1}a_{s0}\binom{k-s}{i-s}(m_2n)^{i-s}$$

$$a_{ij} = 0 \quad \text{for} \quad j \neq 0.$$

Denoting a_{i0} by a_i for all $i = 0, \ldots, k$, we get

$$(2.7) \qquad (E^{m_1 m_2} - a^{m_1 m_2}) \sum_{i=0}^{k} \left[\frac{nx}{m_1}\right]^{k-i} a_i h(x, m_2) = \left[\frac{nx}{m_1}\right]^{k} h(x, m_2).$$

If we assume that for some j,

$$(2.8) \qquad a_j = \binom{k}{j}(m_2 n)^j \sum_{r=0}^{j} (a_{00})^{r+1} r! S_n^j,$$

S_r^j being Sterling's numbers of the second kind [2]. Then equation (2.6) gives

$$\frac{a_{j+1}}{a_{00}} = -\sum_{q=0}^{j} \binom{k-q}{j+1-q}(m_2 n)^{j+1-q}\binom{k}{q}(m_2 n)^q \sum_{r=0}^{q}(a_{00})^{q+1} S_r^q,$$

since

$$\binom{k-q}{j+1-q}\binom{k}{q} = \binom{k}{j+1}\binom{j+1}{q}$$

and

$$\sum_{q=0}^{j} S_p^q \binom{j+1}{q} = (p+1) S_{p+1}^{j+1}$$

and $S_0^{j+1} = 0$ and $S_p^q = 0$ for $q < p$, we obtain

$$a_{j+1} = \binom{k}{j+1}(m_2 n)^{j+1} \sum_{r=0}^{j+1} (a_{00})^{r+1} r! S_r^{j+1}.$$

As equation (2.8) is true for $j = 0$, it must hold for all values of j. Thus, putting equation (2.8) into equation (2.7), we obtain

$$(2.9) \qquad f(x) = \sum_{i=0}^{k} \sum_{r=0}^{i} \left[\frac{nx}{m_1}\right]^{k-i} \binom{k}{i} \frac{(m_2 n)^i r! S_r^i h(x, m)}{(1 - a^{m_1 m_2})^{r+1}}.$$

Thus, we have a general method for solving the required ψ_p difference equations.

3. AN INTERESTING IDENTITY

We find that by considering a special case of equation (2.7) we are able to obtain an identity between combinations of two well-known sequences—the Fibonacci and Lucas sequences. This is given as an example of the technique given above and its applications. We consider the ψ_p difference equation

$$(3.1) \qquad (E - a)(E - b) f(x) = x^k h(x, m).$$

Writing the particular solution of equation (3.1) as $P_k(x)$, we write

$$P_k(x) = (E - a)^{-1}(E - b)^{-1} x^k h(x, m)$$

$$(3.2) \qquad = \left\{\sum_{i=1}^{m} \sum_{j=1}^{m} E^{m-i} a^{i-1} E^{m-j} b^{j-1}\right\}\left\{\sum_{p=0}^{k} \sum_{q=0}^{k} a(k, q) b(k-q, p-q) x^k\right\},$$

where

$$(3.3) \qquad \begin{aligned} a(k, q) &= m^q \binom{k}{q} \sum_{r=0}^{q} \frac{q! S_r^q}{(1 - a^m)^{r+1}} \\ b(k, q) &= m^q \binom{k}{q} \sum_{r=0}^{q} \frac{q! S_r^q}{(1 - b^m)^{r+1}} \end{aligned}$$

Taking $m = 1$ in equation (3.3) and using the requirement that there is symmetry under interchange of a and b, we obtain

$$a(k,\ q)b(k-q,\ p-q) = \frac{1}{2}\{a(k,\ q)b(k-q,\ p-q) + a(k-q,\ p-q)b(k,\ q)\}$$

(3.4)
$$= \frac{1}{2}\binom{k}{q}\binom{p}{q}\left\{\sum_{i=0}^{M}\sum_{j=1}^{M-i} j!(j+1)!(-1)^{i+j+1}\ell_i(S_i^q S_{i+j}^{p-q} + S_{i+j}^q S_i^{p-q})_2\right\}$$

where M is the greater of q and $(p - q)$ and ℓ_i is Lucas' sequence, given by

$$\ell_n = \ell_{n-1} + \ell_{n-2} \quad \text{and} \quad \ell_0 = 2,\ \ell_1 = 1.$$

Also, if a and b are roots of the equation $y^2 - y - 1 = 0$, the lefthand side of equation (3.3) should be [3]

$$-\sum_{i=1}^{q} i!F_{i+1}S(q,\ i)$$

from equation (3.2), F_i being the Fibonacci sequence. Then

(3.5)
$$\sum_{i=0}^{q} i!F_{i+1}S(q,\ i) = -\frac{1}{2}\binom{p}{q}\sum_{i=0}^{M}\sum_{j=0}^{M-i}(-1)^{i+j+1}j!(j+1)!\ell_i(S_j^q S_{i+j}^{p-q} + S_{i+j}^q S_j^{p-q}),$$

which is the required identity.

CONCLUSION

We have defined ψ_p difference equations as generalizations of the periodic difference equations. This is a much wider class of difference equations than the periodic ones, but does not contain all difference equations. We extended Minkowski's operational calculus to deal with a large class (but not all) ψ_p difference equations. This is of interest in itself as a means of solving more difference equations than Minkowski's calculus enabled us to. It is also of interest inasmuch as it provides an independent means of solving periodic difference equations and thereby discovering new identities between combinations of various sequences. Thus, it can also be regarded as being of interest in number theory.

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REFERENCES

1. L. Brand. *Differential and Difference Equations*. New York: John Wiley & Sons, 1966.
2. J. Riordan. *Combinatorial Identities*. New York: John Wiley & Sons, 1968.
3. R. J. Weinshank and V. E. Hoggatt, Jr. "On Solving $C_{n+2} = C_{n+1} + C_n + n^m$ by Expansions and Operators." *The Fibonacci Quarterly* 8 (1970):39-48.

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SOLUTION OF PSEUDO-PERIODIC DIFFERENCE EQUATIONS

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ABSTRACT

A method for getting the particular solution of pseudo-periodic difference equations, by using a discrete periodic function has been given. Some identities, equalities, and inequalities have been derived by using the above-mentioned discrete function.

1. INTRODUCTION

Periodic difference equations have been previously solved [1] by the use of Minkowski's operational calculus. The type of equations solved by this method are

(1.1)
$$P(E)f(x) = (a_1,\ a_2,\ \ldots,\ a_n)_n,$$

where $P(E)$ is a polynomial function of E with constant coefficients, n is the period, and the a_i's are constants. An obvious extension of this would be to make the a 's functions of the variable x. Since the resulting equations would no longer be periodic, we call them pseudo-periodic.

To solve pseudo-periodic cifference equations, we define a discrete function $h(x, m)$,

$$(1.2) \qquad \begin{cases} h(x, m) = 1 & \text{when } m/x \\ \qquad\quad = 0 & \text{when } m\nmid x \end{cases}$$

It can easily be seen that $h(x, m)$ satisfies the following properties:

(1) $\{h(x, m)\}^j = h(x, m)$ for all integers $j > 0$;

(2) $\sum_{j=0}^{m-1} h(x + j, m) = 1$;

(3) $h(x, m_1)h(x, m_2) = h(x, m)$, m being the L.C.M. of m_1 and m_2;

(4) $h(x + mk, m) = h(x, m)$ for all integers $k \geq 0$;

(5) $h(nx, m) = h(x, m)$, n and m being relatively prime.

We shall use these properties to evaluate the expression

$$f(x) = \frac{1}{(E^{m_1 m_2} - a^{m_1 m_2})}\left[\frac{nx}{m_1}\right]^k h(x, m_2), \; a \neq 1,$$

where $\left[\frac{a}{b}\right]$ is the bracket function, being the largest integer less than or equal to $\frac{a}{b}$. This can be used to solve any pseudo-periodic difference equation, in principle. All pseudo-periodic difference equations being periodic difference equations we can, of course, solve periodic difference equations, as well, by this method.

The plan of work is as follows. In Section 2 we solve equation (1.3). As an example of this method, we have solved a previously solved difference equation

$$(E - a)(E - b)f(x) = x^k,$$

where a and b are the roots of the equation $y^2 - y - 1 = 0$. This yields an identity involving Fibonacci numbers and Sterling's numbers of the first and second kind.

In Section 3 we give some equalities and inequalities involving bracket functions that can be derived by using the discrete function $h(x, m)$.

2. SOLUTION OF PSEUDO-PERIODIC DIFFERENCE EQUATIONS

We shall first find the particular solution of the difference equation

$$(1.2) \qquad \Delta f(x) = \left(\frac{x}{m}\right)^k h(x, m).$$

Any polynomial with different periods can be constructed from terms of the type of the righthand side of (2.1) with different values of k and m. We can thus construct artibtary functions and solve an linear, first-order pseudo-periodic difference equation.

Consider the action of the difference operator on the kth Bernoulli polynomial [1, 2] with the argument $\left[\frac{x - 1}{m}\right] + 1$,

$$\Delta B_{k+1}\left(\left[\frac{x - 1}{m}\right] + 1\right) = \sum_{i=0}^{k+1}\binom{k+1}{i}(1 + B)^{k-i+1}\left[\frac{x - 1}{m}\right]^i,$$

where $B^r \equiv B_r$ is the rth Bernoulli number. Using property (2) of $h(x, m)$ and equation (3.2) given in the next section, we get

$$\Delta B_{k+1}\left(\left[\frac{x - 1}{m}\right] + 1\right) = \sum_{i=1}^{k+1}(-1)^{k-i+1}\binom{k+1}{i}B_{k-i+1}\sum_{j=1}^{i}(-1)^{j+1}\binom{i}{j}\left(\frac{x}{m}\right)^{i-j}h(x, m),$$

putting $j = i - r$, changing the order of summation, and then putting $i = s + r$,

$$\Delta B_{k+1}\left(\left[\frac{x - 1}{m}\right] + 1\right) = \sum_{r=0}^{k}\sum_{s=1}^{k-r+1}(-1)^{k-r}\binom{k+1}{s+r}\binom{s+r}{r}B_{k-s-r+1}\left(\frac{x}{m}\right)^r h(x, m).$$

Now
$$\binom{k+1}{s+r}\binom{s+r}{r} = \binom{k+1}{r}\binom{k-r+1}{s}$$

and
$$\sum_{s=1}^{k-r+1} B_{k-s-r+1}\binom{k-s+1}{s} = (1+B)^{k-r+1} - B^{k-r+1},$$

$$\therefore \quad \Delta B_{k+1}\left(\left[\frac{x-1}{m}\right]+1\right) = \sum_{r=0}^{k} (-1)^{k-r}\binom{k+1}{r}\left(\frac{x}{m}\right)^{r}\left\{(1+B)^{k-r+1} - B^{k-r+1}\right\}h(x,\ m)$$

$$= (k+1)\left(\frac{x}{m}\right)^{k} h(x,\ m)$$

$$f(x) = \frac{1}{k+1} B_{k+1}\left(\left[\frac{x-1}{m}\right]+1\right) + c. \tag{2.2}$$

It can be seen that if
$$\Delta^{-1}f(x) = F(x) + c$$
$$\Delta^{-1}f\left(\left[\frac{x}{m}\right]\right) = \sum_{i=0}^{m-1} F\left(\left[\frac{x+i}{m}\right]\right) + c$$

then (2.2) gives
$$\Delta^{-1}\left[\frac{x}{m}\right]^{k} = \frac{1}{k+1}\sum_{r=0}^{m-1} B_{k+1}\left(\left[\frac{x+r}{m}\right]\right)$$
$$= \frac{m}{k+1} B_{k+1}\left(\left[\frac{x}{m}\right]\right) - \frac{1}{k+1}\left(x - m\left[\frac{x}{m}\right]\right)\left[\frac{x}{m}\right]^{k} \tag{2.3}$$

Now we shall consider the difference equation (1.3). To solve this, put $f(x)$ in the form
$$f(x) = \sum_{i=0}^{k}\sum_{j=0}^{k} a_{ij}\left(\left[\frac{nx}{m_1}\right]\right)^{k-i} h(x+j,\ m_2). \tag{2.4}$$

By operating on both sides of (2.4) with $(E^{m_1 m_2} - a^{m_1 m_2})$ and comparing with (1.3), we see that
$$\begin{cases} a_{00} = \dfrac{1}{1 - a^{m_1 m_2}} \\[2ex] a_{i0} = -a_{00}\displaystyle\sum_{s=0}^{i-1} a_{s0}\binom{k-s}{i-s}(m_2 n)^{i-s} \\[2ex] a_{ij} = 0 \quad \text{for } j \neq 0 \end{cases} \tag{2.5}$$

Denoting a_{i0} by a_i, we get
$$(E^{m_1 m_2} - a^{m_1 m_2})\sum_{i=0}^{k} a_i\left[\frac{nx}{m_1}\right]^{k-i} h(x,\ m_2) = \left[\frac{nx}{m_1}\right]^{k} h(x,\ m_2). \tag{2.6}$$

Assume that for some $i = j$
$$a_{ij} = a_j = \frac{(m_2 n)^j k^{(j)}}{j!}\sum_{r=0}^{j} r!\,(a_{00})^{r+1} S_r^j\ . \tag{2.7}$$

Then (2.5) gives
$$\frac{a_{j+1}}{a_{00}} = -\sum_{q=0}^{j}\binom{k-q}{j-q+1}(m_2 n)^{j+1}\binom{k}{q}\sum_{r=0}^{q} r!\,(a_{00})^{r+1} S_r^q$$

where S_r^q are Sterling's numbers of the second kind [2]. Substituting from (2.7)
$$\frac{a_{j+1}}{a_{00}} = -\sum_{q=0}^{j}\binom{k-q}{j-q+1}(m_2 n)^{j+1}\binom{k}{q}\sum_{r=0}^{q} r!\,(a_{00})^{r+1} S_r^q = -(m_2 n)^{j+1}\binom{k}{j+1}\sum_{p=0}^{j}\sum_{q=0}^{j}\binom{j+1}{q} p!\,(a_{00})^{p+1} S_p^q$$
$$= -\binom{k}{j+1}(m_2 n)^{j+1}\sum_{p=0}^{j} p!\,(p+1)(a_{00})^{p+1} S_{p+1}^{j+1}$$

$$\because \quad \sum_{q=0}^{j} \binom{j+1}{q} S_p^q = (j+1) S_{p+1}^{j+1}$$

$$\frac{a_{j+1}}{a_{00}} = \binom{k}{j+1} (m_2 n)^{j+1} \sum_{p=0}^{j} (p+1)!(a_{00})^{p+1} S_{p+1}^{j+1}.$$

Now, $j + 1$ being positive, $S_0^{j+1} = 0$. Hence

$$a_{j+1} = \frac{(m_2 n)^{j+1} K^{(j+1)}}{(j+1)!} \sum_{t=0}^{j+1} (t+1)!(a_{00})^{t+1} S_t^{j+1}.$$

Thus, if (2.7) is true for j, it is also true for $(j + 1)$. It is easily seen that it is true for $j = 1$. Hence, it is true for all j. Putting (2.7) into (2.6), we obtain

$$f(x) = \sum_{i=0}^{k} \sum_{r=0}^{i} (m_2 n) \binom{k}{i} \left[\frac{nx}{m_1}\right]^{k-i} \frac{(r)! S_r^i}{(1 - a^{m_1 m_2})^{r+1}} h(x, m). \tag{2.8}$$

As an example of the method given above, consider the pseudo-periodic difference equation

$$(E - a)(E - b) f(x) = x^k h(x, m), \tag{2.9}$$

writing the particular solution of (2.9) as $P_k(x)$

$$\begin{aligned} P_k(x) &= (E - a)^{-1}(E - b)^{-1} x^k h(x, m) \\ &= \left\{ \sum_{i=1}^{m} \sum_{j=1}^{m} E^{m-i} a^{i-1} E^{m-j} b^{j-1} \right\} \left\{ \sum_{p=0}^{k} \sum_{q=0}^{k} a(k, q) b(k - q, p - q) x^{k-q} \right\}, \end{aligned} \tag{2.10}$$

where

$$\begin{cases} a(k, q) = m^q \binom{k}{q} \displaystyle\sum_{r=0}^{q} \frac{(q)! S_r^q}{(1 - a^m)^{r+1}} \\ b(k, q) = m^q \binom{k}{q} \displaystyle\sum_{r=0}^{q} \frac{(q)! S_r^q}{(1 - b^m)^{r+1}} \end{cases} \tag{2.11}$$

Notice that (2.10) must hold with a and b interchanged, since (2.9) is symmetric in a and b. Thus, we shall interchange and take the sum. For $m = 1$, we get

$$\begin{aligned} \therefore \quad a(k, q) b(k - q, p - q) &= \frac{1}{2} \{ a(k, q) b(k - q, p - q) + b(k, q) a(k - q, p - q) \} \\ &= \frac{1}{2} \binom{k}{p} \binom{p}{q} \left\{ \sum_{i=0}^{M} \sum_{j=0}^{M-i} (j)!(j + i)!(-1)^{j+i+1} \ell_i (S_j S_{i+j}^{p-q} + S_i S_j^{p-q} \right\}. \end{aligned} \tag{2.12}$$

Where M is the greater of P and $P - q$ and $\{\ell_i\}$ is Lucas' sequence [2]

$$\ell_n = \ell_{n-1} + \ell_{n-2}, \quad \ell_0 = 2, \ \ell_1 = 1.$$

Also, if a and b are the roots of the equation $y^2 - y - 1 = 0$, the L.H.S. of (2.12) should be [3]

$$-\sum_{i=1}^{q} (i)! F_{i+1} S(q, i)$$

from (2.10). Thus,

$$\sum_{i=0}^{q} (i)! F_{i+1} S(q, i) = -\frac{1}{2} \binom{p}{q} \sum_{i=0}^{M} \sum_{j=0}^{M-i} (-1)^{i+j+1} (j)!(j + i)! \ell_i (S_j^q S_{i+j}^{p-q} + S_{i+j}^q S_j^{p-q}). \tag{2.13}$$

3. SOME RESULTS OBTAINED BY USING $h(x, m)$

It is easy to see that:

$$\left[\frac{x}{m}\right] = \frac{1}{m} \left\{ x - m + 1 + \sum_{j=0}^{m-1} j h(x + j + 1, m) \right\}; \tag{3.1}$$

$$(3.2)\qquad \left[\frac{x-1}{m}\right]^k = \sum_{j=1}^{m-1}(-1)^{j+1}\binom{k}{j}\left(\frac{x}{m}\right)^{k-j}h(x, m).$$

Putting $k = 1$ in (3.2) and then using (3.1), we get

$$(3.3)\qquad \Delta^{-1}h(x, m) = \frac{1}{m}\left\{x - m + \sum_{j=0}^{m-1} jh(x + j, m)\right\} + c.$$

The bracket function inequality

$$\left[\frac{x+y}{mn}\right] \geq \left[\frac{\left[\frac{x}{m}\right] + \left[\frac{y}{m}\right]}{n}\right]$$

on using (3.1), gives the inequality

$$(3.4)\qquad \begin{cases} \displaystyle\sum_{r=0}^{mn-1} rh(x + y + 1, mn) \geq m\left\{\sum_{r=0}^{n} rh\left(\left[\frac{x}{m}\right] + \left[\frac{y}{m}\right] + r + 1, m\right) - 1\right\} \\ \qquad\qquad \displaystyle + \sum_{r=0}^{n-1}\left\{h(x + r + 1, m) + h(y + r + 1, m)\right\} + 1. \end{cases}$$

Similarly, the bracket function equality

$$\left[\frac{x}{mn}\right] = \left[\frac{\left[\frac{x}{m}\right]}{n}\right] = \left[\frac{\left[\frac{x}{n}\right]}{m}\right]$$

on using (3.1), gives the equality

$$(3.5)\qquad \begin{cases} \displaystyle x + 1 + \sum_{j=0}^{mn} jh(x + j + 1, mn) = n\left\{\left[\frac{x}{n}\right] + 1 + \sum_{j=0}^{m-1} jh\left(\left[\frac{x}{n}\right] + j + 1, m\right)\right\} \\ \qquad\qquad \displaystyle = m\left\{\left[\frac{x}{m}\right] + 1 + \sum_{j=0}^{n-1} jh\left(\left[\frac{x}{m}\right] + j + 1, n\right)\right\}. \end{cases}$$

Now consider

$$\sum_{r=1}^{n-k} h(nx + y + r, m) = \left|\Delta^{-1}h(nx + y + r, m)\right|_{r=1}^{nk+1} = \left|\left[\frac{nx + y + r - 1}{m}\right]\right|_{r=1}^{nk+1}$$

$$= \left[\frac{n(x + k) + y}{m}\right] - \left[\frac{nx + y}{m}\right] = (E^k - 1)\left[\frac{nx + y}{m}\right]$$

$$(3.6)\qquad \therefore\quad \left[\frac{nx + y}{m}\right] = \frac{1}{E^k - 1}\sum_{r=1}^{n-k} h(mx + y + r, m) + c.$$

Putting $k = 1$, we get

$$\left[\frac{nx + y}{m}\right] = \Delta^{-1}\sum_{r=1}^{n} h(nx + y + r, m) + c.$$

If n and m are relatively prime, there will exist two integers, a and b, such that $an + bm = 1$,

$$\therefore\quad \left[\frac{nx + y}{m}\right] = \Delta^{-1}\sum_{r=1}^{n} h(nx + (y + r)(an + bm), m) + c.$$

Using property (4) of $h(x, m)$,

$$\left[\frac{nx + y}{m}\right] = \Delta^{-1}\sum_{r=1}^{n} h(x + a(y + r), m) + c = \sum_{r=1}^{n}\left[\frac{x + (y + r)a - 1}{m}\right] + c.$$

Determine c by putting $x = 0$ in the above equation:

$$C = \left[\frac{y}{m}\right] - \sum_{r=1}^{n}\left[\frac{a(y + r) - 1}{m}\right]$$

(3.7) $$\therefore \quad \left[\frac{nx + y}{m}\right] = \sum_{r=1}^{n}\left[\frac{x + (y + r)a - 1}{m}\right] - \sum_{r=1}^{n}\left[\frac{(y + r)a - 1}{m}\right] + \left[\frac{y}{m}\right].$$

Putting $y = 0$, $m = pq + p + 1$ and $n = pq + 1$ in (3.7) we obtain the equation

$$\left[\frac{(pq + 1)x}{pq + p + 1}\right] = \sum_{r=1}^{pr+1}\left[\frac{x + r(q + 1) - 1}{pq + p + 1}\right] - \sum_{r=1}^{pq+1}\left[\frac{r(q + 1) - 1}{pq + p + 1}\right],$$

and it is easily checked that $pq + 1$ and $pq + p + 1$ are relatively prime. Breaking the summation into the q summations with ranges $r = 1, p$; $r = p + 1, \ldots, 2p, \ldots$; $r = p(q - 1) + 1, \ldots, pq$; and the term for $r = pq + 1$, we can write the expression as a double sum, and obtain the equation

$$\left[\frac{(pq + 1)x}{pq + p + 1}\right] = \sum_{j=0}^{q-1}\sum_{i=0}^{p-1}\left\{\left[\frac{x + (pj + i + 1)(q + 1) - 1}{pq + p + 1}\right] - \left[\frac{(pj + i + 1)(q + 1) - 1}{pq + p + 1}\right]\right\} + \left[\frac{x}{pq + p + 1}\right].$$

Taking multiples of $(pq + p + 1)$ out of the numerators of the two bracket functions, we obtain

$$\left[\frac{(pq + 1)x}{pq + p + 1}\right] = \sum_{j=0}^{q-1}\sum_{i=0}^{p-1}\left\{\left[\frac{x + (q + 1)i + q - j}{pq + p + 1}\right] - \left[\frac{(q + 1)i + q - j}{pq + p + 1}\right]\right\} + \left[\frac{x}{pq + p + 1}\right].$$

Reversing the order of summation of j we notice that $q - j$ is replaced by $j + 1$. Now the maximum value of $(q + 1)i + j + 1$ is $pq + p - 1$, which is less than $pq + p + 1$. Thus, the second bracket expression is always zero. Changing the range of summation of j from 0 to $q - 1$ to 1 to q, we obtain

$$\left[\frac{(pq + 1)x}{pq + p + 1}\right] = \sum_{j=1}^{q}\sum_{i=0}^{p-1}\left[\frac{x + (q + 1)i + j}{pq + p + 1}\right] + \left[\frac{x}{pq + p + 1}\right].$$

Now adding and subtracting the expression for $j = 0$, we get

$$\left[\frac{(pq + 1)x}{pq + p + 1}\right] = \sum_{j=0}^{q}\sum_{i=0}^{p-1}\left[\frac{x + (q + 1)i + j}{pq + p + 1}\right] - \sum_{i=0}^{p-1}\left[\frac{x + (q + 1)i}{pq + p + 1}\right] + \left[\frac{x}{pq + p + 1}\right].$$

Notice that the last bracket expression cancels the bracket expression in the second term on the L.H.S. for $i = 0$. Also, we can replace the summation over i and j by a summation over t, where the range of t is 0 to $pq + p - 1$. We can thus write

(3.8) $$\left[\frac{(pq + 1)x}{pq + p + 1}\right] = \sum_{t=0}^{pq+p}\left[\frac{x + t}{pq + p + 1}\right] - \sum_{i=1}^{p-1}\left[\frac{x + (q + 1)i}{pq + p + 1}\right] - \left[\frac{x + pq + p}{pq + p + 1}\right].$$

To reduce this further, consider the inverse difference operator acting on the equation for property (2) of $h(x, m)$ and the fact that

$$\Delta^{-1}h(x + j, m) = \left[\frac{x + j - 1}{m}\right] + c$$

we obtain the equation

$$x = \sum_{j=0}^{m-1}\left[\frac{x + j - 1}{m}\right] + c.$$

Evaluating c by putting $x = 0$ and absorbing into the summation, we obtain the result that

(3.9) $$x = \sum_{j=0}^{m-1}\left[\frac{x + j}{m}\right].$$

Putting (3.9) into (3.8), we obtain the identity

(3.10) $$\left[\frac{(pq + 1)x}{pq + p + 1}\right] + \sum_{i=1}^{p}\left[\frac{x + (q + 1)i}{pq + p + 1}\right] = x.$$

Similarly, we get the equation

(3.11) $$\left[\frac{nx + y}{m}\right] = \sum_{i=1}^{p}\left[\frac{x + P_r}{m}\right] - \sum_{r=1}^{n}\left[\frac{P_r}{m}\right],$$

where

$$P_r = \frac{(n - y - r)(m^{\phi(n)} - i)}{n},$$

$\phi(n)$ being the number of natural numbers less than or equal to n which are relatively prime with respect to n (i.e., Euler's ϕ-function). Then putting $y = 0$ and replacing x by n^x, we get

$$\Delta\left[\frac{n^x}{m}\right] = \sum_{r=1}^{n-1}\left[\frac{n^x + P_r}{m}\right] - \sum_{r=1}^{n-1}\left[\frac{P_r}{m}\right]$$

(3.12) $$\therefore \quad \Delta^{-1}\sum_{r=1}^{n-1}\left[\frac{n^x + P_r}{m}\right] = \left[\frac{n^x}{m}\right] + x\sum_{r=1}^{n-1}\left[\frac{P_r}{m}\right] + c.$$

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REFERENCES

1. L. Brand. *Differential and Difference Equations*. New York: John Wiley & Sons, 1966.
2. J. Riordan. *Combinatorial Identities*. New York: John Wiley & Sons, 1968.
3. R. J. Weinshenk and V. E. Hoggatt, Jr. "On Solving $C_{n+2} = C_{n+1} + C_n + n^m$ by Expansions and Operators." *The Fibonacci Quarterly* 8 (1970):39-48.

APPENDIX

Let

$$P_k(x) = \frac{1}{E^2 - E - 1}x^k,$$

then

$$P_k(x) = -\sum_{i=0}^{k}(F_{i+1}\Delta^i)x^k = -\sum_{i=0}^{k}F_{i+1}\Delta^i\sum_{j=0}^{k}S_{(k,j)}\,x^{(j)},$$

where $x^{(j)} = x(x-1)(x-2)\ \dots\ (x-j+1)$, the falling factorial, where S_j^k are Sterling's numbers of the second kind, and F_i are Fibonacci numbers, $F_0 = 1 = F_1$, $F_n = F_{n-1} + F_{n-2}$.

$$\therefore \quad P_k(x) = -\sum_{i=0}^{k}\sum_{j=i}^{k}(j)^{(i)}F_{i+1}S_j^k x^{(j-i)}$$

$$= -\sum_{i=0}^{k}\sum_{j=0}^{k-i}(k-j)^{(i)}F_{i+1}S_{(k-j)}^k\,x^{(k-i-j)}.$$

Put $i + j = \ell$. Then $\min(\ell) = 0$ and $\max(\ell) = k$.

$$\therefore \quad P_k(x) = -\sum_{i=0}^{k}\sum_{\ell=0}^{k}(k-\ell+i)^{(i)}F_{i+1}S_{(k-\ell+i)}^k x^{(k-\ell)}$$

$$= -\sum_{i=0}^{k}\sum_{\ell=0}^{k}\sum_{j=0}^{k}(k-\ell+i)^{(i)}F_{i+1}S_{(k-\ell,\,j)}S_{(k-\ell+i)}^k x^j,$$

where $S_{(k-1,\,j)}$ are Sterling's numbers of the first kind.

Put $j + 1 = m$. Then $\min(m) = 0$ and $\max(m) = k$,

$$\therefore \quad P_k(x) = -\sum_{i=0}^{k}\sum_{\ell=0}^{k}\sum_{m=0}^{k}(k-\ell+i)^{(i)}F_{i+1}S_{(k-\ell+i)}^k S_{(k-1,\,k-m)}x^{k-m}.$$

Now consider the coefficients of x^{k-m}. By reversing the order of summation of 1, we can replace $k - \ell$ by ℓ. Also note that

$$(\ell + i)^{(i)} = \binom{\ell + i}{i},$$

and also that

$$\sum_{\ell=k}^{k-i}S_{(i+\ell)}^k S_{(\ell,\,k-m)}\binom{\ell+i}{i} = \binom{k}{k-m}S_{(m,\,i)}.$$

Since the expression is zero for $\ell < k - s$ and for $1 > k - i$,

$$\therefore \quad P_k(x) = -\sum_{i=0}^{k}\sum_{m=0}^{k} i!\binom{k}{k-m}S_{(m,i)}F_{i+1}x^{k-m}$$

$$= \sum_{i=0}^{k}\sum_{m=0}^{k}\binom{k}{m}S_{(m,i)}\, i!F_{i+1}x^{k-m}.$$

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A CLASS OF DIOPHANTINE EQUATIONS*

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ABSTRACT

In this paper, we prove a theorem: If $k \equiv 5 \pmod 8$ and $f \not\equiv 2^{3t-1} \pmod{2^{3t}}$ for all positive integers t, then $c(3a^2b + kb^3) + d(a^3 + 3kab^2) = 16f$ has no solutions in integers $ab \neq 0$ if c and d are both odd integers. Then it is shown how this theorem enables us to solve the diophantine equations $y^2 - k = x^3$, $k \equiv 5 \pmod 8$. In the end, we give solutions for $k = 109, 116, 125, 133, 149, 157, 165, 173, 180$, and 181.

The Mordell equation $y^2 - k = x^3$, the simplest of all nontrivial diophantine equations of degree greater than 2, has interested mathematicians for more than three centuries, and has played an important role in the development of Number Theory.

We already know the complete solutions for $y^2 - k = x^3$, $|k| \leq 100$. The author in his doctroal dissertation (UCLA, 1971) has treated the range $100 < k \leq 200$. The present paper treats 10 particular cases in the above range.

First we prove two lemmas to prove the theorem.

Theorem 1: If $k \equiv 5 \pmod 8$, $f \not\equiv 2^{3t-1} \pmod{2^{3t}}$ for all positive integers t, then $c(3a^2b + kb^3) + d(a^3 + 3kab^2) = 16f$ has no solution in integers $ab \neq 0$ if c and d are both odd integers.

Lemma 1: Let $k \equiv 5 \pmod 8$ and c and d be odd integers. Then $c(3a^2b + kb^3) + d(a^3 + 3kab^2) = 0$ has only solution $a = 0$ and $b = 0$ in integers.

Proof: Suppose $a \neq 0$, $b \neq 0$ is a solution of

(1) $$c(3a^2b + kb^3) + d(a^3 + 3kab^2) = 0$$

in integers. ($a = 0$ implies $b = 0$, and conversely.) We see from (1) that $a \neq b$ and $a \equiv b \pmod 2$. Then $3a^2b + kb^3 = b(3a^2 + kb^2) \equiv 0 \pmod 8$ and $a^3 + 3kab^2 = a(a^2 + 3kb^2) \equiv 0 \pmod 8$, since $k \equiv 5 \pmod 8$.

Hence, $c(3a^2b + kb^3) + d(a^3 + 3kab^2) \equiv (3a^2b + kb^3) + (a^3 + 3kab^2) \pmod{16}$ as both c and d are odd integers. Then, from (1), we deduce that

(2) $$(3a^2b + kb^3) + (a^3 + 3kab^2) \equiv 0 \pmod{16}.$$

But

(3) $$a^3 + 3a^2b + kb^3 + 3kab^2 = (a + b)^3 + (k - 1)b^2(a + b) + 2(k - 1)ab^2.$$

Inserting $a + b = 2r$ and $k = 8\ell + 5$ in (3), we obtain

$$\begin{aligned} a^3 + 3a^2b + kb^3 + 3kab^2 &\equiv 8r(r^2 + b^2) + 8ab^2 \pmod{16} \\ &\equiv 8 \pmod{16} \text{ when both } a \text{ and } b \text{ are odd;} \\ &\equiv 0 \text{ or } 8 \pmod{16} \text{ when both } a \text{ and } b \text{ are even.} \end{aligned}$$

Then (2) implies that a and b are both even. Since $a \neq b$, suppose $a = 2^p m$ and $b = 2^q n$ where m and n are odd integers.

Now (a,b) is a solution of (1) implies that (a_1,b_1) is a solution of

$$c(3a_1^2b_1 + kb_1^3) + d(a_1^3 + 3ka_1b_1^2) = 0,$$

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where $a_1 = 2^{p-1}m$ and $b_1 = 2^{q-1}n$. Arguing as before, we have both a_1 and b_1 even. If

(i) $p < q$, then $a_p = m$, $b_p = 2^{q-p}n$;

(ii) $p = q$, then $a_p = m$, $b_p = n$;

(iii) $p > q$, then $a_p = 2^{p-q}m$, $b_p = n$.

In all these cases we see that a_p and b_p are not both even. So we have a contradiction.

Lemma 2: Suppose $k \equiv 5 \pmod 8$ and c and d are odd integers. Then the necessary condition for the equation $c(3a^2b + kb^3) + d(a^3 + 3kab^2) = 16f$ to be solvable in integers is

$$f \equiv 2^{3t-1} \pmod{2^{3t}} \text{ or } f \equiv 0 \pmod{2^{3t}}.$$

Proof: In the proof of Lemma 1 we have shown that

$$c(3a^2b + kb^3) + d(a^3 + 3kab^2) \equiv 8 \pmod{16} \text{ when } a \text{ and } b \text{ are odd,}$$

$$\equiv 0 \text{ or } 8 \pmod{16} \text{ when } a \text{ and } b \text{ are even.}$$

From

(4) $$c(3a^2b + kb^3) + d(a^3 + 3kab^2) = 16f$$

we see that a and b are even. Suppose $a = 2a_1$ and $b = 2b_1$. Then we have

(5) $$c(3a_1^2b_1 + kb_1^3) + d(a_1^3 + 3ka_1b_1^2) = 2f.$$

The necessary condition for (5) to be solvable in integers is $f \equiv 0$ or $4 \pmod 8$, for in (5) $a_1 \equiv b_1 \pmod 2$, i.e., $f = 8f_1$ or $4 + 8f_1$. Hence, the lemma is true for $t = 1$. If $f \not\equiv 4 \pmod 8$ then $f = 8f_1$ and

(5') $$c(3a_1^2b_1 + kb_1^3) + d(a_1^3 + 3ka_1b_1^2) = 16f_1.$$

Arguing as before, $f_1 \equiv 0$ or $4 \pmod 8$, whence $f \equiv 0$ or $32 \pmod{64}$ and the proof follows by induction.

Proof of Theorem 1: By Lemma 2, the necessary condition for equation (4) to be solvable in integers is either $f \equiv 2^{3t-1} \pmod{2^{3t}}$ or $f \equiv 0 \pmod{2^{3t}}$. By hypothesis $f \not\equiv 2^{3t-1} \pmod{2^{3t}}$ for all positive integers t. Now $f \neq 0$ for $f = 0$ implies $a = b = 0$. Then there exists t_1 such that $f \not\equiv 0 \pmod{2^{3t_1}}$ for $f \equiv 0 \pmod{2^{3t}}$ for all t implies $f = 0$. Again, by hypothesis $f \not\equiv 2^{3t_1-1} \pmod{2^{3t_1}}$. Hence the equation is insoluble in integers $ab \neq 0$. If $f = 0$, then the equation has no solution in integers.

We need the following theorem, due to Hemer [1].

Theorem 2: If $2f$ has no prime factor which splits into two different prime ideals in the field $Q(\sqrt{k})$, then all the integer solutions of the equation $y^2 - kf^2 = x^3$ can be obtained by solving the $(3^{e+1} + 1)/2$ equations $N(\beta_i)\ (\pm y + f\sqrt{k}) = \eta \cdot \beta_i \cdot \alpha^3$ where (β_i) $(i = 1, 2, \ldots, 3^e)$ are the cubes of arbitrary ideals, one from each of the 3^e classes c_i with the property $c_1^3 = (1)$, α is an integer in $Q(\sqrt{k})$ and $\eta = 1$ or ε, where ε is the fundamental unit of $Q(\sqrt{k})$ (or an arbitrary unit which is not a cube). Here e is the basis number for 3 in the group of ideal classes.

If the class number h of $Q(\sqrt{k})$ is not divisible by 3, we have $e = 0$ and we get $\pm y + f\sqrt{k} = \eta \cdot \alpha^3$ where $\eta = 1$ or ε, if $k > 1$ and $\eta = 1$ if $k < 0$ and $\eta = \sqrt{-1}$ and $(1 + \sqrt{-3})/2$ if $k = -1$ or -3, respectively. Again we have $e = 1$ if the group of ideal classes is cyclic and if $h \equiv 0 \pmod 3$.

Now consider

(6) $$y^2 - kf^2 = x^3.$$

For $100 < k \leq 200$, k square free, only $Q(\sqrt{142})$ has class number $h \equiv 0 \pmod 3$. If we take $f = 1$, $k \equiv 5 \pmod 8$, then $2f = 2$ does not split into two different primes in $Q(\sqrt{k})$. Hence, by Theorem 2, all the integer solutions of (6) $[f = 1, k \equiv 5 \pmod 8]$ can be obtained from

$$\pm y + \sqrt{k} = \left(\frac{a + b\sqrt{k}}{2}\right)^3$$

$$\pm y + \sqrt{k} = \left(\frac{c + d\sqrt{k}}{2}\right)\left(\frac{a + b\sqrt{k}}{2}\right)^3$$

where the fundamental unit $\eta = \dfrac{c + d\sqrt{k}}{2}$ and $\dfrac{a + b\sqrt{k}}{2}$ is an integer in the field.

Now $c \equiv d \pmod 2$ and $a \equiv b \pmod 2$ for $k \equiv 1 \pmod 4$. On equating irrational parts, we get, respectively,

(7) $$b(3a^2 + kb^2) = 8,$$

and

(8) $$c(3a^2b + kb^3) + d(a^3 + 3kab^2) = 16.$$

Equation (7) can be completely solved for a given k. In particular, if $k \equiv 5 \pmod 8$, $k > 0$, $k \neq 5$, then (7) has no solution in integers. If c and d are odd, then by Theorem 1, (8) has no solution in integers. Whence (6) is not solvable in integers.

In particular, $y^2 - k = x^3$ is without integer solutions for the following k's:

$k = 109$, $\eta = (261 + 25\sqrt{109})/2$
$k = 133$, $\eta = (173 + 15\sqrt{133})/2$
$k = 149$, $\eta = (61 + 5\sqrt{149})/2$
$k = 157$, $\eta = (213 + 17\sqrt{157})/2$
$k = 165$, $\eta = (13 + \sqrt{165})/2$
$k = 173$, $\eta = (13 + \sqrt{173})/2$
$k = 181$, $\eta = (1305 + 97\sqrt{181})/2$

Below we consider three cases where $f \neq 1$.

Case 1:

(9) $$y^2 - 116 = x^3$$

The equation can be written as $y^2 - 2^2 \cdot 29 = x^3$. Here $k = 29 \equiv 5 \pmod 8$ and $f = 2$. The fundamental unit of $Q(\sqrt{29})$ is $\eta = (5 + \sqrt{29})/2$ and $h[Q(\sqrt{29})] = 1$. Since 2 remains a prime in $Q(\sqrt{29})$, by Theorem 2, all the solutions of (7) can be obtained from

(10) $$\pm y + 2\sqrt{29} = \left(\frac{a + b\sqrt{29}}{2}\right)^3,$$

and

(11) $$\pm y + 2\sqrt{29} = \left(\frac{5 + \sqrt{29}}{2}\right)\left(\frac{a + b\sqrt{29}}{2}\right)^3.$$

On equating irrational parts, we have, respectively,

(12) $$b(3a^2 + 29b^2) = 16,$$

and

(13) $$5(3a^2b + 29b^3) + (a^3 + 3 \cdot 29ab^2) = 16 \cdot 2.$$

(12) is easily seen to be insoluble in integers and (13) has no solution in integers by Theorem 1.

Case 2:

(14) $$y^2 - 180 - x^3$$

Here $k = 5$ and $f = 6$ in $y^2 - kf^2 = x^3$. The fundamental unit of $Q(\sqrt{5})$ is $\eta = (1 + \sqrt{5})/2$ and $h[Q(\sqrt{5})] = 1 \not\equiv 0 \pmod 3$. Again $2f = 12$ has 2 prime divisors 2 and 3. Since (2) = (2) and (3) = (3) in $Q(\sqrt{5})$, we need examine the following two equations by Theorem 2.

(15) $$\pm y + 6\sqrt{5} = \left(\frac{a + b\sqrt{5}}{2}\right)^3,$$

and

(16) $$\pm y + 6\sqrt{5} = \left(\frac{1 + \sqrt{5}}{2}\right)\left(\frac{a + b\sqrt{5}}{2}\right)^3.$$

From (15) and (16), we obtain, respectively,

(17) $$b(3a^2 + 5b^2) = 48,$$

and

(18) $$(a^3 + 3 \cdot 5ab^2) + (3a^2b + 5b^3) = 96.$$

From (17), we see that $b \equiv 0 \pmod 3$. Then $b(3a^2 + 5b^2) \equiv 0 \pmod 9$, while $48 \equiv 3 \pmod 9$. Hence (17) has no solution in integers. Again (18) has no solution in integers by Theorem 1.

Case 3:

(19) $$y^2 - 125 = x^3$$

By Theorem 2, we gat all the solutions of (19) from

(20) $$3a^2b + 5b^3 = 40,$$

and

(21) $$(a^3 + 15ab^2) + (3a^2b + 5b^3) = 80.$$

It is easy to see that (20) has only one solution given by $a = 0$ and $b = 2$. From this solution we find $x = (1/4)/(a^2 - 5b^2) = -5$ and hence $y = 0$. Since, by Theorem 1, (21) has no solution in integers, we have exactly one integral solution for $y^2 - 125 = x^3$, namely

$$x = -5,\ y = 0.$$

REFERENCES

1. O. Hemer. "On the Diophantine Equation $y^2 - k = x^3$." Unpublished doctoral dissertation, Uppsala, 1952.
2. S. P. Mohanty. "On the Diophantine Equation $y^2 - k = x^3$." Unpublished doctoral dissertation, UCLA, 1971.
3. L. J. Mordell. "The Diophantine Equation $y^2 - k = x^3$." *Proc. London Math. Soc.* (2) 13 (1913):60-80.

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A DIVISIBILITY PROPERTY CONCERNING BINOMIAL COEFFICIENTS

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I

The following observation was made by P. Erdős. The exponent of 2 in the canonical decomposition[1] of

$$\binom{2^{n+1}}{2^n} - \binom{2^n}{2^{n-1}}$$

is $3n$ for $n \geq 2$. He conjectured that this is always true.[2] I succeeded in proving his conjecture, which raised the analogous question for odd primes instead of 2.

For the solution of this problem, I can prove the following.

Theorem: The exponent of the prime number p in the canonical decomposition of the difference

$$\binom{p^{n+1}}{p^n} - \binom{p^n}{p^{n-1}}$$

is

(i) $3n$ for $p = 2$,

(ii) $3n + 1$ for $p = 3$,

(iii) at least $3n + 2$ for $p > 3$.

More generally, I will investigate, for integers K, M divisible by p ($K = kp$, $M = mp$), the difference

$$A = A_p(K, M) = \binom{K}{M} - \binom{k}{m}.$$

By an algebraic transformation, we will be led to the following question: If p is a prime and $m(p - 1)$ is even, which power of p divides the sum

$$\sum_{j=1,\, p\nmid j}^{m(p-1)/2} \frac{\prod_{r=1,\ p\nmid j}^{(mp-1)/2} r(mp - r)}{j(mp - j)}\ ?$$

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[1] I.e., the decomposition into the product of powers of different prime numbers.

[2] Oral communication, July 1976.

We obtain a fairly good answer to this question. However, the determination of the exact value of the exponent for $p > 3$ seems to me as hopeless at present as deciding for which primes $(p-1)!+1$ is divisible by p^2.

II

Simplifying the factors divisible by p of the numerator and denominator of $\binom{K}{M}$, we can write A as follows:

$$A = \binom{K}{M} - \binom{k}{m} = \binom{k}{m}\left(\prod_{\substack{j=1\\ p\nmid j}}^{M-1} \frac{K-j}{M-j} - 1\right) = \binom{k}{m}\left\{\prod_{\substack{j=1\\ p\nmid j}}^{M-1} (K-j) - \prod_{\substack{j=1\\ p\nmid j}}^{M-1} j\right\} \Bigg/ \prod_{\substack{j=1\\ p\nmid j}}^{M-1} j \tag{1}$$

The difference $D = D_p(K, M)$ within brackets can now be transformed.

If the number of factors $M - m = m(p-1)$ is even, we arrange the "symmetrical" factors corresponding to values j and $M - j$ in pairs and get

$$\begin{aligned} D &= \prod_{j=1,\, p\nmid j}^{(M-1)/2} (K-j)(K-M+j) - \prod_{j=1,\, p\nmid j}^{(M-1)/2} j(M-j) \\ &= \prod_{j=1,\, p\nmid j}^{(M-1)/2} (K(K-M) + j(M-j)) - \prod_{j=1,\, p\nmid j}^{(M-1)/2} j(M-j) \\ &= \sum_{r=1}^{m(p-1)/2} (K(K-M))^r B_r, \end{aligned} \tag{2}$$

where $B_r = B_r(p, M)$ denotes the following expression:

$$\sum_{\substack{1 \le j_1 < \cdots < j_\mu \le \frac{M-1}{2} \\ p \nmid j_s}} \prod_{s=1}^{\mu} j_s (M - j_s) \qquad \left(1 \le r \le \frac{m(p-1)}{2} - 1\right)$$

with $\mu = \frac{m(p-1)}{2} - r$; finally,

$$B_{\frac{m(p-1)}{2}} = 1.$$

III

If $m(p-1)$ is odd, then the prime p must be 2 and m must be odd. In this case, we can proceed similarly after separating the middle factor corresponding to $j = m$ and obtain

$$\begin{aligned} D &= (K-m) \prod_{j=1,\, 2\nmid j}^{m-1} (K-j)(K-M+j) - m \prod_{j=1,\, 2\nmid j}^{m-1} j(M-j) \\ &= (K-m) \prod_{j=1,\, 2\nmid j}^{m-1} (K(K-M) + j(M-j)) - m \prod_{j=1,\, 2\nmid j}^{m-1} j(M-j) \\ &= (K-m) \sum_{r=1}^{(m-1)/2} (K(K-M)^r B_r + 2(k-m)) \prod_{j=1,\, 2\nmid j}^{m-1} j(M-j) \end{aligned}$$

where $B_r = B_r(2, M)$ denotes the expression:

$$\sum_{\substack{1 \le j_1 < \cdots < j_\mu \le m-1 \\ 2 \nmid j_s}} \prod_{s=1}^{\mu} j_s (M - j_s) \qquad \left(1 \le r \le \frac{m-1}{2} - 1\right)$$

with $\mu = \frac{m-1}{2} - r$; finally,

$$B_{\frac{m-1}{2}} = 1.$$

Here $K - m = 2k - m$ is odd because m is now odd and each term of the sum is divisible by $K(K - M) = 4k(k - m)$, whereas the last term is an odd multiple of $2(k - m)$.

Returning to the expression (1), the denominator in the last form is odd for $p = 2$ (and for every prime p relatively prime to p), so that the exponent of 2 in the canonical decomposition of A is the sum of its exponent in the binomial coefficient $\binom{k}{m}$ and in $2(k - m)$.

IV

In the case when $m(p - 1)$ is even we are led to the determination of the exponent of p in the canonical decomposition of $B_1(p, M)$, as indicated at the beginning. Let us write M in the form $M = m_1 p^a$ where $a \geq 1$ and $p \nmid m_1$.

Each factor $j_s(m_1 p^a - j_s)$ of B_1 is congruent to the opposite of a square mod p^a, so each term of the sum is congruent to $(-1)^{\frac{m(p-1)}{2} - 1}$ times a square.

V

First we consider the case $p = 2$, $a \geq 2$. For $a = 2$, $B_1(2, 4) = 1$, by definition, and if $m_1 > 1$, then B_1 is the sum of m_1 odd integers; thus, B_1 is also odd.

Let us have $a \geq 3$ and j_1, j_2 be two odd integers with

$$0 \leq u2^{a-2} < j_1 < j_2 < (u + 1)2^{a-2} (\leq m_1 2^{a-1}).$$

The terms of B_1 belonging to j_1 and j_2 are incongruent mod p^a. Their difference is the product of a common odd factor of the two terms and of

$$j_2(M - j_2) - j_1(M - j_1) = (j_2 - j_1)(M - j_1 - j_2). \tag{3}$$

(The common factor for any other prime p is always coprime with p.)

Here both factors are even, one of them is not divisible by 4 because of

$$j_1 + j_2 - (j_2 - j_1) = 2j_1,$$

and none of them is divisible by 2^{a-1}, as we have

$$0 < j_2 - j_1 < 2^{a-2}$$

and

$$u2^{a-1} < j_1 + j_2 < (u + 1)2^{a-1}.$$

Now the squares of the odd numbers of such an interval represent a system of all quadratic residues (coprime with p^a) because of

$$c^2 \equiv (2^{a-1} \pm c)^2 \equiv (2^a - c)^2 \pmod{2^a}.$$

The interval $[1, m_1 p^{a-1} - 1]$ consists of $2m_1$ intervals of length 2^{a-2}; thus, we obtain

$$B_1(2, m_1 2^a) \equiv -2m_1 \left(\sum_{u=1}^{2^{a-2}} u^2 - 4 \sum_{u=1}^{2^{a-3}} u^2 \right) = - \frac{2^{a-2}(2^{a-1} - 1)m_1}{3} \pmod{2^a}.$$

The exponent of 2 in the canonical factorization of $B_1(2, m_1 2^a)$ is therefore $a - 2$, and this holds even for $a = 2$.

VI

In the case of an odd prime p, the terms corresponding to intervals of the length $p^a/2$ are pairwise incongruent modulo p^a; thus, they give complete systems of quadratic residues modulo p^a. Namely, $p > 2$, and

$$(0 \leq) \frac{u}{2} p^a < j_1 < j_2 < \frac{u + 1}{2} p^a \leq \frac{m_1}{2} p, \quad p \nmid j_1, j_2,$$

so both factors of (3) cannot be divisible by p and none is divisible by p^a, because of

$$0 < j_2 - j_1 < \frac{1}{2} p^a, \ up^a < j_1 + j_2 < (u + 1)p^a.$$

Thus, we have in this case,

$$B_1(p, M) = (-1)^{\frac{m_1 p^{a-1}(p-1)}{2} - 1} m_1 \left(\sum_{u=1}^{(p^a - 1)/2} u^2 - p^2 \sum_{u=1}^{(p^{a-1} - 1)/2} u^2 \right)$$

(continued)

$$= \frac{(-1)^{\frac{m_1 p^{a-1}(p-1)}{2}-1} p^a(p^{2a-1}+1)(p-1)m_1}{24}, \qquad (\text{mod } p^a).$$

This means that the exponent of 3 in the canonical expansion of $B_1(3, m_1 3^a)$ $(3 \nmid m_1)$ is $a - 1$, and

$$p^a \mid B_1(p, m_1 p^a) \quad (p \nmid m_1) \quad \text{for} \quad p > 3.$$

In the last case, we do not know, however, the exact exponent of p in the canonical factorization of B_1.

VII

Returning to the difference

$$\binom{p^{n+1}}{p^n} - \binom{p^n}{p^{n-1}} = A_p(p^{n+1}, p^n) = \binom{p^n}{p^{n-1}} \left\{ \sum_{r=1}^{m(p-1)/2} \left(p^{2n+1}(p-1)\right)^r B_r \right\} \Bigg/ \prod_{j=1, p \nmid j}^{p^n - 1} j,$$

we know that

$$\binom{p^n}{p^{n-1}} = p \prod_{j=1}^{p^{n-1}} \frac{p^n - j}{p^{n-j} - j}$$

is divisible by p but not by p^2. Thus, the results concerning the divisibility of B_1 give immediately the results ennounced in the theorem. More generally, if

$$M = m_1 p^a,\ K = k_1 p^b,\ \min(a, b) = c,\ (p \nmid m_1, k_1), \text{ and } 2/M\left(1 - \frac{1}{p}\right),$$

then

$$p^{a+b+c+d} / D_p(K, M),$$

where

$$d = \begin{cases} -2 & \text{for } p = 2,\ a \geq 2, \\ -1 & \text{for } p = 3, \\ 0 & \text{for } p > 3. \end{cases}$$

As for $A_p(K, M)$, we have to multiply this by the power of p in the factorization of $\binom{K}{M}$ which can be calculated by the theorem of Lagrange.

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FORMATION OF GENERALIZED F-L IDENTITIES OF THE FORM $\sum_{r=1}^{n} r^{\overline{s}} F_{kr+m}$

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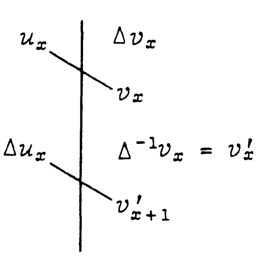
PRELIMINARIES

$r^{\overline{s}} = r(r+1) \cdots (r+s-1)$. The various identities will be formed by using, if necessary, an iterated integration by parts formula for finite differences. $r^{\overline{s}}$ is convenient since $\Delta r^{\overline{s}} = s(r+1)^{\overline{s-1}}$, $\Delta^2 r^{\overline{s}} = s(s-1)(r+2)^{\overline{s-2}}$, $\cdots$, $\Delta r^{\overline{s}} = s!$

$$\Delta^{-1}[u_x \Delta v_x] = u_x v_x - \Delta^{-1}[(\Delta u_x)(v_{x+1})].$$

This formula can be iterated:



(continued)

$$\Delta^2 u_x \quad \Big| \quad \Delta^{-1}v'_{x+1} = v''_{x+1}$$

$$\searrow v''_{x+2}$$

$$\Delta^3 u_x \quad \Big| \quad \Delta^{-1}v''_{x+2} = v'''_{x+2}$$

$$\searrow v'''_{x+3}$$

Starting with the second term, after each finite integration, the subscript x is replaced by $x + 1$.

$$\Delta^{-1}[u_x v_x] = u_x v_x - (\Delta u_x)(v'_{x+1}) + (\Delta^2 u_x)(v''_{x+2}) - (\Delta^3 u_x)(v'''_{x+3}) + \cdots .$$

1. PURE FIBONACCI-LUCAS IDENTITIES

$$\Delta F_x = F_{x+1} - F_x = F_{x-1}, \quad \Delta L_x = L_{x+1} - L_x = L_{x-1}, \quad \Delta^{-1}F_x = F_{x+1}, \quad \Delta^{-1}L_x = L_{x+1}.$$

$$(1) \qquad \sum_1^n F_x = [\Delta^{-1}F_x]_1^{n+1} = [F_{x+1}]_1^{n+1} = F_{n+2} - F_2$$

$$(2) \qquad \sum_1^n L_x = [\Delta^{-1}L_x]_1^{n+1} = [L_{x+1}]_1^{n+1} = L_{n+2} - L_2$$

$$\sum_1^n xF_x = [xF_{x+1} - F_{x+3}]_1^{n+1} = (n+1)F_{n+2} - F_{n+4} - F_2 + F_4$$

$$(3) \qquad \sum_1^n xF_x = (n+1)F_{n+2} - F_{n+4} + F_3$$

$$(4) \qquad \sum_1^n xL_x = (n+1)L_{n+2} - L_{n+4} + L_3$$

$$\sum_1^n x^{\overline{2}}F_x = [x^{\overline{2}}F_{x+1} - 2(x+1)F_{x+3} + 2F_{x+5}]_1^{n+1}$$
$$= (n+1)(n+2)F_{n+2} - 2(n+2)F_{n+4} + 2F_{n+6} - 2!(F_2 - 2F_4 + F_6)$$

$$(5) \qquad \sum_1^n x^{\overline{2}}F_x = (n+1)(n+2)F_{n+2} - 2(n+2)F_{n+4} + 2F_{n+6} - 2!F_4$$

$$(6) \qquad \sum_1^n x^{\overline{2}}L_x = (n+1)(n+2)L_{n+2} - 2(n+2)L_{n+4} + 2L_{n+6} - 2!L_4$$

$$\sum_1^n x^{\overline{3}}F_x = [x^{\overline{3}}F_{x+1} - 3(x+1)^{\overline{2}}F_{x+3} + 6(x+2)F_{x+5} - 6F_{x+7}]_1^{n+1}$$
$$= (n+1)^{\overline{3}}F_{n+2} - 3(n+2)^{\overline{2}}F_{n+4} + 6(n+3)F_{n+6} - 6F_{n+8} - 3!(F_2 - 3F_4 + 3F_6 - F_8)$$

$$(7) \qquad \sum_1^n x^{\overline{3}}F_x = (n+1)(n+2)(n+3)F_{n+2} - 3(n+2)(n+3)F_{n+4} + 6(n+3)F_{n+6} - 6F_{n+8} + 3!F_5$$

$$(8) \qquad \sum_1^n x^{\overline{3}}L_x = (n+1)^{\overline{3}}L_{n+2} - 3(n+2)^{\overline{2}}L_{n+4} + 6(n+3)L_{n+6} - 6L_{n+8} + 3!L_5$$

$$(9) \qquad \sum_1^n x^{\overline{s}}F_x = (n+1)^{\overline{s}}F_{n+2} - s(n+2)^{\overline{s-1}}F_{n+4} + s(s-1)(n+3)^{\overline{s-2}}F_{n+6} \cdots$$
$$+ (-1)^s s!F_{n+2s+2} + (-1)^{s+1}s!F_{s+2}$$

$$(10) \qquad \sum_1^n x^{\overline{s}}L_x = (n+1)^{\overline{s}}L_{n+2} - s(n+2)^{\overline{s-1}}L_{n+4} + s(s-1)(n+3)^{\overline{s-2}}L_{n+6} - \cdots$$
$$+ (-1)^s s!L_{n+2s+2} + (-1)^{s+1}s!L_{s+2}$$

2. PURE FIBONACCI-LUCAS IDENTITIES

$$\Delta F_{2x} = F_{2x+2} - F_{2x} = F_{2x+1}$$
$$\Delta^{-1}F_{2x} = F_{2x-1}, \; \Delta^{-1}L_{2x} = L_{2x-1}$$

(11) $$\sum_1^n F_{2x} = [\Delta^{-1}F_{2x}]_1^{n+1} = [F_{2x-1}]_1^{n+1} = F_{2n+1} - F_1$$

(12) $$\sum_1^n L_{2x} = L_{2n+1} - L_1$$

(13) $$\sum_1^n F_{2x+1} = F_{2n+2} - F_2$$

(14) $$\sum_1^n L_{2x+1} = L_{2n+2} - L_2$$

$$\sum_1^n xF_{2x} = [xF_{2x-1} - F_{2x}]_1^{n+1} = (n+1)F_{2n+1} - F_{2n+2} - (F_1 - F_2)$$

(15) $$\sum_1^n xF_{2x} = (n+1)F_{2n+1} - F_{2n+2} + F_0$$

(16) $$\sum_1^n xL_{2x} = (n+1)L_{2n+1} - L_{2n+2} + L_0$$

(17) $$\sum_1^n xF_{2x+1} = (n+1)F_{2n+2} - F_{2n+3} + F_1$$

(18) $$\sum_1^n xL_{2x+1} = (n+1)L_{2n+2} - L_{2n+3} + L_1$$

$$\sum_1^n x^{\overline{2}}F_{2x} = [x^{\overline{2}}F_{2x-1} - 2(x+1)F_{2x} + 2F_{2x+1}]_1^{n+1}$$
$$= (n+1)(n+2)F_{2n+1} - 2(n+2)F_{2n+2} + 2F_{2n+3} - 2!(F_1 - 2F_2 + F_3)$$

(19) $$\sum_1^n x^{\overline{2}}F_{2x} = (n+1)(n+2)F_{2n+1} - 2(n+2)F_{2n+2} + 2F_{2n+3} - 2!F_1$$

(20) $$\sum_1^n x^{\overline{2}}L_{2x} = (n+1)(n+2)L_{2n+1} - 2(n+2)L_{2n+2} + 2L_{2n+3} - 2!L_1$$

(21) $$\sum_1^n x^{\overline{2}}F_{2x-1} = (n+1)(n+2)F_{2n} - 2(n+2)F_{2n+1} + 2F_{2n+2} - 2!F_2$$

(22) $$\sum_1^n x^{\overline{2}}L_{2x-1} = (n+1)(n+2)L_{2n} - 2(n+2)L_{2n+1} + 2L_{2n+2} - 2!L_2$$

$$\sum_1^n x^{\overline{3}}F_{2x} = [x^{\overline{3}}F_{2x-1} - 3(x+1)^{\overline{2}}F_{2x} + 6(x+2)F_{2x+1} - 6F_{2x+2}]_1^{n+1}$$
$$= (n+1)^{\overline{3}}F_{2n+1} - 3(n+2)^{\overline{2}}F_{2n+2} + 6(n+3)F_{2n+3} - 6F_{2n+4} - 3!(F_1 - 3F_2 + 3F_3 - F_4)$$

(23) $$\sum_1^n x^{\overline{3}}F_{2x} = (n+1)^{\overline{3}}F_{2n+1} - 3(n+2)^{\overline{2}}F_{2n+2} + 6(n+3)F_{2n+3} - 6F_{2n+4} + 3!F_{-2}$$

(24) $$\sum_1^n x^{\overline{3}}L_{2x} = (n+1)^{\overline{3}}L_{2n+1} - 3(n+2)^{\overline{2}}L_{2n+2} + 6(n+3)L_{2n+3} - 6L_{2n+4} + 3!L_{-2}$$

(25) $$\sum_1^n x^{\overline{3}}F_{2x-1} = (n+1)^{\overline{3}}F_{2n} - 3(n+2)^{\overline{2}}F_{2n+1} + 6(n+3)F_{2n+2} - 6F_{2n+3} + 3!F_{-3},$$

(26) $$\sum_1^n x^{\overline{3}} L_{2x-1} = (n+1)^{\overline{3}} L_{2n} - 3(n+2)^{\overline{2}} L_{2n+1} + 6(n+3) L_{2n+2} - 6L_{2n+3} + 3! L_{-3}$$

. . . .

(27) $$\sum_1^n x^{\overline{s}} F_{2x} = (n+1)^{\overline{3}} F_{2n+1} - s(n+2)^{\overline{s-1}} F_{2n+2} + s(s-1)(n+3)^{\overline{s-2}} F_{2n+3} - \cdots$$
$$(-1)^s s! F_{2n+s+1} + (-1)^{s+1} s! F_{-(s-1)}$$

(28) $$\sum_1^n x^{\overline{s}} L_{2x} = (n+1)^{\overline{s}} L_{2n+1} - s(n+2)^{\overline{s-1}} L_{2n+2} + s(s-1)(n+3)^{\overline{s-2}} L_{2n+3} \cdots$$
$$(-1)^s s! L_{2n+s+1} + (-1)^{s+1} s! L_{-(s-1)}$$

3. PURE FIBONACCI-LUCAS IDENTITIES

$$F_{3x} = F_{3x+3} - F_{3x} = \frac{1}{\sqrt{5}}[\alpha^{3x}(\alpha^3 - 1) - \beta^{3x}(\beta^3 - 1)]$$
$$= \frac{1}{\sqrt{5}}[\alpha^{3x}(2\alpha) - \beta^{3x}(2\beta)] = \frac{2}{\sqrt{5}}(\alpha^{3x+1} - \beta^{3x+1}) = 2F_{3x+1}$$

$$\Delta^{-1} F_{3x} = \frac{1}{2} F_{3x-1},\ \Delta^{-1} L_{3x} = \frac{1}{2} L_{3x-1}$$

(29) $$\sum_1^n F_{3x} = [\Delta^{-1} F_{3x}]^{n+1} = \left[\frac{1}{2} F_{3x-1}\right]_1^{n+1} = \frac{1}{2} F_{3n+2} - \frac{1}{2} F_2$$

(30) $$\sum_1^n F_{3x+1} = \frac{1}{2} F_{3n+3} - \frac{1}{2} F_3$$

(31) $$\sum_1^n F_{3x+2} = \frac{1}{2} F_{3n+4} - \frac{1}{2} F_4$$

(32) $$\sum_1^n L_{3x} = \frac{1}{2} L_{3n+2} - \frac{1}{2} L_2$$

(33) $$\sum_1^n L_{3x+1} = \frac{1}{2} L_{3n+3} - \frac{1}{2} L_3$$

(34) $$\sum_1^n L_{3x+2} = \frac{1}{2} L_{3n+4} - \frac{1}{2} L_4$$

$$\sum_1^n xF_{3x} = \left[\frac{1}{2} xF_{3x-1} - \frac{1}{4} F_{3x+1}\right]_1^{n+1}$$

(35) $$\sum_1^n xF_{3x} = \frac{1}{2}(n+1)F_{3n+2} - \frac{1}{4} F_{3n+4} + \frac{F_1}{4}$$

(36) $$\sum_1^n xF_{3x+1} = \frac{1}{2}(n+1)F_{3n+3} - \frac{1}{4} F_{3n+5} + \frac{1}{4} F_2$$

(37) $$\sum_1^n xF_{3x+2} = \frac{1}{2}(n+1)F_{3n+4} - \frac{1}{4} F_{3n+6} + \frac{1}{4} F_3$$

(38) $$\sum_1^n xL_{3x} = \frac{1}{2}(n+1)L_{3n+2} - \frac{1}{4} L_{3n+4} + \frac{1}{4} L_1$$

(39) $$\sum_1^n xL_{3x+1} = \frac{1}{2}(n+1)L_{3n+3} - \frac{1}{4} L_{3n+5} + \frac{1}{4} L_2$$

(40) $$\sum_1^n xL_{3x+2} = \frac{1}{2}(n+1)L_{3n+4} - \frac{1}{4} L_{3n+6} + \frac{1}{4} L_3$$

$$\sum_1^n x^{\overline{2}} F_{3x} = \left[\frac{1}{2} x^{\overline{2}} F_{3x-1} - \frac{1}{4}\, 2(x+1) F_{3x+1} + \frac{2}{8} F_{3x+3}\right]_1^{n+1}$$

(41) $$\sum_{1}^{n} x^{\overline{2}}F_{3x} = \frac{1}{2}(n+1)(n+2)F_{3n+2} - \frac{2}{4}(n+2)F_{3n+4} + \frac{2}{8}F_{3n+6} - \frac{2!}{8}F_0$$

(42) $$\sum_{1}^{n} x^{\overline{2}}F_{3x+1} = \frac{1}{2}(n+1)(n+2)F_{3n+3} - \frac{2}{4}(n+2)F_{3n+5} + \frac{2}{8}F_{3n+7} - \frac{2!}{8}F_1$$

(43) $$\sum_{1}^{n} x^{\overline{2}}F_{3x+2} = \frac{1}{2}(n+1)(n+2)F_{3n+4} - \frac{2}{4}(n+2)F_{3n+6} + \frac{2}{8}F_{3n+8} - \frac{2!}{8}F_2$$

(44) $$\sum_{1}^{n} x^{\overline{2}}L_{3x} = \frac{1}{2}(n+1)(n+2)L_{3n+2} - \frac{2}{4}(n+2)L_{3n+4} + \frac{2}{8}L_{3n+6} - \frac{2!}{8}L_0$$

(45) $$\sum_{1}^{n} x^{\overline{2}}L_{3x+1} = \frac{1}{2}(n+1)(n+2)L_{3n+3} - \frac{2}{4}(n+2)L_{3n+5} + \frac{2}{8}L_{3n+7} - \frac{2!}{8}L_1$$

(46) $$\sum_{1}^{n} x^{\overline{2}}L_{3x+2} = \frac{1}{2}(n+1)(n+2)L_{3n+4} - \frac{2}{4}(n+2)L_{3n+6} + \frac{2}{8}L_{3n+8} - \frac{2!}{8}L_2$$

It is evident that the formulas for $\sum_{1}^{n} x^{\overline{2}}F_{3x+a}$, $a = 1, 2, \ldots,$ follow directly from $\sum_{1}^{n} x^{\overline{2}}F_{3x}$. However, they are very useful for determining lower limit algorithms, if they exist.

$$\sum_{1}^{n} x^{\overline{3}}F_{3x} = \left[\frac{1}{2}x^{\overline{3}}F_{3x-1} - \frac{3}{4}(x+1)^{\overline{2}}F_{3x+1} + \frac{6}{8}(x+2)F_{3x+3} - \frac{6}{16}F_{3x+5}\right]_1^{n+1}$$

(47) $$\sum_{1}^{n} x^{\overline{3}}F_{3x} = \frac{1}{2}(n+1)^{\overline{3}}F_{3n+2} - \frac{3}{4}(n+2)^{\overline{2}}F_{3n+4} + \frac{6}{8}(n+3)F_{3n+6} - \frac{6}{16}F_{3n+8} + \frac{3!}{16}F_{-1}$$

(48) $$\sum_{1}^{n} x^{\overline{3}}L_{3x} = \frac{1}{2}(n+1)^{\overline{3}}L_{3n+2} - \frac{3}{4}(n+2)^{\overline{2}}L_{3n+4} + \frac{6}{8}(n+3)L_{3n+6} - \frac{6}{16}L_{3n+8} + \frac{3!}{16}L_{-1}$$

$$\sum_{1}^{n} x^{\overline{s}}F_{3x} = \left[\frac{1}{2}x^{\overline{s}}F_{3x-1} - \frac{s}{4}(x+1)^{\overline{s-1}}F_{3x+1} + \frac{s(s-1)}{8}(x+2)^{\overline{s-2}}F_{3x+3} - \cdots + (-1)^s s!F_{3x+2s-1}\right]_1^{n+1}$$

(49) $$\sum_{1}^{n} x^{\overline{s}}F_{3x} = \frac{1}{2}(n+1)^{\overline{s}}F_{3n+2} - \frac{s}{4}(n+2)^{\overline{s-1}}F_{3n+4} + \frac{s(s-1)}{8}(n+3)^{\overline{s-2}}F_{3n+6} - \cdots + (-1)^s\frac{s!}{2^{s+1}}F_{3n+2s+2} + (-1)^{s+1}\frac{s!}{2^{s+1}}F_{-s+2}$$

(50) $$\sum_{1}^{n} x^{\overline{s}}L_{3x} = \frac{(n+1)^{\overline{s}}}{2}L_{3n+2} - \frac{s}{4}(n+2)^{s-1}L_{3n+4} + \frac{s(s-1)}{8}(n+s)^{s-2}L_{3n+6} - \cdots + \frac{(-1)^s s!}{2^{s+1}}L_{3n+2s+2} + (-1)^{s+1}\frac{s!}{2^{s+1}}L_{-s+2}$$

Further Remarks: The 50 identities in Sections 1, 2, and 3 involved Fibonacci sequence properties. The following identities involve Type 1 primitive unit properties. Let $(a + b\sqrt{D})/2$ be a primitive unit in the real quadratic field $(\sqrt{D})$, $D \equiv 5 \pmod 8$, $a^2 - b^2D = -4$. Let

$$\alpha = \frac{a + b\sqrt{D}}{2},\ \beta = \frac{a - b\sqrt{D}}{2},\ \left(\frac{a + b\sqrt{D}}{2}\right)^n = \frac{L_n + F_n\sqrt{D}}{2},\ F_n = \frac{1}{\sqrt{D}}(\alpha^n - \beta^n),\ L_n = \alpha^n + \beta^n,\ \alpha\beta = -1.$$

F_n and L_n are also given by the finite difference equations

(*) $$F_{n+2} = aF_{n+1} + F_n,\ F_1 = b,\ F_2 = ab$$
$$L_{n+2} = aL_{n+1} + L_n,\ L_1 = a,\ L_2 = a^2 + 2$$

Examples:

$$D = 5, \quad \text{with primitive unit } \frac{1 + \sqrt{5}}{2}$$

$$D = 13, \quad \text{with primitive unit } \frac{3 + \sqrt{5}}{2}$$

$$D = 61, \quad \text{with primitive unity } \frac{39 + 5\sqrt{61}}{2}$$

4. TYPE 1 PRIMITIVE UNIT IDENTITIES

$$\Delta F_{4rx} = F_{4r(x+1)} - F_{4rx} = \frac{1}{\sqrt{D}}[\alpha^{4rx}(\alpha^{4r} - 1) - \beta^{4rx}(\beta^{4r} - 1)]$$

$$= \frac{1}{\sqrt{D}}[\alpha^{4rx}\alpha^{2r}(\alpha^{2r} - \beta^{2r}) + \beta^{4rx}\beta^{2r}(\alpha^{2r} - \beta^{2r})]$$

$$\Delta F_{4rx} = F_{2r}L_{2r(2x+1)}$$

$$\Delta^{-1}L_{4rx} = \frac{1}{F_{2r}} F_{2r(2x-1)}$$

$$\Delta L_{4rx} = \alpha^{4rx}\alpha^{2r}(\alpha^{2r} - \beta^{2r}) - \beta^{4rx}\beta^{2r}(\alpha^{2r} - \beta^{2r})$$

$$\Delta L_{4rx} = DF_{2r}F_{2r(2x+1)}$$

$$\Delta^{-1}F_{4rx} = \frac{1}{DF_{2r}} L_{2r(2x-1)}$$

(51) $$\sum_1^n F_{4rx} = [\Delta^{-1}F_{4rx}]_1^{n+1} = \frac{1}{DF_{2r}} L_{2r(2n+1)} - \frac{1}{DF_{2r}} L_{2r}$$

(52) $$\sum_1^n L_{4rx} = \left[\frac{1}{F_{2r}} F_{2r(2x-1)}\right]_1^{n+1} = \frac{F_{2r(2n+1)}}{F_{2r}} - \frac{F_{2r}}{F_{2r}}$$

$$\sum_1^n xF_{4rx} = \left[\frac{x}{DF_{2r}} L_{2r(2x-1)} - \frac{1}{DF_{2r}^2} F_{2r(2x)}\right]_1^{n+1}$$

(53) $$\sum_1^n xF_{4rx} = \frac{(n+1)}{DF_{2r}} L_{2r(2n+1)} - \frac{1}{DF_{2r}^2} F_{2r(2n+2)} + \frac{1}{DF_{2r}^2} F_0$$

(54) $$\sum_1^n xL_{4rx} = \frac{(n+1)}{F_{2r}} F_{2r(2n+1)} - \frac{1}{DF_{2r}^2} L_{2r(2n+2)} + \frac{1}{DF_{2r}^2} L_0$$

$$\sum_1^n x^2F_{4rx} = \left[\frac{x^2}{DF_{2r}} L_{2r(2x-1)} - \frac{2(x+1)}{DF_{2r}^2} F_{2r(2x)} + \frac{2}{D^2F_{2r}^3} L_{2r(2x+1)}\right]_1^{n+1}$$

$$\sum_1^n x^2L_{4rx} = \left[\frac{x^2}{F_{2r}} F_{2r(2x-1)} - \frac{2(x+1)}{DF_{2r}^2} L_{2r(2x)} + \frac{2}{DF_{2r}^3} F_{2r(2x+1)}\right]_1^{n+1}$$

(55) $$\sum_1^n x^2F_{4rx} = \frac{(n+1)(n+2)}{DF_{2r}} L_{2r(2n+1)} - \frac{2(n+2)}{DF_{2r}^2} F_{2r(2n+2)} + \frac{2}{D^2F_{2r}^3} F_{2r(2n+3)} - \frac{2!}{D^2F_{2r}^3} L_{-2}$$

(56) $$\sum_1^n x^2L_{4rx} = \frac{(n+1)(n+2)}{F_{2r}} F_{2r(2n+1)} - \frac{2(n+2)}{DF_{2r}^2} L_{2r(2n+2)} + \frac{2}{DF_{2r}^3} F_{2r(2n+3)} - \frac{2!}{DF_{2r}^3} F_{-2}$$

$$\sum_1^n x^3F_{4rx} = \left[\frac{x^3}{DF_{2r}} L_{2r(2x-1)} - \frac{3(x+1)^2}{DF_{2r}^2} F_{2r(2x)} + \frac{6(x+2)}{D^2F_{2r}^3} L_{2r(2x+1)} - \frac{6}{D^2F_{2r}^4} F_{2r(2x+2)}\right]_1^{n+1}$$

$$\sum_1^n x^3L_{4rx} = \left[\frac{x^3}{F_{2r}} F_{2r(2x-1)} - \frac{3(x+1)^2}{DF_{2r}^2} L_{2r(2x)} + \frac{6(x+2)}{D^2F_{2r}^3} F_{2r(2x+1)} - \frac{6}{D^2F_{2r}^4} L_{2r(2x+2)}\right]_1^{n+1}$$

$$(57)\qquad \sum_{1} x^{\overline{3}} F_{4rx} = \frac{(n+1)(n+2)(n+3)}{DF_{2r}} L_{2r(2n+1)} - \frac{3(n+2)(n+3)}{DF_{2r}^2} F_{2r(2n+2)}$$

$$+ \frac{6(n+3)}{D^2F_{2r}^3} L_{2r(2n+3)} - \frac{6}{D^2F_{2r}^4} F_{2r(2n+4)} + \frac{3!}{D^2F_{2r}^4} F_{-4r}$$

$$(58)\qquad \sum_{1}^{n} x^{\overline{3}} L_{4rx} = \frac{(n+1)(n+2)(n+3)}{F_{2r}} F_{2r(2n+1)} - \frac{3(n+2)(n+3)}{DF_{2r}^2} L_{2r(2n+2)}$$

$$+ \frac{6(n+3)}{DF_{2r}^3} F_{2r(2n+3)} - \frac{6}{D^2F_{2r}^4} L_{2r(2n+4)} + \frac{3!}{D^2F_{2r}^4} L_{-4r}$$

$$(59)\qquad \sum_{1}^{n} x^{\overline{4}} F_{4rx} = \frac{(n+1)^{\overline{4}}}{DF_{2r}} L_{2r(2n+1)} - \frac{4(n+2)^{\overline{3}}}{DF_{2r}^2} F_{2r(2n+2)} + \frac{12(n+3)^{\overline{2}}}{D^2F_{2r}^3} L_{2r(2n+3)}$$

$$- \frac{24(n+4)F_{2r(2n+4)}}{D^2F_{2r}^4} + \frac{24}{D^3F_{2r}^5} L_{2r(2n+5)} - \frac{4!}{D^3F_{2r}^5} L_{-6r}$$

$$(60)\qquad \sum_{1}^{n} x^{\overline{4}} L_{4rx} = \frac{(n+1)^{\overline{4}}}{F_{2r}} F_{2r(2n+1)} - \frac{4(n+2)^{\overline{3}}}{DF_{2r}^2} L_{2r(2n+2)} + \frac{12(n+3)^{\overline{2}}}{DF_{2r}^3} F_{2r(2n+3)}$$

$$- \frac{24(n+4)}{D^2F_{2r}^4} L_{2r(2n+4)} + \frac{24}{D^2F_{2r}^5} F_{2r(2n+5)} - \frac{4!}{D^2F_{2r}^5} F_{-6r}$$

$$(61)\qquad \sum_{1}^{n} x^{\overline{2s}} F_{4rx} = \frac{(n+1)^{\overline{2s}}}{DF_{2r}} L_{2r(2n+1)} - \frac{2s(n+2)^{\overline{2s-1}}}{DF_{2r}^2} F_{2r(2n+2)}$$

$$+ \frac{2s(2s-1)(n+3)^{\overline{2s-2}}}{D^2F_{2r}^3} L_{2r(2n+3)} - \frac{2s(2s-1)(2s-2)(n+4)^{2s-3}}{D^2F_{2r}^4} F_{2r(2n+4)}$$

$$+ \cdots + \frac{(-1)^{2s}(2s)!L_{2r(2n+2s+1)}}{D^{s+1}F_{2r}^{2s+1}} + \frac{(-1)^{2s+1}L_{2r(2s-1)}(2s)!}{D^{s+1}F_{2r}^{2s+1}}$$

$$(62)\qquad \sum_{1}^{n} x^{\overline{2s}} L_{4rx} = \frac{(n+1)^{\overline{2s}}F_{2r(2n+1)}}{F_{2r}} - \frac{2s(n+2)^{\overline{2s-1}}}{DF_{2r}^2} L_{2r(2n+2)} + \frac{2s(2s-1)(n+3)^{\overline{2s-2}}}{D^2F_{2r}^3}$$

$$F_{2r(2n+3)} - \frac{(2s)(2s-1)(2s-2)(n+4)^{\overline{2s-3}}}{D^2F_{2r}^4} L_{2r(2n+4)} + \cdots$$

$$+ \frac{(-1)^{2s}(2s)!}{D^2F_{2r}^{2s+1}} F_{2r(2n+2s+1)} + \frac{(-1)^{2s\ \ 1}(2s)!F_{-2r(2s-1)}}{D^2F_{2}^{2s+1}}$$

$$(63)\qquad \sum_{1}^{n} x^{\overline{2s+1}} F_{4rx} = \frac{(n+1)^{\overline{2s+1}}}{DF_{2r}} L_{2r(2n+1)} - \frac{(2s+1)(n+2)^{\overline{2s}}F_{2r(2n+2)}}{DF_{2r}^2}$$

$$+ \frac{(2s+1)(2s)(n+3)^{\overline{2s-1}}}{D^2F_{2r}^3} L_{2r(2n+3)} - \frac{(2s+1)(2s)(2s-1)(n+4)^{\overline{2s-2}}}{D^4F_{2r}^4}$$

$$F_{2r(2n+4)} + \cdots + \frac{(-1)^{2s+1}(2s+1)}{D^{s+1}F_{2r}^{2s\ \ 2}} F_{2r(2n+2s+2)} + \frac{(-1)^{2s}(2s+1)!F_{-2r(2s)}}{D^{s+1}F_{2r}^{2s+2}}$$

(64) $$\sum_1^n x^{2s+1}L_{4rx} = \frac{(n+1)^{2s+1}F_{2r(2n+1)}}{F_{2r}} - \frac{(2s+1)(n+2)^{2s}L_{2r(2n+2)}}{DF_{2r}^2} + \frac{(2s+1)(2s)(n+3)^{2s-1}}{DF_{2r}^3}F_{2r(2n+3)} - \frac{(2s+1)(2s)(2s-1)(n+4)^{2s-2}}{D^2F_{2r}^4}L_{2r(2n+4)} + \cdots + \frac{(-1)^{2s+1}(2s+1)!}{D^sF_{2r}^{2s+2}}L_{2r(2n+2s+2)} + \frac{(-1)^{2s}(2s+1)!L_{-2r(2s)}}{D^{s+1}F_{2r}^{2s+2}}$$

5. TYPE 1 PRIMITIVE UNIT IDENTITIES

$$\Delta F_{2x(2r+1)} = F_{(2x+2)(2r+1)} - F_{2x(2r+1)}$$
$$= \frac{1}{\sqrt{D}}[\alpha^{x(4r+2)}\alpha^{2r+1}(\alpha^{2r+1}+\beta^{2r+1}) - \beta^{x(4r+2)}\beta^{2r+1}(\alpha^{2r+1}+\beta^{2r+1})]$$
$$= L_{2r+1}F_{(2r+1)(2x+1)}$$

$$\Delta^{-1}F_{(2r+1)2x} = \frac{1}{L_{2r+1}}F_{(2r+1)(2x-1)}, \quad \Delta^{-1}L_{(2r+1)2x} = \frac{1}{L_{2r+1}}L_{(2r+1)(2x-1)}$$

(65) $$\sum_1^n F_{2x(2r+1)} = \left[\frac{1}{L_{2r+1}}F_{(2r+1)(2x-1)}\right]_1^{n+1} = \frac{1}{L_{2r+1}}F_{(2r+1)(2n+1)} - \frac{F_{2r+1}}{L_{2r+1}}$$

(66) $$\sum_1^n L_{2x(2r+1)} = \left[\frac{1}{L_{2r+1}}L_{(2r+1)(2x-1)}\right] = \frac{1}{L_{2r+1}}L_{(2r+1)(2n+1)} - 1$$

$$\sum_1^n xF_{2x(2r+1)} = \left[\frac{x}{L_{2r+1}}F_{(2r+1)(2x-1)} - \frac{F_{(2r+1)2x}}{L_{2r+1}^2}\right]_1^{n+1}$$

$$\sum_1^n xL_{2x(2r+1)} = \left[\frac{x}{L_{2n+1}}L_{(2r+1)(2x-1)} - \frac{1}{L_{2r+1}^2}L_{(2r+1)(2x)}\right]_1^{n+1}$$

(67) $$\sum_1^n xF_{2x(2r+1)} = \frac{(n+1)}{L_{2r+1}}F_{(2r+1)(2n+1)} - \frac{1}{L_{2r+1}^2}F_{(2r+1)(2n+2)} + \frac{F_0}{L_{2r+1}^2}$$

(68) $$\sum_1^n xL_{2x(2r+1)} = \frac{(n+1)}{L_{2r+1}}L_{(2r+1)(2n+1)} - \frac{1}{L_{2r+1}^2}L_{(2r+1)(2n+2)} + \frac{L_0}{L_{2r+1}^2}$$

$$\sum_1^n x^2F_{2x(2r+1)} = \left[\frac{x^2}{L_{2r+1}}F_{(2r+1)(2x-1)} - \frac{2(x+1)}{L_{2r+1}^2}F_{(2r+1)2x} + \frac{2}{L_{2r+1}^3}F_{(2r+1)(2x+1)}\right]_1^{n+1}$$

$$\sum_1^n x^2L_{2x(2r+1)} = \left[\frac{x^2}{L_{2r+1}}L_{(2r+1)(2x-1)} - \frac{2(x+1)}{L_{2r+1}^2}L_{(2r+1)2x} + \frac{2}{L_{2r+1}^3}L_{(2r+1)(2x+1)}\right]_1^{n+1}$$

(69) $$\sum_1^n x^2F_{2x(2r+1)} = \frac{(n+1)(n+2)}{L_{2r+1}}F_{(2r+1)(2n+1)} - \frac{2(n+2)}{L_{2r+1}^2}F_{(2r+1)(2n+2)} + \frac{2}{L_{2r+1}^3}F_{(2r+1)(2n+3)} - \frac{2!}{L_{2r+1}^3}L_{-(2r+1)}$$

(70) $$\sum_{1}^{n} x^{\overline{2}} L_{2x(2r+1)} = \frac{(n+1)(n+2)}{L_{2r+1}} L_{(2r+1)(2n+1)} - \frac{2(n+2)}{L^2_{2r+1}} L_{(2r+1)(2n+2)} + \frac{2}{L^3_{2r+1}} L_{(2r+1)(2n+3)} - \frac{2!}{L^3_{2r+1}} L_{-(2r+1)}$$

(71) $$\sum_{1}^{n} x^{\overline{s}} F_{(2r+1)2x} = \frac{(n+1)^{\overline{s}}}{L_{2r+1}} F_{(2r+1)(2n+1)} - \frac{s(n+2)^{\overline{s-1}}}{L^2_{2r+1}} F_{(2r+1)(2n+2)} + \frac{s(s-1)(n+3)^{\overline{s-2}}}{L^3_{2r+1}} F_{(2r+1)(2n+3)} + \cdots + \frac{(-1)^s s!}{L^{s+1}_{2r+1}} F_{(2r+1)(2n+s+1)} + \frac{(-1)^{s+1} s!}{L^{s+1}_{2r+1}} F_{-(s-1)(2r+1)}$$

(72) $$\sum_{1}^{n} x^{\overline{s}} L_{(2r+1)2x} = \frac{(n+1)^{\overline{s}}}{L_{2r+1}} L_{(2r+1)(2n+1)} - \frac{s(n+2)^{\overline{s-1}}}{L^2_{2r+1}} L_{(2r+1)(2n+2)} + \frac{s(s-1)(n+3)^{\overline{s-2}}}{L^3_{2r+1}} L_{(2r+1)(2n+3)} - \cdots + \frac{(-1)^s s!}{L^{s+1}_{2r+1}} L_{(2r+1)(2n+s+1)} + \frac{(-1)^{s+1} s!}{L^{s+1}_{2r+1}} L_{-(s-1)(2r+1)}$$

SECTION 6

$t = 2r + 1.$ $$\sum_{1}^{n} F_{xt} = \frac{1}{\sqrt{D}}[\alpha^t + \cdots + \alpha^{nt} - (\beta^t + \cdots + \beta^{nt})]$$

$$= \frac{1}{\sqrt{D}}\left[\alpha^t \frac{(\alpha^{nt} - 1)}{(\alpha^t - 1)} \frac{(\alpha^t + 1)}{(\alpha^t + 1)} - \frac{\beta^t(\beta^{nt} - 1)}{(\beta^t - 1)} \frac{(\beta^t + 1)}{(\beta^t + 1)}\right]$$

$$= \frac{1}{\sqrt{D}}\left[\frac{(\alpha^{nt} - 1)(\alpha^t + 1)}{\alpha^t + \beta^t} - \frac{(\beta^{nt} - 1)(\beta^t + 1)}{\alpha^t + \beta^t}\right]$$

$$= \frac{1}{\sqrt{D} L_t}[\alpha^{t(n+1)} + \alpha^{nt} - \alpha^t - \beta^{t(n+1)} - \beta^{nt} + \beta^t]$$

$$\sum_{1}^{n} F_{xt} = [\Delta^{-1} F_{xt}]_1^{n+1} = \frac{1}{L_t}(F_{t(n+1)} + F_{nt} - F_t)$$

$$\Delta^{-1} F_{xt} = \frac{1}{L_t}(F_{tx} + F_{t(x-1)}), \quad \Delta^{-1} L_{xt} = \frac{1}{L_t}(L_{tx} + L_{t(x-1)})$$

(73) $$\sum_{1}^{n} F_{xt} = \frac{1}{L_t}(F_{t(n+1)} + F_{tn} - F_t)$$

(74) $$\sum_{1}^{n} L_{xt} = \frac{1}{L_t}(L_{t(n+1)} + L_{tn} - L_t - L_0)$$

(75) $$\sum_{1}^{n} x F_{xt} = \left[\frac{x}{L_t}(F_{tx} + F_{t(x-1)}) - \frac{1}{L_t^2}(F_{t(x+1)} + 2F_{tx} + F_{t(x-1)})\right]_1^{n+1}$$

$$(76)\quad \sum_{1}^{n} xL_{xt} = \left[\frac{x}{L_t}(L_{tx} + L_{t(x-1)}) - \frac{1}{L_t^2}(L_{t(x+1)} + 2L_{tx} + L_{t(x-1)})\right]_1^{n+1}$$

$$(77)\quad \sum_{1}^{n} x^{\overline{2}}F_{tx} = \left[\frac{x^{\overline{2}}}{L_t}(F_{tx} + F_{t(x-1)}) - \frac{2(x+1)}{L_t^2}(F_{t(x+1)} + 2F_{tx} + F_{t(x-1)}) + \frac{1}{L_t^3}(F_{t(x+2)} + 3F_{t(x+1)} + 3F_{tx} + F_{t(x-1)})\right]_1^{n+1}$$

$$(78)\quad \sum_{1}^{n} x^{\overline{2}}L_{tx} = \left[\frac{x^{\overline{2}}}{L_t}(L_{tx} + L_{t(x-1)}) - \frac{2(x+1)}{L_t^2}(L_{t(x+1)} + 2L_{tx} + L_{t(x-1)}) + \frac{1}{L_t^3}(L_{t(x+2)} + 3L_{t(x+1)} + 3L_{tx} + L_{t(x-1)})\right]_1^{n+1}$$

$$(79)\quad \sum_{1}^{n} x^{\overline{s}}L_{tx} = \left[\frac{x^{\overline{s}}}{L_t}(L_{tx} + L_{t(x-1)}) - \frac{s(x+1)^{\overline{s-1}}}{L_t^2}(F_{(t+1)x} + 2F_{tx} + F_{t(x-1)}) + \frac{s(s-1)(x+2)^{\overline{s-2}}}{L_t^3}(F_{(t+2)x} + 3F_{(t+1)x} + 3F_{(t+1)x} + 3F_{tx} + F_{t(x-1)}) - \cdots + \frac{(-1)^s s!}{L_t^{s+1}}\left\{F_{t(x+s)} + \binom{s+1}{1}F_{t(x+s-1)} + \binom{s+1}{2}F_{t(x+s-2)} + \cdots + F_{t(x-1)}\right\}\right]_1^{n+1}$$

$$(80)\quad \sum_{1}^{n} x^{\overline{s}}L_{tx} = \left[\frac{x^{\overline{s}}}{L_t}(L_{tx} + L_{t(x-1)}) - \frac{s(x+1)^{\overline{s-1}}}{L_t^2}(L_{t(x+1)} + 2L_{tx} + L_{t(x-1)} + \frac{s(s-1)(x+2)^{\overline{s-2}}}{L_t^3}(L_{t(x+2)} + 3L_{t(x+1)} + 3L_{tx} + L_{t(x-1)}) + \cdots + \frac{(-1)^s s!}{L_t^{s+1}}\left\{L_{t(x+s)} + \binom{s+1}{1}L_{t(x+s-1)} + \binom{s+1}{2}L_{t(x+s-2)} + \cdots + L_{t(x-1)}\right\}\right]_1^{n+1}$$

Note 1: The author has a slightly larger collection of corresponding formulas for

$$\sum_{1}^{2m \text{ or } 2m+1} (-1)^{x+1}x^{\overline{s}}F_{rx}.$$

The union of these formula sets makes possible the formation of identities for $\sum_{1}^{n} P(x)F_{rx}^{m}$.

Note 2: The author has a table of Type 1 and Type 2 primitive units for quadratic domains 5 to 9997 and a second table that includes primitive units for quadratic domains from 2 to 9999. Current on file computer programs can extend these tables to 999999. A true tested but unused program can be used for integers with more than six digits.

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GENERALIZED FIBONACCI-LUCAS DIFFERENCE EQUATIONS

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Let p be an integer of the form $(2m+1)^2+4$, $m = 0, 1, 2, \ldots$. The finite difference equation

(1) $$P_{n+2} = (2m+1)P_{n+1} + P_n;\; P_1 = 1,\; P_2 = 2m+1$$

has solutions given by

$$P_n = \frac{1}{\sqrt{p}}(\alpha^n - \beta^n);\; \alpha = \frac{2m+1+\sqrt{p}}{2},\; \beta = \frac{2m+1-\sqrt{p}}{2}.$$

The finite difference equation

(2) $$Q_{n+2} = (2m+1)Q_{n+1} + Q_n;\; Q_1 = 2m+1,\; Q_2 = p-2$$

has solutions given by

$$Q_n = \alpha^n + \beta^n;\; \alpha = \frac{2m+1+\sqrt{p}}{2},\; \beta = \frac{2m+1-\sqrt{p}}{2}.$$

The following relations can be found.

$$\alpha\beta = -1$$

$$\alpha = \frac{2m+1+\sqrt{p}}{2} = \frac{Q_1 + P_1\sqrt{p}}{2}$$

$$\alpha^2 = \frac{p-2+(2m+1)\sqrt{p}}{2} = \frac{Q_2 + P_2\sqrt{p}}{2}$$

$$\alpha^3 = \frac{(2m+1)(p-1)+(p-3)\sqrt{p}}{2} = \frac{Q_3 + P_3\sqrt{p}}{2}$$

$$\alpha^4 = \frac{Q_4 + P_4\sqrt{p}}{2}$$

$$\alpha^5 = \frac{Q_5 + P_5\sqrt{p}}{2}$$

$$\alpha^6 = \frac{Q_6 + P_6\sqrt{p}}{2}$$

P_n and Q_n are both even if $n \equiv 0 \pmod 3$; otherwise they are both odd. The basic Fibonacci-Lucas identities can also be generalized.

1. $\sum_{i=1}^{n} F_i = F_{n+2} - 1$

1′ $2(2m+1)\sum_{i=1}^{n} P_i = Q_{n+1} - (2m-1)P_{n+1} - 2$

2. $\sum_{i=1}^{n} L_i = L_{n+2} - 3$

2′ $2(2m+1)\sum_{i=1}^{n} Q_i = pP_{n+1} - (2m-1)Q_{n+1} - 4m - 6$

3. $F_{n+1}F_{n-1} - F_n^2 = (-1)^n$ 3′ $P_{n+1}P_{n-1} - P_n^2 = (-1)^n$

4. $L_{n+1}L_{n-1} - L_n^2 = 5(-1)^{n+1}$ 4′ $Q_{n+1}Q_{n-1} - Q_n^2 = p(-1)^{n+1}$

5. $L_n = F_{n+1} + F_{n-1}$ 5′ $Q_n = P_{n+1} + P_{n-1}$

6.	$F_{2n+1} = F^2_{2n+1} + F^2_n$	6′	$P_{2n+1} = P^2_{n+1} + P^2_n$
7.	$F_{2n} = F^2_{n+1} - F^2_{n-1}$	7′	$(2m+1)P_{2n} = P^2_{n+1} - P^2_{n-1}$
8.	$F_{2n} = F_n L_n$	8′	$P_{2n} = P_n Q_n$
9.	$F_{n+p+1} = F_{n+1}F_{p+1} + F_n F_p$	9′	$P_{n+t+1} = P_{n+1}P_{t+1} + P_n P_t$
10.	$\sum_{i=1}^{n} F_i^2 = F_n F_{n+1}$	10′	$(2m+1)\sum_{i=1}^{n} P_i^2 = P_n P_{n+1}$
11.	$L_n^2 - 5F_n^2 = 4(-1)^n$	11′	$Q_n^2 - pP_n^2 = 4(-1)^n$
12.	$F_{-n} = (-1)^{n+1}F_n$	12′	$P_{-n} = (-1)^{n+1}P_n$

$$\left.\begin{array}{ll} x = Q_1, & y = P_1 \\ x = Q_3, & y = P_3 \\ x = Q_5, & y = P_5 \end{array}\right\} \text{ are particular solutions of } x^2 - py^2 = -4$$

$$\left.\begin{array}{ll} x = Q_2, & y = P_2 \\ x = Q_4, & y = P_4 \\ x = Q_6, & y = P_6 \end{array}\right\} \text{ are particular solutions of } x^2 - py^2 = 4$$

Since $x = Q_1$, $y = P_1$ is not a solution of $x^2 - py^2 = -1$ but $x = \frac{1}{2}Q_3$, $y = \frac{1}{2}P_3$ is, $x = \frac{1}{2}Q_3$, $y = \frac{1}{2}P_3$ is the primitive solution of $x^2 - py^2 = -1$ and $x = \frac{1}{2}Q_6$, $y = \frac{1}{2}P_6$ is the primitive solution of $x^2 - py^2 = 1$. Also,

$$\alpha^r \cdot \alpha^s = \alpha^{r+s} = \frac{Q_{r+s} + P_{r+s}\sqrt{p}}{2}.$$

Theorem 1: An integer y is an integer of the sequence $P_1, P_2, \ldots$, if and only if $py^2 - 4$ or $py^2 + 4$ is an integer square.

Proof (1):

(3) $$x^2 - py^2 = -4$$

The three solution chains of $x^2 - py^2 = -4$ are given by

$$(Q_1 + P_1\sqrt{p})(\tfrac{1}{2}Q_6 + \tfrac{1}{2}P_6\sqrt{p})^t = Q_{6t+1} + P_{6t+1}\sqrt{p}$$

$$(Q_3 + P_3\sqrt{p})(\tfrac{1}{2}Q_6 + \tfrac{1}{2}P_6\sqrt{p})^t = Q_{6t+3} + P_{6t+3}\sqrt{p}$$

$$(Q_5 + P_5\sqrt{p})(\tfrac{1}{2}Q_6 + \tfrac{1}{2}P_6\sqrt{p})^t = Q_{6t+5} + P_{6t+5}\sqrt{p}, \quad t = 0, 1, 2, \ldots .$$

Letting $t = 0, 1, 2, \ldots$, the y integer values are $P_1, P_3, P_5, \ldots$, the successive odd P numbers.

(4) $$x^2 - py^2 = 4$$

The three solution chains of $x^2 - py^2 = 4$ are given by

$$(Q_2 + P_2\sqrt{p})(\tfrac{1}{2}Q_6 + \tfrac{1}{2}P_6\sqrt{p})^t = Q_{6t+2} + P_{6t+2}\sqrt{p}$$

$$(Q_4 + P_4\sqrt{p})(\tfrac{1}{2}Q_6 + \tfrac{1}{2}P_6\sqrt{p})^t = Q_{6t+4} + P_{6t+4}\sqrt{p}$$

$$(Q_6 + P_6\sqrt{p})(\tfrac{1}{2}Q_6 + \tfrac{1}{2}P_6\sqrt{p})^t = Q_{6t+6} + P_{6t+6}\sqrt{p}, \quad t = 0, 1, 2, \ldots .$$

Letting $t = 0, 1, 2, \ldots$, the y integer values are $P_2, P_4, P_6, \ldots$, the successive even P numbers.

Proof (2):

(5) $$y = P_{2m+1}$$

$$py^2 - 4 = pP_{2m+1}^2 - 4 = (\alpha^{2m+1} - \beta^{2m+1})^2 - 4$$

$$= \alpha^{4m+2} + 2(\alpha\beta)^{2m+1} + \beta^{4m+2} = (\alpha^{2m+1} + \beta^{2m+1})^2$$

$$= Q_{2m+1}^2, \text{ an integer square.}$$

(6) $$y = P_{2m}$$

$$py^2 + 4 = pP_{2m}^2 + 4 = (\alpha^{2m} - \beta^{2m})^2 + 4$$

$$= \alpha^{4m} + 2(\alpha\beta)^{2m} + \beta^{4m} = (\alpha^{2m} + \beta^{2m})^2$$

$$= Q_{2m}^2, \text{ an integer square.}$$

Theorem 2: An integer y is an integer of the sequence $Q_1, Q_2, Q_3, \ldots,$ if and only if $pn^2 - 4p$ or $pn^2 + 4p$ is an integer square.

Proof (3):

(7) $$x^2 - py^2 = -4p$$

The three solution chains of $x^2 - py^2 = -4p$ are given by

$$(pP_1 + Q_1\sqrt{p})(\tfrac{1}{2}Q_6 + \tfrac{1}{2}P_6\sqrt{p})^t = pP_{6t+1} + Q_{6t+1}\sqrt{p}$$

$$(pP_3 + Q_3\sqrt{p})(\tfrac{1}{2}Q_6 + \tfrac{1}{2}P_6\sqrt{p})^t = pP_{6t+3} + Q_{6t+3}\sqrt{p}$$

$$(pP_5 + Q_5\sqrt{p})(\tfrac{1}{2}Q_6 + \tfrac{1}{2}P_6\sqrt{p})^t = pP_{6t+5} + Q_{6t+5}\sqrt{p}, \quad t = 0, 1, 2, 3, \ldots .$$

Letting $t = 0, 1, 2, 3, \ldots$, the y integer values are $Q_1, Q_3, Q_5, \ldots$, the successive odd Q numbers.

(8) $$x^2 - py^2 = 4p$$

The three solution chains of $x^2 - py^2 = 4p$ are given by

$$(pP_2 + Q_2\sqrt{p})(\tfrac{1}{2}Q_6 + \tfrac{1}{2}P_6\sqrt{p})^t = pP_{6t+2} + Q_{6t+2}\sqrt{p}$$

$$(pP_4 + Q_4\sqrt{p})(\tfrac{1}{2}Q_6 + \tfrac{1}{2}P_6\sqrt{p})^t = pP_{6t+4} + Q_{6t+4}\sqrt{p}$$

$$(pP_6 + Q_6\sqrt{p})(\tfrac{1}{2}Q_6 + \tfrac{1}{2}P_6\sqrt{p})^t = pP_{6t+6} + Q_{6t+6}\sqrt{p}, \quad t = 0, 1, 2, \ldots .$$

Letting $t = 0, 1, 2, 3, \ldots$, the y integer values will be $Q_2, Q_4, Q_6, \ldots$, the successive even Q numbers.

Proof (4):

(9) $$y = Q_{2m+1}$$

$$py^2 + 4p = p(Q_{2m+1}^2 + 4) = p[\alpha^{4m+2} + 2(\alpha\beta)^{2m+1} + \beta^{4m+2} + 4]$$

$$= p[\alpha^{4m+2} - 2(\alpha\beta)^{2m+1} + \beta^{4m+2}]$$

$$= p(\alpha^{2m+1} - \beta^{2m+1})^2$$

$$= p^2P_{2m+1}^2, \text{ an integer square.}$$

(10) $$y = Q_{2m}$$

$$pn^2 - 4p = p[(\alpha^{2m} + \beta^{2m})^2 - 4]$$

$$= p(\alpha^{2m} - \beta^{2m})^2$$

$$= p^2P_{2m}^2, \text{ an integer square.}$$

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CONDITIONS FOR $\phi(N)$ TO PROPERLY DIVIDE $N - 1$

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This paper is concerned with limitations upon solutions in integers $k > 1$ and $n > 0$ to the equation

(1) $$k\phi(n) = n - 1,$$

where ϕ is the Euler phi-function. The question of whether or not (1) has a solution was first raised by Lehmer [1] and, more recently, was proposed as an "elementary problem" by Marshall [4] and as a research problem by Alter [5].

Here we review some previous results (Theorems A and B below) and then derive additional limitations (Theorems 1-4) on possible solutions (k,n) to (1).

In all that follows, we assume that n is a composite positive integer for which $k\phi(n) = (n - 1)$, k integral and at least 2. We represent n as the product $p_1p_2p_3 \ldots p_r$ of r positive primes. It is occasionally convenient to express n as $(t_1 + 1)(t_2 + 1) \ldots (t_r + 1)$, where $t_i + 1 = p_i$ for $1 \le i \le r$.

We begin with a few basic results which have appeared previously in various places.

Theorem A:

(i) If n satisfies (1), then n is odd, square-free, and the product of at least three primes.

(ii) If n satisfies (1) and p is a prime in n, then n has no prime of the form $px + 1$ where x is a positive integer.

Part (i) was first demonstrated by Lehmer [1]; part (ii) by Schuh [2]. Both are fairly direct consequences of the formula

$$\phi(m) = m\left(1 - \frac{1}{q_1}\right)\left(1 - \frac{1}{q_2}\right) \ldots \left(1 - \frac{1}{q_r}\right)$$

where $m = (q_1^{e_1})(q_2^{e_2}) \ldots (q_r^{e_r})$ is the representation of m as the product of powers of distinct primes.

Indeed, from this formula we see that if n satisfies (1), then

$$\begin{aligned}\phi(n) &= p_1p_2 \ldots p_r(1 - 1/p_1)(1 - 1/p_2) \ldots (1 - 1/p_r)\\ &= (p_1 - 1)(p_2 - 1) \ldots (p_r - 1)\\ &= t_1t_2 \ldots t_r\end{aligned}$$

and that

$$\begin{aligned}k &= \frac{p_1p_2 \cdots p_r - 1}{(p_1 - 1) \ldots (p_r - 1)} = \frac{(t_1 + 1)(t_2 + 1) \ldots (t_r + 1) - 1}{t_1t_2 \ldots t_r}\\ &= 1 + \sum\nolimits_1 + \sum\nolimits_2 + \cdots + \sum\nolimits_{r-1}\end{aligned}$$

where $\sum_j$ is the sum of the products of the inverses of the t_i taken j at a time. This immediately implies the following result, noted by Lieuwens [3].

Theorem A:

(iii) If in the index set $\{1, 2, \ldots, r\}$ an index j exists such that $q_j < q_j'$ and if $q_i \le q_i'$ for all other indices i, then

$$\frac{q_1q_2 \ldots q_r - 1}{(q_1 - 1) \ldots (q_r - 1)} > \frac{q_1'q_2' \ldots q_r' - 1}{(q_1' - 1) \ldots (q_r' - 1)}.$$

Thus, increasing some or all of the primes in n acts to decrease $(n - 1)/\phi(n)$.

Lieuwens showed in addition that if the smallest prime factor of n is not 5 then n is the product of at least 13 primes, and that if 3 is a factor of n then n is the product of at least 213 primes. Watterberg [6] showed that if 5 is a factor of n then n is still the product of at least 13 primes. We offer yet another addition to this set of results with the following.

Theorem 1: If n satisfies (1) and the smallest prime in n is at least 7, then n is the product of at least 26 primes.

Proof: Since for a given number r of primes in n, increasing any one of the primes decreases the value of $(n - 1)/\phi(n)$, it follows that we can bound this ratio above by making n the product of the first r primes. A better bound is possible, however, since we know that 2 is not in n and that both p and $mp + 1$ cannot be in n at the same time.

In seeking this upper bound, it is necessary to achieve a balance between these considerations. For example, it should be better to use 7 in n instead of 29, because that will give the higher ratio. But by including 7 in n, we must exclude 43, 71, 113, 127, ... as well as 29. Thus, if we accept 7 as a factor of n, how many of the $7m + 1$ can we exclude from n and still be guaranteed an upper bound?

Now,

$$(n - 1)/\phi(n) = \frac{p_1p_2 \cdots p_r - 1}{(p_1 - 1) \cdots (p_r - 1)} < \frac{p_1 \cdots p_r}{(p_1 - 1) \cdots (p_r - 1)} = \frac{n}{\phi(n)}.$$

If we use this last ratio as an upper bound, one approach might be to calculate $p/(p - 1)$ along with the product of as many $(mp + 1)/(mp)$ as we need to consider and simply see which is larger. This lends itself to useful results in specific cases, but a more general approach follows.

To begin with, we need only consider $mp + 1$ with m even, since we want $mp + 1$ to be an odd prime. Since

$$\begin{aligned}(2p + 1)(4p + 1)(6p + 1)(p - 1) &= 48p^4 - 4p^3 - 32p^2 - 11p - 1\\ &< 48p^4 \text{ since } p > 0,\end{aligned}$$

$$\frac{(2p + 1)}{2p}\,\frac{(4p + 1)}{4p}\,\frac{(6p + 1)}{6p} = \frac{(2p + 1)(4p + 1)(6p + 1)}{48p^3} < \frac{p}{p - 1}.$$

Hence, we get a higher value of $n/\phi(n)$ by using p and omitting three $mp + 1$, regardless of the values of p and the $mp + 1$.

Considering the next case,

$$\begin{aligned}(2p + 1)(4p + 1)(6p + 1)(8p + 1)(p - 1) &= 384p^5 + 16p^4 - 260p^3 - 120p^2 - 19p - 1\\ &< 384p^5 + 16p^4 - 260p^3 - 120p^2\\ &= 384p^5 + 4p^2(4p^2 - 65p - 30).\end{aligned}$$

For positive p, $4p^2 - 65p - 30$ is negative if p is less than 16. Hence,

$$(2p + 1)(4p + 1)(6p + 1)(8p + 1)(p - 1) < 384p^5 \text{ if } p < 16, \text{ or}$$

$$\frac{2p + 1}{2p}\,\frac{4p + 1}{4p}\,\frac{6p + 1}{6p}\,\frac{8p + 1}{8p} < \frac{p}{p - 1} \text{ if } p < 16.$$

By the same reasoning as before, then, we can eliminate four $mp + 1$ when p is in n if p is 3, 5, 7, 11, or 13, and still be guaranteed an upper bound for $(n - 1)/\phi(n)$.

Applying this result for primes at least 7, we derive the sequence of 25 integers 7, 11, 13, 17, 19, 31, 37, 41, 47, 59, 61, 73, 97, 101, 107, 109, 127, 139, 151, 163, 167, 173, 179, 181, 193 which, when multiplied together to produce n, give

$$\frac{n}{\phi(n)} = \frac{1683\ 931359\ 756224\ 971448\ 190042\ 001610\ 486666\ 623927}{842\ 103229\ 776040\ 364896\ 736617\ 728835\ 584000\ 000000} < 2.$$

But this ratio is an upper bound of $(n - 1)/\phi(n)$ for all n with fewer than 26 primes. Since it is less than 2, n cannot satisfy (1) if it is the product of fewer than 26 primes. Hence, if all prime factors of n are 7 or greater, n is the product of at least 26 primes.

We next look at an unrelated result which deals with the powers of two in $\phi(n)$. Define $e(p)$ to be the largest j such that 2 divides $p - 1$. We have seen that all primes in n are odd, and thus that all the $t = p - 1$ are even. Hence $e(p)$ is at least 1 for all p in n. The following interesting result then emerges.

Theorem 2: If n satisfies (1), then $e(p)$ is minimal for an even number of primes p in n.

Proof: Let $n = p_1p_2p_3 \ldots p_r$ and let m be the smallest value of $e(p_i)$ over $1 \le i \le r$. Suppose without loss of generality that p_1 satisfies $e(p_1) = m$. Since $k\phi(n) = n - 1$,

$$k(p_1 - 1) \ldots (p_r - 1) = p_1p_2 \ldots p_r - 1$$

or

$$kt_1t_2 \ldots t_r = (t_1 + 1) \ldots (t_r + 1) - 1$$

$$= t_1t_2 \ldots t_r + \sum t_{i_1} \ldots t_{i_{r-1}} + \cdots + \sum t_{i_1}t_{i_2} + \sum t_i.$$

Since m is the minimum $e(p)$ and m is at least 1, any product of two or more t_i is a multiple of 2^{m+1}. Thus, taking residuals modulo 2^{m+1} in the preceding equation, we see

$$0 \equiv 0 + 0 + \cdots + 0 + \sum t_i \pmod{2^{m+1}},$$

i.e.,

$$0 \equiv \sum t_i \pmod{2^{m+1}}.$$

Some terms in $\sum t_i$ are themselves multiples of 2^{m+1}—specifically, all those t_i for which $e(p_i)$ is at least $m + 1$. These terms also vanish modulo 2^{m+1}, leaving only those t_i for which $e(p_i) = m$. The sum of all such t_i must thus be a multiple of 2^{m+1}. Since each of these t_i are *odd* multiples of 2^m, there must be an even number of them to produce as a sum a multiple of 2^{m+1}.

Hence $e(p)$ is minimal with a value of m for an even number of primes p in n.

Lastly, we consider an extension of the technique involved in the following theorem of Schuh.

Theorem B: If 3 divides n, then k is of the form $3x + 1$.

Proof (from Schuh): Suppose $n = 3p_2p_3 \ldots p_r$. No prime p_i is of the form $3x$, since it is then either 3 or not prime, and by Theorem A we see that in this case no prime in n can be of the form $3x + 1$. Hence, all the p_i must be of the form $3x + 2$. Since $k\phi(n) = n - 1$,

$$k(2)(p_2 - 1) \ldots (p_r - 1) = 3p_2p_3 \ldots p_r - 1.$$

Taking residuals modulo 3 in this equation, we find that

$$(k)(2)(1)(1) \ldots (1) \equiv 0 - 1 \pmod 3$$

or

$$2k \equiv -1 \pmod 3$$

and thus $k \equiv 1 \pmod 3$; i.e., k is of the form $3x + 1$.

This result cannot be extended in the same form, as the limitation upon the form of the p_i becomes less specific as the known factor of n (in this case, 3) increases. However, certain combinations of k and the p_i can be shown to be incompatible, and we can tabulate the possible combinations, in the following manner:

If p is prime, then the set $\{1, 2, 3, \ldots, p - 1\}$ is a group under multiplication modulo p. In particular, every member of the set has an inverse in the set, and (since no prime except p is divisible by p) all the other $p_2, p_3, \ldots, p_r$ in n are congruent modulo p to members of this set. Suppose then that p is a prime in n and that $n = pp_2p_3 \ldots p_r$. Then we can associate those primes in n which are inverses modulo p, and from this extract a few results.

At this point it becomes clearer to consider specific cases. Suppose $n = 5p_2p_3 \ldots p_r$. Then the p_i may be congruent to 2, 3, or 4 modulo 5. If i_2, i_3, i_4 are the number of primes in n congruent to 2, 3, or 4, respectively, $k\phi(n) = n - 1$ implies that

$$k(2 - 1)^{i_2}(3 - 1)^{i_3}(4 - 1)^{i_4} \equiv -1 \pmod 5$$

or

$$k(1^{i_2})(2^{i_3})(3^{i_4}) \equiv k(2^{i_3})(3^{i_4}) \equiv -1 \pmod 5$$

Now, 2 and 3 are inverses and of order 4 under multiplication modulo 5, so

$$(2^{i_3})(3^{i_4}) \equiv 2 \pmod 5$$

and this is congruent to 2^j for some $j = 0, 1, 2,$ or 3. Hence, we have the following.

Theorem 3: $k(2^j) \equiv -1$ (modulo 5), where j is the number in $\{0, 1, 2, 3\}$ that is congruent modulo 4 to $i_3 - i_4$.

This relation between j and k gives rise to Table 1.

TABLE 1

$i_3 - i_4$ (mod 4)	k (mod 5)
0	4
1	2
2	3
3	1

The next case, when 7 divides n, is naturally a bit more complicated. Defining i_2, i_3, i_4, i_5, i_6 in the same manner as before, we obtain

$$k(1^{i_2})(2^{i_3})(3^{i_4})(4^{i_5})(5^{i_6}) \equiv -1 \quad (\text{modulo } 7).$$

Inverse pairs are 2, 4 (of order 3) and 3, 5 (of order 6, so we have

Theorem 4: $k(2^{i_3 - i_5})(3^{i_4 - i_6}) \equiv -1 \pmod 7$, where $i_3 - i_5$ may be reduced modulo 3 and $i_4 - i_6$ may be reduced modulo 6.

This relationship is shown in Table 2.

TABLE 2

$i_3 - i_5$ (mod 3) \ $i_4 - i_6$ (mod 6); k (mod 7)	0	1	2	3	4	5
0	6	2	3	1	5	4
1	3	1	5	4	6	2
2	5	4	6	2	3	1

The same method can be applied to whatever case is desired: the next case, when 11 divides n, yields a four-dimensional table with 2500 entries.

REFERENCES

1. D. H. Lehmer. "On Euler's Totient Function." *Bull. Amer. Math. Soc.* 38 (1932):745-751.
2. Fr. Schuh. "Do There Exist Composite Numbers m for Which $\phi(m) | m - 1$?" (Dutch) *Mathematica Zutphen* B 13 (1944):102-107.
3. E. Lieuwens. "Do There Exist Composite Numbers M for Which $k\phi(M) = M - 1$ Holds?" *Nieuw Arch. voor Wiskunde* (3), 18 (1970):165-169.
4. Arthur Marshall. Problem E 2337. *American Math. Monthly* 77, No. 5 (1970):522.
5. Ronald Alter. "Can $\phi(n)$ Properly Divide $n - 1$?" *American Math. Monthly* 80, No. 2 (1973):192-193.
6. P. A. Watterberg. "Conditions for $\phi(n)$ To Properly Divide $n - 1$." Abstract 73T-A252, *Notices of the American Math. Society* (August 1973).

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ON FIBONACCI NUMBERS OF THE FORM $x^2 + 1$

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Let F_n (n nonnegative) be the nth term of the Fibonacci sequence, defined by $F_0 = 0$, $F_1 = 1$, $F_{n+2} = F_{n+1} + F_n$, and let L_n (n nonnegative) be the nth term of the Lucas sequence, defined by $L_0 = 2$, $L_1 = 1$, $L_{n+2} = L_{n+1} + L_n$. In a previous paper [3], we showed that the equation

(1) $$F_n = y^2 + 1$$

holds only for $n = 1, 2, 3$, and 5. However, the proof given was quite complicated and depended upon some deep properties of units in quartic fields. Recently, Williams [4] has given a simpler solution of (1) which depends on some very pretty identities involving the Fibonacci and Lucas numbers. In this note, we present a completely elementary solution of

(1) which uses neither algebraic number theory nor the identities employed by Williams. In the course of our investigation we shall use the following theorems, which we state without proof.

Theorem 1: L_n and F_n satisfy the relation

(2) $$L_n^2 - 5F_n^2 = 4(-1)^n.$$

Also, if $x^2 - 5y^2 = 4$, then $x = L_{2n}$ and $y = F_{2n}$ for some n.

Theorem 2 [1]:

(a) If $L_n = x^2$, then $n = 1$ or 3.
(b) If $L_n = 2x^2$, then $n = 0$ or 6.
(c) If $F_n = x^2$, then $n = 0, 1, 2,$ or 12.
(d) If $F_n = 2x^2$, then $n = 0, 3,$ or 6.

Theorem 3 [2]: The only nonnegative integer solutions of the equation $x^2 - 5y^4 = 4$ are $(x, y) = (2, 0), (3, 1)$, and $(322, 12)$.

We now return to our problem and first prove

Lemma 1: If $F_m = 3y^2$, then $y = 0$ or 1.

Proof: If m is odd, Theorem 1 yields $L_m^2 - 45y^4 = -4$, which is impossible mod 3. If m is even and not divisible by 3, then $F_{2m} = F_m L_m$ and $(F_m, L_m) = 1$. Thus, either $F_m = u^2$, $L_m = 3v^2$, or $F_m = 3u^2$, $L_m = v^2$. By Theorem 2 the first case holds only for $m = 1$ or 2 since $3 \nmid m$. If $m = 1$, we get $L_m = 1 \neq 3v^2$. If $m = 2$, we get $F_m = 1$, $L_m = 3$, $y = 1$. Finally, $L_m = v^2$ implies $m = 1$ (by Theorem 2) but then $F_1 \neq 3u^2$.

Next, suppose m is even and $3 \mid m$. Then $(F_m, L_m) = 2$ and $F_{2m} = F_m L_m$ implies $F_m = 2u^2$, $L_m = 6v^2$, or $F_m = 6u^2$, $L_m = 2v^2$. By Theorem 2, the first case only holds for $m = 0, 3,$ or 6, but then $L_m \neq 6v^2$. Finally, the second case implies $m = 0$ or 6, but $F_m = 6v^2$ only for $m = 0$.

Corollary 2: The only nonnegative integer solutions of the equation $x^2 - 45y^4 = 4$ are $(x, y) = (2, 0)$ and $(7, 1)$.

Proof: The equation $x^2 - 45y^4 = 4$ implies $F_{2m} = 3y^2$ for some m. By Lemma 1, $y = 0$ and $y = 1$ are the only possible solutions.

Now we return to (1). If n is odd (1) and (2) yield

(3) $$5x^4 + 10x^2 + 1 = y^2,$$

where x and y are nonnegative integers.

Here $(x, y) = 1$ and we may write (3) as

(4) $$\frac{(5x^2 + 1 + y)}{2}\,\frac{(5x^2 + 1 - y)}{2} = 5x^4.$$

Any common factor of the two expressions on the left-hand side of (4) divides y and $5x^4$ and, thus, since $5 \nmid y$ these two factors are relatively prime. Thus, we conclude that

$$\frac{5x^2 + 1 + y}{2} = u^4,\ \frac{5x^2 + 1 - y}{2} = 5v^4, \text{ i.e., } u^4 + 5v^4 = 5x^2 + 1,\ uv = x.$$

This yields $u^4 + 5v^4 - 5u^2v^2 = 1$, which may be written $(2u^2 - 5v^2)^2 - 5v^4 = 4$. By Theorem 3, the only nonnegative integers v satisfying this equation are $v = 0, 1,$ or 12. The solution $v = 12$ yields $u^2 = 521$, which is impossible. The solution $v = 0$ yields $u = 1$. Finally, $v = 1$ yields $u = 1$ and $u = 2$. Thus, if n is odd (1) holds only for $n = 1, 3,$ and 5.

Next, suppose n is even. Then (1) and (2) yield

(5) $$5x^4 + 10x^2 + 9 = y^2.$$

To solve (5) we note, first of all, that $3 \mid y$. Next, we prove that $3 \mid x$. To see this, we compute F_n mod 16 and find that (1) can hold only when $n \equiv 2$ or $8 \pmod{24}$. But if $n \equiv 8 \pmod{24}$, then $4 \mid n$, $3 \mid F_n$ and F_n cannot be of the form $x^2 + 1$. If $n \equiv 2 \pmod{24}$, then $F_n \equiv 1 \pmod 3$ and if $F_n = x^2 + 1$, then $3 \mid x$. Thus, (5) reduces to

(6) $$45X^4 + 10X^2 + 1 = Y^2.$$

Equation (6) may be written

$$\frac{(Y - 5X^2 - 1)}{2}\,\frac{(Y + 5X^2 + 1)}{2} = 5X^4,$$

and it is easily shown that the two factors on the left-hand side of this equation are relatively prime. Thus, we conclude

$$\frac{Y + 5X^2 + 1}{2} = u^4, \quad \frac{Y - 5X^2 - 1}{2} = 5v^4,$$

which yields

$$u^4 - 5v^4 = 5X^2 + 1, \; uv = X, \text{ i.e., } u^4 - 5u^2v^2 - 5v^4 = 1.$$

This equation may be written $(2u^2 - 5v^2)^2 - 45v^4 = 4$. By Corollary 2, this equation holds only for $v = 0$ and 1, but only $v = 0$ yields a solution, namely $u = 1$. Thus, the only solution of (4) is $x = 3$, $y = 0$. So if n is even and $F_n = x^2 + 1$, then $n = 2$.

In conclusion, we note that Williams [4] has shown that the complete solution of (1) implies that the only integer solutions of the equation $(x - y)^7 = x^5 - y^5$ with $x > y$ are (1, 0) and (0, -1).

REFERENCES

1. J. H. E. Cohn. "Square Fibonacci Numbers, Etc." *The Fibonacci Quarterly* 2 (1964):109-113.
2. J. H. E. Cohn. "Lucas and Fibonacci Numbers and Some Diophantine Equations." *Proc. Glasgow Math. Soc.* 7 (1965):24-28.
3. R. Finkelstein. "On Fibonacci Numbers which Are One More Than a Square." *J. für die reine und angew. Math.* 262/263 (1973):171-178.
4. H. C. Williams. "On Fibonacci Numbers of the Form $k^2 + 1$. *The Fibonacci Quarterly* 13 (1975):213-214.

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FIBONACCI SEQUENCE CAN SERVE PHYSICIANS AND BIOLOGISTS

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PART I: SOME OF THEIR SPECIAL NEEDS

All of earth's living beings can react suitably to a range of circumstances. They react to certain stimuli. How much they react is related to how much is the stimulus. Biologists have found that over a wide range of intensity, a proportionate change in stimulus calls forth the same change in response.

Two examples of this relationship are:

1. the nervous system of an animal recognizes as increase in stimulus the same proportional change across most of the range of stimulus;
2. the immune system of an animal responds to the same proportional change in challenge across most of a very wide range.

From (1) derives the Weber-Fechner law of sensory perception. It is a generalization from a wealth of data. Thus, a certain person feels as heavier as little as 11 ounces compared to 10 ounces, and 11 grams compared to 10 grams, and 11 pounds compared to 10 pounds, as well as 110 pounds compared to 100 pounds. Across a range of five-thousandfold, that person distinguishes the same proportionate difference of 10 percent. Note that the basic distinction is not 1 gram nor 1 pound nor 10 pounds, but remains 10 percent.

From (1) likewise derives that some person can hear one musical note as sharper or flatter than another when it is as little as 0.5 percent sharper or flatter, whether the note is tested at the basso's CC, or the coloratura's ccc, which is five octaves higher with soundwaves vibrating 32 times faster than CC. Indeed, the person's range of perception of musical notes may extend beyond two-hundredfold, much more than 32.

With example (2), the immune system, the range often stretches beyond a millionfold. Throughout that range, the amount of offending protein (antigen) that elicits a given rise in the body's immune substance (antibody) stays proportional to the amount of antibody already present.

So in the workaday world of (1) a biologist measuring changes of taste-sensitivity, or of (2) a physician treating a patient's allergic disorder by regular periodic injections of a solution of an offending protein allergen, either scientist should seek to lay out a schedule of ever-increasing strength of solutions, the increase usually being at a constant proportional rate.

Either scientist, therefore, must use a very long geometric series of increments of strength of solution. They measure out the amounts of solution and of diluent with standard laboratory glassware that is calibrated at arithmetic intervals.

A mathematician can solve the problem that has arisen, which is this. How should they measure out the usual long geometric series using only an arithmetic scale on their glassware tools? They cannot measure out, for instance, ten steps of 5 percent increase with their tools, for it would call for measuring out a series,

$$1,\ 1.05,\ (1.05)^2 \text{ which is } 1.1025,\ \ldots,\ (1.05)^{10} \text{ which is longer than } 1.628{,}895,\ \ldots\ .$$

Yet the needs of their experiment or of their treatment of a patient may make them aim to use about ten steps of about 5 percent increase.

Here we may stop to ponder an example of serious error that was made in tools and schedules used to treat patients for allergic disorders. For many years, a fine old drug firm made high-quality extracts of protein allergens that physicians need and often use. The extracts were supplied sterile in a multiple-dose syringe, but the results of the treatment seemed poor despite the finest of syringes and extracts. The reason was finally found to be the build-in Procrustean schedule of dosages that was based on erroneous mathematics, so that the unhappy patient was sure never to get the right dose as needed. The syringes came calibrated in ten equal divisions. Schedule of dosage was naively planned as 1 such unit first, next 2 units, then 3 units, lastly 4 units from the first syringe; from the second syringe that contained a tenfold concentrate of the solution in the first, the same arithmetic scale of dosages was calibrated for. Each successive syringe contained a tenfold concentrate of the solution in the last former one used.

Increments of dosage thus ran 100 percent, 50 percent, $33\frac{1}{3}$ percent, 250 percent, then again 100 percent, 50 percent, and so on. Increments grew on an arithmetic scale during use of each syringe. They only grew on a geometric scale from one syringe to the next, the step always being tenfold. Mathematicians will at once note that to increase doses tenfold over four even steps of increment, each increment should remain $\sqrt[4]{10} - 1$, which = 77.95... percent.

Besides its nonproportionate increments, the system had another major built-in fault. It was inflexible. It made John Doe's dosages all the same as Jane Doe's and as Richard Roe's. Of course, in real life, each would do best on his or her own schedule; and at special times that schedule should vary, as when John Doe is having a chest cold, or after Richard Roe moves away from an area low in ragweed pollen into an area medium-high in both ragweed and tumbleweed pollen. The faults of the system could have been corrected by proper calibrations of the syringes along with correct mathematical design and directions. Instead, the drug firm's accountants' balances led it to abandon the business, which otherwise had provided unexcelled quality of syringe materials and of allergen extracts.

During the past three decades, biologists and physicians among other scientists have grown to accept that they routinely need to seek the skills and insights of statisticians, both early when designing their work and later when drawing conclusions from their efforts. We write this and some follow-up articles to show that on these and other occasions, certain biologists and physicians need an expert in mathematics to plan and to adapt a schedule of dosages of allergen for a patient, and to plan mathematical details for measuring sensory thresholds of taste.

In this first article, let us peek ahead at coming attractions. Let us watch a medical case being treated.

The patient, a woman of 24, had married an American soldier while he was serving in her homeland in Europe, and had immigrated to the U.S. A. when he finished his military service. They moved to his home state in the upper mid-West. She took ill during her first summer in the United States with sneezing, itching eyes, stuffed and drizzling nostrils, loss of sleep, failing appetite, loss of weight, and so on. The physician's tests of allergens in skin and eyes showed marked hypersensitivity to ragweed pollens that were infesting the air throughout the state and region. He planned treatment that included visiting a cousin who lived in ragweed-free Arizona. After heavy frosts of late autumn cleared the ragweed pollens from the ambient air, she returned as planned to her home already well scrubbed and with contents well laundered. She remained free of distressing symptoms. To prepare her resistance and to lower her hypersensitivity toward the ragweed pollens of the next year's late summer and early autumn, the physician treated her by a long series of injections, one every seven days, of extract containing the proteins of ragweed pollens.

Initial dosage contains only one millionth of the estimated final effective dosage. Dosages grow each step at a rate of about 62 percent. (We shall see later that this is larger than most patients usually need as increment.) The constant increment is pared down near the end of the months of treatment so that the last eight injections increase only by the amount that four of the earlier doses did. All doses were measured out in a standard syringe of the "tuberculin" type, which contains one milliliter (formerly styled "cubic centimeter") and is calibrated at hundredths, 0.01, 0.02, ..., 0.99, 1.00 ml. The first five injections were administered one every seven days beginning later November through December. The first five amounts measured were 0.08, 0.13, 0.21, 0.34, and 0.55. The greatest and least increments between any of these doses were 62.5 percent and 61.5385... percent. Q.E.D.

For another patient, a sequence of dosages growing stronger at a rate of approximately 27 percent might be appropriate. Such a sequence could be given using two solutions, A and B, with B approximately 27 percent stronger than A and alternating the dosages between the solutions as follows:

0.08A, 0.08B, 0.13A, 0.13B, 0.21A, 0.21B, ...

We shall follow up the progress of these patients and shall review other patients' allergic problems in a later paper.

So far, we have considered the mathematical problems met in sensory biology and in allergy-immunology but not often solved by nonmathematicians. We have touched upon the special limitations imposed by our tools that measure out amounts of liquids, and have groped toward adapting these tools for best results. We shall aim to get results that can be safe, simple, and on the mark. Our quest will lead us through continued fractions, and sometimes through Fibonacci-ratio fractional approximations.

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VALUES OF CIRCULANTS WITH INTEGER ENTRIES

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It is well known that the differences of squares $m = x^2 - y^2$, with x and y integers, are the integers satisfying $2 \nmid m$ or $4 \mid m$. It is not difficult to show that the integers m of the form $x^3 + y^3 + z^3 - 3xyz$, with x, y, and z integers, are those integers satisfying $3 \nmid m$ or $9 \mid m$. This paper generalizes on those results.

Let $C_n(x_1, \ldots, x_n)$ be the determinant of the circulant matrix (a_{ij}) in which $a_{ij} = x_k$ when $j - i + 1 \equiv k \pmod{n}$. Note that $C_2(x, y) = x^2 - y^2$ and $C_3(x, y, z) = x^3 + y^3 + z^3 - 3xyz$.

Let V_n be the set of values of C_n when the domain is the set of all ordered n-tuples $(x_1, \ldots, x_n)$ with integer entries x_k. We will show below that, for odd primes p, V_p consists of the integers m with either $p \nmid m$ or $p^2 \mid m$, and that V_{2p} consists of the integers m satisfying either $p \nmid m$ or $p^2 \mid m$ and also satisfying either $2 \nmid m$ or $4 \mid m$, i.e.,

$$V_{2p} = [\{m : p \nmid m\} \cup \{m : p^2 \mid m\}] \cap [\{m : 2 \nmid m\} \cup \{m : 4 \mid m\}].$$

1. GENERAL N

In this section, the x_k may be any complex numbers. It is well known (see [1]) that

$$C_n(x_1, \ldots, x_n) = \prod_{h=0}^{n-1} \left(\sum_{k=1}^{n} x_k \exp[2\pi h(k - 1)i/n] \right). \tag{1.1}$$

We use this to establish the following.

<u>*Theorem 1*</u>: $C_n(x_1 + a, x_2 + a, \ldots, x_n + a) = R \cdot C_n(x_1, x_2, \ldots, x_n)$ where $R = (na + x_1 + x_2 + \cdots + x_n)/(x_1 + x_2 + \cdots + x_n)$.

<u>*Proof*</u>:

$$\begin{aligned} C_n(x_1 + a, \ldots, x_n + a) &= \prod_{h=0}^{n-1} \left(\sum_{k=1}^{n} (x_k + a) \exp[2\pi h(k - 1)i/n] \right) \\ &= \prod_{h=0}^{n-1} \left(\sum_{k=1}^{n} x_k \exp[2\pi h(k - 1)i/n] + a \sum_{k=1}^{n} \exp[2\pi h(k - 1)i/n] \right). \end{aligned}$$

Now

$$\sum_{k=1}^{n} \exp[2\pi h(k - 1)i/n] = \begin{cases} n & \text{for } h = 0 \\ \dfrac{1 - \exp(2\pi hni/n)}{1 - \exp(2\pi hi/n)} = 0 & \text{for } 1 \le h \le n - 1. \end{cases}$$

Thus

$$C_n(x_1 + a, \ldots, x_n + a) = \left(na + \sum_{k=1}^{n} x_k\right) \cdot \prod_{h=0}^{n-1} \left(\sum_{k=1}^{n} x_k \exp[2\pi h(k-1)i/n]\right)$$

$$= C_n(x_1, \ldots, x_n)\left(na + \sum_{k=1}^{n} x_k\right) \Big/ \left(\sum_{k=1}^{n} x_k\right).$$

Another result to be used later is the following.

Theorem 2: Let $n = rs$ where r and s are relatively prime. Then

$$C_n(x_1, \ldots, x_n) = \prod_{g=0}^{s-1} C_r(y_{g1}, y_{g2}, \ldots, y_{gr})$$

where

$$y_{gj} = \sum_{k=0}^{s-1} x_{kr+j} \exp[2\pi g(kr + j - 1)i/s].$$

Proof:

$$\prod_{g=0}^{s-1} C_r(y_{g1}, \ldots, y_{gr}) = \prod_{g=0}^{s-1} \prod_{h=0}^{r-1} \left(\sum_{j=1}^{r} y_{gj} \exp[2\pi h(j-1)i/r]\right)$$

$$= \prod_{g=0}^{s-1} \prod_{h=0}^{r-1} \left(\sum_{j=1}^{r} \sum_{k=0}^{s-1} x_{kr+j} \exp\{2\pi i[g(kr + j - 1)/s + h(j-1)/r]\}\right)$$

$$= \prod_{g=0}^{s-1} \prod_{h=0}^{r-1} \left(\sum_{k=0}^{s-1} \sum_{j=1}^{r} x_{kr+j} \exp\left\{\frac{2\pi i}{n}[(rg + sh)(j-1) + r^2gk]\right\}\right)$$

$$= \prod_{g=0}^{s-1} \prod_{h=0}^{r-1} D(g, h).$$

In $D(g, h)$ each variable x_t appears once and only once. Let d_t be the coefficient of x_t. Then

$$\frac{d_{t+1}}{d_t} = \frac{\exp\{2\pi i[(rg + sh)j + r^2gk]/n\}}{\exp\{2\pi i[(rg + sh)(j-1) + r^2gk]/n\}} = \exp\{2\pi i(rg + sh)/n\}$$

or

$$\frac{d_{t+1}}{d_t} = \frac{\exp\{2\pi i[(rg + sh)(1-1) + r^2g(k+1)]/n\}}{\exp\{2\pi i[(rg + sh)(r-1) + r^2gk]/n\}}$$

$$= \exp[2\pi i(rg + sh - rsh)/n]$$

$$= \exp[2\pi i(rg + sh)/n] \cdot \exp(-2\pi ih)$$

$$= \exp[2\pi i(rg + sh)/n].$$

Also, d_1 occurs when $k = 0$ and $j = 1$. So $d_1 = \exp 0 = 1$. Thus

$$D(g, h) = \sum_{t=1}^{n} x_t \exp[2\pi i(rg + sh)(t-1)/n].$$

Now as g goes from 0 to $s - 1$ and h goes from 0 to $r - 1$, $(rg + sh)$ (mod n) takes on n values. To see that these n values are distinct, one has that if $rg_1 + sh_1 \equiv rg_2 + sh_2$ (mod n) then $r(g_1 - g_2) \equiv s(h_2 - h_1)$ (mod n). As $\gcd(r, s) = 1$, one then has $r \mid (h_2 - h_1)$ and $s \mid (g_1 - g_2)$. But $0 \le g \le s - 1$ and $0 \le h \le r - 1$, so $h_2 - h_1 = g_1 - g_2 = 0$. Thus $h_1 = h_2$ and $g_1 = g_2$. Hence $(rg + sh)$ (mod n) achieves every value from 0 to $n - 1$. Thus

$$\prod_{g=0}^{s-1} C_r(y_{g1}, \ldots, y_{gr}) = \prod_{u=0}^{n-1} \left(\sum_{t=1}^{n} x_t \exp[2\pi iu(t-1)/n]\right) = C_n(x_1, \ldots, x_n).$$

Corollary: Let $n = 2r$ where r is an odd integer. Then $C_n(x_1, \ldots, x_n)$

$$= C_r(x_1 + x_{r+1}, x_2 + x_{r+2}, \ldots, x_r + x_{2r}) \cdot C_r(x_1 - x_{r+1}, x_{r+2} - x_2, \ldots, x_r - x_{2r}).$$

Proof: $C_n(x_1, \ldots, x_n) = C_r(y_{01}, \ldots, y_{0r})C(y_{11}, \ldots, y_{1r})$ where

$$y_{0j} = x_j e^0 + x_{r+j}e^0 = x_j + x_{r+j}$$

and

$$y_{1j} = x_j(-1)^{j-1} + x_{r+j}(-1)^{(r+j-1)} = (-1)^{j-1}(x_j - x_{r+j})$$

since r is an odd integer.

It is now useful to obtain some n-tuples that produce various values in V_n.

Lemma 1: $C_n(2, 0, 1, 1, 1, \ldots, 1) = n^2$.

Proof: By adding every row to the first row and every column to the first column in the determinant form of $C_n(2, 0, 1, 1, 1, \ldots, 1)$ one has

$$C_n(2, 0, 1, 1, 1, \ldots, 1) = \begin{vmatrix} n^2 & n & n & n & \ldots & n \\ n & 2 & 0 & 1 & \ldots & 1 \\ n & 1 & 2 & 0 & \ldots & 1 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ n & 1 & 1 & 1 & \ldots & 2 \end{vmatrix}$$

One can then factor n from both the first column and the first row. One then adds the negative of the first row to each of the succeeding rows to obtain an upper triangular determinant all of whose diagonal elements are 1. Thus, $C_n(2, 0, 1, 1, 1, \ldots, 1) = n^2$.

By Theorem 1, with $a = j - 1$, one has the following.

Theorem 3: $C_n(j + 1, j - 1, j, j, j, \ldots, j) = jn^2$.

Lemma 2: Let $A(n, r) = C_n(1, 1, \ldots, 1, 0, \ldots, 0)$ where the first r elements of the C_n are one and the rest are zero. One has, if $\gcd(n, r) > 1$, then $A(n, r) = 0$, and if $\gcd(n, r) = 1$, then $A(n, r) = r$.

Proof: From (1.1) one has

$$A(n, r) = \prod_{h=0}^{n-1}\left(\sum_{k=1}^{r}\exp[2\pi h(k - 1)i/n]\right) = r\prod_{h=1}^{n-1}\left(\frac{1 - \exp(2\pi hri/n)}{1 - \exp(2\pi hi/n)}\right)$$

If $\gcd(n, r) = j > 1$, then when $h = n/j$ (which is $\leq n - 1$) one has $1 - \exp(2\pi hri/n) = 0$ and $A(n, r) = 0$. If $\gcd(n, r) = 1$, then letting $\theta = \exp(2\pi i/n)$ one has

$$A(n, r) = r\prod_{h=1}^{n-1}[(1 - \theta^{hr})/(1 - \theta^h)].$$

Suppose $\theta^{hr} = 1$. Then $hr \equiv 0 \pmod n$ and as $\gcd(n, r) = 1$, one has $h \equiv 0 \pmod n$ which has no solutions when $1 \leq h \leq n - 1$. Suppose $\theta^{jr} = \theta^{kr}$. Then $jr \equiv kr \pmod n$ and as $\gcd(n, r) = 1$, one has $j \equiv k \pmod n$. The only solution to this when $1 \leq j \leq n - 1$ and $1 \leq k \leq n - 1$ is $j = k$. Thus the $n - 1$ terms $(1 - \theta^{hr})$, $1 \leq h \leq n - 1$, are all different and nonzero. As there are only $n - 1$ different nonzero terms of the form $(1 - \theta^k)$, one has that these $n - 1$ terms are the same as the $n - 1$ terms $(1 - \theta^k)$ with $1 \leq k \leq n - 1$. Thus, if $\gcd(n, r) = 1$, then $A(n, r) = r$.

By Theorem 1, one has the following.

Theorem 4: Let $A(n, r, j) = C_n(x_1, \ldots, x_n)$ where x_1 through x_r equal $j + 1$ and x_{r+1} through x_n equal j. One has, if $\gcd(n, r) > 1$, then $A(n, r, j) = 0$ and if $\gcd(n, r) = 1$, then $A(n, r, j) = nj + r$.

2. THE CASE $N = P$, AN ODD PRIME

Consider $C_p(x_1, x_2, \ldots, x_p)$ where p is an odd prime. Also let the x_k be integers from now on. Let the corresponding matrix be (a_{ij}) where $a_{ij} = x_k$ when $j - i + 1 \equiv k \pmod p$.

Lemma 3: $C_p(x_1, \ldots, x_p) \equiv x_1 + x_2 + \cdots + x_p \pmod p$.

Proof: Consider any term $\prod_{k=1}^{p} a_{k,i_k}$ in the expansion of the determinant. Consider all terms $\prod_{k=1}^{p} a_{k-j,i_k-j}$ for all integers $j \geq 0$ where subscripts are taken mod p. This method divides all terms in the expansion of the determinant into equivalence classes. If the initial term equals x_i^p, then the class consists of just one member. In any other case, the class consists of p members of equal values in terms of the x_k's. In addition, the sign of the permutation corresponding to the product is the same for each member of a class. This follows by induction, since

$$[(i_3 - i_2)(i_4 - i_2) \ldots (i_p - i_2)(i_1 - i_2)] \cdot [(i_4 - i_3) \ldots (i_p - i_3)(i_1 - i_3)]$$
$$\ldots [(i_p - i_{p-1})(i_1 - i_{p-1})] \cdot [(i_1 - i_p)]$$
$$= (-1)^{p-1}[(i_2 - i_1)(i_3 - i_1) \ldots (i_p - i_1)] \cdot [(i_3 - i_2) \ldots (i_p - i_2)] \ldots [(i_p - i_{p-1})]$$
$$= [(i_2 - i_1)(i_3 - i_1) \ldots (i_p - i_1)] \ldots [(i_p - i_{p-1})]$$

since p is odd. Thus in the expansion of the determinant, all of the p terms have the same sign and the same value. This implies that

(2.1) $$C_p(x_1, \ldots, x_p) \equiv x_1^p + x_2^p + \cdots + x_p^p \pmod{p}.$$

Using Fermat's Theorem, (2.1) implies

(2.2) $$C_p(x_1, \ldots, x_p) \equiv x_1 + x_2 + \cdots + x_p \pmod{p}.$$

One now has the following result.

Theorem 5: If $C_p(x_1, \ldots, x_p)$ is divisible by p, it is divisible by p^2.

Proof: If $C_p(x_1, \ldots, x_p)$ is divisible by p, then (2.2) tells us that $\sum_{j=1}^{p} x_j$ is divisible by p. Also

$$C_p(x_1, \ldots, x_p) = \left(\sum_{j=1}^{p} x_j\right) \begin{vmatrix} 1 & x_2 & x_3 & \ldots & x_p \\ 1 & x_1 & x_2 & \ldots & x_{p-1} \\ 1 & x_p & x_1 & \ldots & x_{p-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_3 & x_4 & \ldots & x_1 \end{vmatrix}$$

$$= \left(\sum_{j=1}^{p} x_j\right) \begin{vmatrix} p & \sum_{j=1}^{p} x_j & \sum_{j=1}^{p} x_j & \ldots & \sum_{j=1}^{p} x_j \\ 1 & x_1 & x_2 & \ldots & x_{p-1} \\ 1 & x_p & x_1 & \ldots & x_{p-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_3 & x_4 & \ldots & x_1 \end{vmatrix}$$

Thus, if $\sum_{j=1}^{p} x_j$ is divisible by p, one can factor an additional p from each entry of the first row of the last determinant and $C_p(x_1, \ldots, x_p)$ is divisible by p^2. This proves the theorem.

Now by using Theorems 3 and 4, one sees that V_p consists of the integers m satisfying either $p \nmid m$ or $p^2 \mid m$.

3. THE CASE $N = 2P$

Consider $C_{2p}(x_1, \ldots, x_{2p})$ where p is an odd prime. By the Corollary to Theorem 2, one has

(3.1) $$C_{2p}(x_1, \ldots, x_{2p}) = C_p(y_1, \ldots, y_p) \cdot C_p(z_1, \ldots, z_p)$$

where $y_j = x_j + x_{p+j}$ and $z_j = (-1)^{j-1}(x_j - x_{p+j})$. One now has the following.

Theorem 6: If $C_{2p}(x_1, \ldots, x_{2p})$ is divisible by p, it is divisible by p^2.

Proof: If $p \mid C_{2p}(x_1, \ldots, x_{2p})$, then $p \mid C_p(y_1, \ldots, y_p)$ or $p \mid C_p(z_1, \ldots, z_p)$. But then $p^2 \mid C_p(y_1, \ldots, y_p)$ or $p^2 \mid C_p(z_1, \ldots, z_p)$ and $p^2 \mid C_{2p}(x_1, \ldots, x_{2p})$.

Theorem 7: If $C_{2p}(x_1, \ldots, x_{2p})$ is divisible by 2, it is divisible by 4.

Proof: One has that $x_j + x_{p+j}$ is of the same parity as $\pm(x_j - x_{p+j})$. Since the calculation of a determinant involves only addition, subtraction, and multiplication, this implies that $C_p(y_1, \ldots, y_p)$ and $C_p(z_1, \ldots, z_p)$ are both odd or are both even. Hence, their product $C_{2p}(x_1, \ldots, x_{2p})$ is either odd or a multiple of 4.

We next turn to some particular results. Let

$$B(2p, r) = C_{2p}(x_1, \ldots, x_{2p}) = C_p(y_1, \ldots, y_p) \cdot C_p(z_1, \ldots, z_p) \tag{3.2}$$

where $y_1 = 1 = y_{r+1}$, $y_2 = y_3 = \cdots = y_r = 2$, $y_{r+2} = \cdots = y_p = 0$, $z_1 = 1 = z_{r+1}$, $z_2 = \cdots = z_r = z_{r+2} = \cdots = z_p = 0$, and $x_{2j+1} = (y_{2j+1} + z_{2j+1})/2$, $x_{2j} = (y_{2j} - z_{2j})/2$ where the subscripts on the y's and z's are taken mod p. Note that since p is odd, $y_j = x_j + x_{p+j}$ and $z_j = (-1)^{(j-1)}(x_j - x_{p+j})$. Also, the x's are integers since $y_j \equiv z_j \pmod 2$. One has the following result.

Lemma 4: $B(2p, r) = 4r$ for $1 \le r \le p - 1$.

Proof: $$C_p(y_1, \ldots, y_p) = \prod_{h=0}^{p-1}\left(1 + 2\sum_{k=2}^{r}\exp[2\pi h(k-1)i/p] + \exp[2\pi hri/p]\right)$$

$$= 2r\prod_{h=1}^{p-1}\left\{\frac{(1 + 3\text{xp}[2\pi hi/p])(1 - \exp 2\pi hri/p])}{(1 - \exp[2\pi hi/p])}\right\}.$$

Now

$$C_p(1, 1, 0, 0, \ldots, 0) = 2 = 2\prod_{h=1}^{p-1}(1 + \exp[2\pi hi/p]). \tag{3.3}$$

Hence,

$$C_p(y_1, \ldots, y_p) = 2r\prod_{h=1}^{p-1}\frac{1 - \exp[2\pi hri/p]}{1 - \exp[2\pi hi/p]} = 2A(p, r). \tag{3.4}$$

As $1 \le r \le p - 1$, $\gcd(p, r) = 1$ so $A(p, r) = r$ and $C\ (y_1, \ldots, y_p) = 2r$. Now

$$C_p(z_1, \ldots, z_p) = \prod_{h=0}^{p-1}(1 + \exp[2\pi hri/p]).$$

But the p terms $1 + \exp[2\pi hri/p]$ are all different, as p and r are relatively prime. Hence, they equal the p terms $1 + \exp[2\pi hi/p]$ in the expansion of $C_p(1, 1, 0, 0, \ldots, 0)$ and one has

$$C_p(z_1, \ldots, z\ ) = \prod_{h=0}^{p-1}(1 + \exp[2\pi hi/p]) = C_p(1, 1, 0, 0, \ldots, 0) = 2. \tag{3.5}$$

Thus, $B(2p, r) = 2r \cdot 2 = 4r$.

By letting $B(2p, r, j)$ be the C_{2p} obtained from $B(2p, r)$ by increasing each x_k by j, one has the following.

Theorem 8: $B(2p, r, j) = 4r + 4pj$ for $1 \le r \le p - 1$.

Proof: $B(2p, r, j) = 4r(2pj + 2r)/(2r) = 4r + 4pj$ by Theorem 1.

Lemma 5: When $x_1 = 1$, $x_2 = 0$, $x_3 = x_4 = \cdots = x_{p+1} = 1$, $x_{p+2} = \cdots = x_{2p} = 0$, one has $C_{2p}(x_1, \ldots, x_{2p}) = p^2$.

Proof: $$C_{2p}(x_1, \ldots, x_{2p}) = C_p(2, 0, 1, 1, \ldots, 1) \cdot C_p(0, 0, 1, -1, 1, \ldots, -1, 1)$$
$$= p^2 C_p(0, 0, 1, -1, 1, \ldots, -1, 1)$$

by Lemma 1. Letting $\theta = \exp(2\pi i/p)$ one has

$$C_p(0, 0, 1, -1, 1, \ldots, -1, 1) = 1 \cdot \prod_{h=1}^{p-1}(\theta^{2h} - \theta^{3h} + \theta^{4h} - \cdots - \theta^{(p-2)h} + \theta^{(p-1)h}) \tag{3.6}$$

$$= \prod_{h=1}^{p-1}\frac{(\theta^{2h} + \theta^{ph})}{(1 + \theta^h)} = \prod_{h=1}^{p-1}\frac{(1 + \theta^{2h})}{(1 + \theta^h)}$$

$$= \frac{C_p(1, 0, 1, 0, 0, \ldots, 0)/2}{C_p(1, 1, 0, 0, 0, \ldots, 0)/2} = \frac{2/2}{2/2} = 1.$$

Thus, $C_{2p}(x_1, \ldots, x_{2p}) = p^2$.

Theorem 9: When $x_1 = j + 1$, $x_2 = j$, $x_3 = x_4 = \cdots = x_{p+1} = j + 1$, $x_{p+2} = \cdots = x_{2p} = j$, one has $C_{2p}(x_1, \ldots, x_{2p}) = (2j + 1)p^2$.

Proof: $C_{2p}(x_1, \ldots, x_{2p}) = p^2(2pj + p)/p = (2j + 1)p^2$, by using Lemma 5 and Theorem 1.

Theorem 10: $V_{2p} = [\{m:p \nmid m\} \cup \{m:p^2 \mid m\}] \cap [\{m:2 \nmid m\} \cup \{m:4 \mid m\}]$.

Proof: By Theorems 6 and 7, no other values are possible. The only possible values are the integers not divisible by 2 or p [by using $A(2p, r, j)$ with $\gcd(2p, r) = 1$], the multiples of 4 that are not divisible by p (by using Theorem 8), the multiples of p^2 that are not divisible by 2 (by using Theorem 9), and the multiples of $4p^2$ (by using Theorem 3). Thus, V_{2p} consists of the integers m satisfying either $p \nmid m$ or $p^2 \mid m$ and also satisfying either $2 \nmid m$ or $4 \mid m$.

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REFERENCE

1. Roger Chalkley. "Circulant Matrices and Algebraic Equations." *Math. Magazine* 48, No. 2 (1975):73-80.

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POWERS OF MATRICES AND RECURRENCE RELATIONS

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0. INTRODUCTION

This article arose out of the desire to demonstrate an interesting and perhaps initially surprising application of the theory of matrices to final year high school students. Thus, we consider a matrix-theoretic approach to firstly the solution of two simultaneous first-order recurrence relations and secondly to the solution of a single second-order recurrence relation, together with the proofs of a few identities.

It is well known that the solution of an mth order linear homogeneous recurrence relation can be found by means of the theory of matrices. Indeed, Rosenbaum [4] gave an approach which is based on the Jordan normal form; the reader should also see the recent article [5] of Ryavec. The technique used in Section 1 of this paper is based upon the Cayley-Hamilton theorem for 2 x 2 matrices and is particularly elementary. A novel feature of Section 2 is the use of 2 x 2 matrices to obtain generalizations of a few well-known identities which interrelate the Fibonacci and Lucas numbers.

1. POWERS OF 2 x 2 MATRICES

Let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ be a 2 x 2 matrix whose entries are real, or even complex, numbers.

The *characteristic polynomial* of A is

$$\det(\lambda I - A) = \begin{vmatrix} \lambda - a_{11} & -a_{12} \\ -a_{21} & \lambda - a_{22} \end{vmatrix} = \lambda^2 - (a_{11} + a_{22}) + (a_{11}a_{22} - a_{12}a_{21}).$$

It can be verified by direct computation that

$$A^2 - (a_{11} + a_{22})A + (a_{11}a_{22} - a_{12}a_{21})I = 0.$$

This is a special case of the famous Cayley-Hamilton theorem which says that if

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mm} \end{pmatrix} \text{ is an } m \times m \text{ matrix } (m \geq 1)$$

and $c(\lambda) = \det(\lambda I - A) = \lambda^m + c_{m-1}\lambda^{m-1} + \cdots + c_1\lambda + c_0$, then $c(A) = 0$ in the sense that $A^m + c_{m-1}A^{m-1} + \cdots + c_1 A + c_0 I = 0$. Here, $c_{m-1} = -(a_{11} + a_{22} + \cdots + a_{mm})$ and $c_0 = (-1)^m$ det A, as is consistent with the above case when $m = 2$.

Let μ_1 and μ_2 be the roots of the *characteristic equation*

$$\lambda^2 - (a_{11} + a_{22}) + (a_{11}a_{22} - a_{12}a_{21}) = 0.$$

Hence, $\mu_1 + \mu_2 = a_{11} + a_{22}$ and $\mu_1\mu_2 = a_{11}a_{22} - a_{12}a_{21}$. Thus,

$$(A - \mu_1 I)(A - \mu_2 I) = A^2 - (\mu_1 + \mu_2)A + \mu_1\mu_2 I = 0 = (A - \mu_2 I)(A - \mu_1 I)$$

by the Cayley-Hamilton theorem; we will use these relations in the proofs of equations 1.2 and 1.7 below.

Case 1: $\mu_1 \neq \mu_2$.

Firstly, let us assume that the roots μ_1, μ_2 of the characteristic equation are distinct. Then, we may meaningfully introduce the matrices

$$E_1 = \frac{1}{\mu_1 - \mu_2}(A - \mu_2 I), \quad E_2 = \frac{1}{\mu_2 - \mu_1}(A - \mu_1 I),$$

which have the following properties:

(1.1) $$E_1 + E_2 = I.$$

The proof is a direct computation:

(1.2) $$E_1E_2 = 0 = E_2E_1;$$

(1.3) $$E_1^2 = E_1 \text{ and } E_2^2 = E_2.$$

Proof: By (1.1) and (1.2), $E_1 = E_1 I = E_1(E_1 + E_2) = E_1^2 + E_1E_2 = E_1^2 + 0 = E_1^2$. Similarly, $E_2^2 = E_2$.

(1.4) $$A = \mu_1 E_1 + \mu_2 E_2.$$

By (1.2), E_1 and E_2 commute, hence (1.4) and the "binomial theorem" yield

$$(\mu_1E_1 + \mu_2E_2)^n = \mu_1^nE_1^n + \binom{n}{1}\mu_1^{n-1}\mu_2E_1^{n-1}E_2 + \cdots + \mu_2^nE_2^n \text{ for } n \geq 1,$$

and so

$$A^n = \mu_1^nE_1 + \mu_2^nE_2 \text{ for } n \geq 1.$$

To take into account the case of $n = 0$ and the possibility that one of μ_1 and μ_2 is zero, *we adopt the definitions*

$$A^0 = I \text{ and } 0^0 = 1;$$

the latter definition is not so common, yet it will be useful in all that follows. Thus, these definitions and (1.1) allow us to assert that

$$A^n = \mu_1^nE_1 + \mu_2^nE_2 \text{ for } n \geq 0$$

and then substituting for E_1 and E_2, we obtain

(1.5) $$A^n = \frac{1}{\mu_1 - \mu_2}\begin{pmatrix} a_{11}(\mu_1^n - \mu_2^n) + (\mu_1\mu_2^n - \mu_2\mu_1^n) & a_{12}(\mu_1^n - \mu_2^n) \\ a_{21}(\mu_1^n - \mu_2^n) & a_{22}(\mu_1^n - \mu_2^n) + (\mu_1\mu_2^n - \mu_2\mu_1^n) \end{pmatrix}, \quad n \geq 0.$$

Case 2: $\mu_1 = \mu_2 = \mu$.

Secondly, we assume that the characteristic equation has a repeated root $\mu = \mu_1 = \mu_2$. Let $H = A - \mu I$. Then we have the following properties:

(1.6) $$A = \mu I + H;$$

(1.7) $$H^2 = 0.$$

Proof: $H^2 = (A - \mu I)^2 = (A - \mu_1 I)(A - \mu_2 I) = 0.$

Because μI and H commute, the "binomial theorem," (1.6) and (1.7) give

(1.8) $$A^n = \mu^n I + n\mu^{n-1}H, \quad n \geq 1.$$

Substituting for H in terms of A we obtain

(1.9) $$A^n = \mu^{n-1}\begin{pmatrix} n(a_{11} - \mu) + \mu & na_{12} \\ na_{21} & n(a_{22} - \mu) + \mu \end{pmatrix}, \quad n \geq 1.$$

We now consider the simultaneous first-order linear recurrence relations

(1.10) $$\begin{aligned} y_{n+1} &= a_{11}y_n + a_{12}z_n \\ z_{n+1} &= a_{21}y_n + a_{22}z_n \end{aligned}$$

which hold for $n \geq 0$, and wherein the coefficients a_{ij} are independent of n.

In terms of matrices, (1.10) can be expressed in the form

(1.11) $$\begin{pmatrix} y_n & y_{n+1} \\ z_n & z_{n+1} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} y_{n-1} & y_n \\ z_{n-1} & z_n \end{pmatrix}, \quad n \geq 1.$$

Hence,

(1.12) $$\begin{pmatrix} y_n & y_{n+1} \\ z_n & z_{n+1} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}^n \begin{pmatrix} y_0 & y_1 \\ z_0 & z_1 \end{pmatrix}, \quad n \geq 0,$$

or, more briefly,

(1.13) $$Y_n = A^n Y_0, \quad n \geq 0,$$

where

$$Y_n = \begin{pmatrix} y_n & y_{n+1} \\ z_n & z_{n+1} \end{pmatrix} \text{ for } n \geq 0 \text{ and } A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

Applying (1.5) and (1.9) to yield expressions for A^n and then equating the elements in the first row and column of left-hand and right-hand sides of (1.13) gives

(1.14) *Theorem*: Let μ_1 and μ_2 be the roots of the characteristic equation of the matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

and consider the recurrence relations

$$\begin{aligned} y_{n+1} &= a_{11}y_n + a_{12}z_n \\ z_{n+1} &= a_{21}y_n + a_{22}z_n, \quad n \geq 0. \end{aligned}$$

If μ_1 and μ_2 are distinct, then for any $n \geq 0$,

$$y_n = \frac{(a_{11}y_0 + a_{12}z_0)(\mu_1^n - \mu_2^n) + y_0(\mu_1\mu_2^n - \mu_2\mu_1^n)}{\mu_1 - \mu_2}$$

$$z_n = \frac{(a_{21}y_0 + a_{22}z_0)(\mu_1^n - \mu_2^n) + z_0(\mu_1\mu_2^n - \mu_2\mu_1^n)}{\mu_1 - \mu_2}.$$

If μ_1 and μ_2 are both equal to μ, then for any $n \geq 1$,

$$y_n = \mu^{n-1}(n(a_{11}y_0 + a_{12}z_0 - \mu y_0) + \mu y_0)$$
$$z_n = \mu^{n-1}(n(a_{21}y_0 + a_{22}z_0 - \mu z_0) + \mu z_0).$$

We now consider the second-order linear recurrence relation

(1.15) $$u_{n+2} = au_{n+1} + bu_n,$$

which holds for $n \geq 0$, and wherein a and b are independent of n.

Of course, (1.15) can be regarded as a special case of (1.10) if we set $y_n = u_n$ and $z_n = u_{n+1}$ for all $n \geq 0$, $a_{11} = 0$, $a_{12} = 1$, $a_{21} = b$, and $a_{22} = a$. Then (1.13) gives

(1.16) $$U_n = B^n U_0,\ n \geq 0,$$

where

$$U_n = \begin{pmatrix} u_n & u_{n+1} \\ u_{n+1} & u_{n+2} \end{pmatrix} \text{ for } n \geq 0 \text{ and } B = \begin{pmatrix} 0 & 1 \\ b & a \end{pmatrix}.$$

Moreover, Theorem 1 specializes to yield the following important and well-known result.

(1.17) *Corollary*: Let μ_1 and μ_2 be the roots of the characteristic equation of the matrix

$$B = \begin{pmatrix} 0 & 1 \\ b & a \end{pmatrix}$$

and consider the recurrence relation

$$u_{n+2} = au_{n+1} + bu_n,\ n \geq 0.$$

If μ_1 and μ_2 are distinct, then for any $n \geq 0$,

$$u_n = \frac{u_1(\mu_1^n - \mu_2^n) + u_0(\mu_1\mu_2^n - \mu_2\mu_1^n)}{\mu_1 - \mu_2}.$$

If μ_1 and μ_2 are both equal to μ, then for any $n \geq 1$,

$$u_n = \mu^{n-1}(n(u_1 - \mu u_0) + u\mu_0).$$

We close this section with an example which occurs as Exercise 3-9 in [3, p. 92]. Beforehand, we note two consequences of (1.16) and (1.17) for the special case of the Fibonacci numbers, which are defined by the recurrence $f_{n+2} = f_{n+1} + f_n$ for $n \geq 0$, where $f_0 = 0$ and $f_1 = 1$. Then (1.16) shows that

$$\begin{pmatrix} f_n & f_{n+1} \\ f_{n+1} & f_{n+2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^{n+1},$$

and (1.17) gives Binet's formula

$$f_n = \frac{\mu_1^n - \mu_2^n}{\sqrt{5}}, \text{ where } \mu_1 = \frac{1+\sqrt{5}}{2} \text{ and } \mu_2 = \frac{1-\sqrt{5}}{2} = -\frac{1}{\mu_1}.$$

Define two sequences $\{y_n\}$, $\{z_n\}$ in terms of binomial coefficients by

$$y_n = \sum_{k=0}^{n-1} \binom{n+k}{2k+1} \text{ when } n \geq 1, \text{ and } y_0 = 0,$$

$$z_n = \sum_{k=0}^{n} \binom{n+k}{2k} \text{ when } n \geq 1, \text{ and } z_0 = 1.$$

Using Pascal's relation $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$, it readily follows that

$$y_{n+1} = y_n + z_n$$
$$z_{n+1} = y_{n+1} + z_n, \text{ for } n \geq 0.$$

Whence we obtain the special case

$$y_{n+1} = y_n + z_n,\ n \geq 0,\ y_0 = 0$$
$$z_{n+1} = y_n + 2z_n, \qquad z_0 = 1$$

of (1.10). Here

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} f_1 & f_2 \\ f_2 & f_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^2$$

in terms of the Fibonacci numbers. Then (1.13) yields

$$\begin{pmatrix} y_n & y_{n+1} \\ z_n & z_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^{2n} \begin{pmatrix} y_0 & y_1 \\ z_0 & z_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^{2n} \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} f_{2n-1} & f_{2n} \\ f_{2n} & f_{2n+1} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} \text{ for } n \geq 1.$$

Hence, $y_n = f_{2n}$, $z_n = f_{2n+1}$ for $n \geq 0$, and Binet's formula provides closed expressions for y_n and z_n.

2. SOME IDENTITIES

In this section we restrict ourselves to demonstrating a few identities concerning sequences $\{u_n\}$ which satisfy (1.15) and which are suggested by (1.16) and (1.17). Basin and Hoggatt [1] and Bicknell [2] have previously used matrix techniques to establish identities satisfied by sequences which are defined by specializations of (1.15) and we follow their techniques. It is worth noting that in [6] Waddill used different matrix techniques to obtain identities.

(2.1) *Proposition:* Let $\{u_n\}$ be a sequence satisfying the recursion formula $u_{n+2} = au_{n+1} + bu_n$ for $n \geq 0$. Then, for $n \geq 1$,

$$u_{n-1}u_{n+1} - u_n^2 = (-b)^{n-1}(u_0u_2 - u_1^2) \text{ and } u_{2n} = \sum_{r=0}^{n} \binom{n}{r} a^{n-r}b^r u_r.$$

Proof: Taking determinants in (1.16) gives det U_n = (det $B)^n$ det U_0, i.e.,

$$u_nu_{n+2} - u_{n+1}^2 = (-b)^n(u_0u_2 - u_1^2) \text{ for } u \geq 0.$$

Replacing n by $n - 1$, we get the first identity of (2.1).

The Cayley-Hamilton theorem implies that $B^2 = aB + bI$ and so (1.16) gives

$$U_{2n} = B^{2n}U_0 = (aB + bI)^n U_0 = \left(\sum_{r=0}^{n} \binom{n}{r} a^r b^{n-r} B^r\right) U_0$$

and the second identity follows.

When $b = 1$, i.e., the sequence $\{u_n\}$ is given by $u_{n+2} = au_{n+1} + u_n$, another type of identity is easily derived from (1.16). For then $B = \begin{pmatrix} 0 & 1 \\ 1 & a \end{pmatrix}$ and the set of matrices which commute with $\begin{pmatrix} 0 & 1 \\ 1 & a \end{pmatrix}$ is $\left\{\begin{pmatrix} x & y \\ y & ay+x \end{pmatrix} : x, y \text{ arbitrary}\right\}$. In particular, U_0 commutes with B and so

$$U_mU_n = B^mU_0B^nU_0 = B^{m+n}U_0^2 = U_{m+n}U_0.$$

Hence, we obtain

(2.2) *Proposition:* Let $\{u_n\}$ be a sequence satisfying the recursion formula $u_{n+2} = au_{n+1} + u_n$ for $n \geq 0$. Then, for any m, $n \geq 0$,

$$u_mu_n + u_{m+1}u_{n+1} = u_{m+n}u_0 + u_{m+n+1}u_1,$$

and

$$u_mu_{n+1} + u_{m+1}u_{n+2} = u_{m+n}u_1 + u_{m+n+1}u_2.$$

(2.3) *Lemma:* Let x and y be arbitrary and $n \geq 0$. Then,

$$\begin{pmatrix} \frac{x+y}{2} & \frac{x-y}{2} \\ \frac{x-y}{2} & \frac{x+y}{2} \end{pmatrix}^n = \begin{pmatrix} \frac{x^n+y^n}{2} & \frac{x^n-y^n}{2} \\ \frac{x^n-y^n}{2} & \frac{x^n+y^n}{2} \end{pmatrix}$$

Proof: Of course (2.3) can be proved by induction. However, we will give a proof which is in the spirit of this paper. Let

$$E = \tfrac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \text{ and } F = \tfrac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

Then, $E^2 = E$, $F^2 = F$, $EF = 0 = FE$, $E + F = I$. In addition,

$$\begin{pmatrix} \frac{x+y}{2} & \frac{x-y}{2} \\ \frac{x-y}{2} & \frac{x+y}{2} \end{pmatrix} = xE + yF. \text{ Hence, } \begin{pmatrix} \frac{x+y}{2} & \frac{x-y}{2} \\ \frac{x-y}{2} & \frac{x+y}{2} \end{pmatrix} = x^nE + y^nF,$$

and the lemma follows.

(2.4) *Proposition:* Suppose the roots μ_1, μ_2 of the characteristic equation of the matrix

$$B = \begin{pmatrix} 0 & 1 \\ b & a \end{pmatrix}$$

are distinct. Let $\{p_n\}$ and $\{q_n\}$ be two sequences satisfying the recursion formula $u_{n+2} = au_{n+1} + bu_n$ and such that $p_0 = 0$, $p_1 = 1$, $q_0 = 2$, $q_1 = a$. Then, for $m, n \geq 0$,

$$\begin{aligned} q_n^2 - (a^2 + 4b)p_n^2 &= 4(-b)^n, \\ q_mq_n + (a^2 + 4b)p_mp_n &= 2q_{m+n}, \\ q_mp_n + q_np_m &= 2p_{m+n}. \end{aligned}$$

Proof: Applying (1.17) and simplifying, we obtain $p_n = \dfrac{\mu_1^n - \mu_2^n}{\mu_1 - \mu_2}$ and $q_n = \mu_1^n + \mu_2^n$ for any $n \geq 0$. Hence, (2.3) yields

$$\tfrac{1}{2}\begin{pmatrix} q_n & (\mu_1 - \mu_2)p_n \\ (\mu_1 - \mu_2)p_n & q_n \end{pmatrix} = \begin{pmatrix} \frac{\mu_1+\mu_2}{2} & \frac{\mu_1-\mu_2}{2} \\ \frac{\mu_1-\mu_2}{2} & \frac{\mu_1+\mu_2}{2} \end{pmatrix}^n, \; n \geq 0. \tag{2.5}$$

Since $\mu_1\mu_2 = -b$ and $\mu_1 + \mu_2 = a$, $(\mu_1 - \mu_2)^2 = (\mu_1 + \mu_2)^2 - 4\mu_1\mu_2 = a^2 + 4b$. Using this observation and taking determinants in (2.5), we obtain the first identity of (2.4).

Equation (2.5) implies that for $m, n \geq 0$,

$$\begin{pmatrix} q_m & (\mu_1 - \mu_2)p_m \\ (\mu_1 - \mu_2)p_m & q_m \end{pmatrix}\begin{pmatrix} q_n & (\mu_1 - \mu_2)p_n \\ (\mu_1 - \mu_2)p_n & q_n \end{pmatrix} = 2\begin{pmatrix} q_{m+n} & (\mu_1 - \mu_2)p_{m+n} \\ (\mu_1 - \mu_2)p_{m+n} & q_{m+n} \end{pmatrix}. \tag{2.6}$$

Equating the elements in the first row and column of the left and right of (2.6) and also doing the same for the elements in the first row and second column gives, after simplification, the remaining two identities of (2.4).

REFERENCES

1. S. L. Basin and Verner E. Hoggatt, Jr. "A Primer on the Fibonacci Sequence—Part II." *The Fibonacci Quarterly* 1, No. 2 (1963):61-68.
2. Marjorie Bicknell. "A Primer on the Pell Sequence and Related Sequences." *The Fibonacci Quarterly* 13, No. 4 (1975):345-349.

3. C. L. Liu. *Introduction to Combinatorial Mathematics*. New York: McGraw-Hill Book Company, Inc., 1968.
4. R. A. Rosenbaum. "An Application of Matrices to Linear Recursion Relations." *American Math. Monthly* 66 (1959):792-793.
5. C. Ryavec. "An Application of Spectral Theory to Fibonacci Numbers." *The Fibonacci Quarterly* 13, No. 4 (1975):307-308.
6. Marcellus E. Waddill. "Matrices and Generalized Fibonacci Sequences." *The Fibonacci Quarterly* 12, No. 4 (1974):381-386.

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CONVERGENCE PROPERTIES OF LINEAR RECURSION SEQUENCES

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1. INTRODUCTION

The object of this paper is to examine convergence properties of linear recursion sequences of complex numbers. Included are several theorems providing necessary and sufficient conditions, in terms of solutions of an associated auxiliary equation, for various cases and types of convergence.

The question of convergence of linear recursion sequences was raised by Singmaster in Advanced Problem H-179 [6]. The articles of Raphael [4], Shannon [5], and Jarden [3] give representations for linear recursion sequences of integers which are valid also for complex number sequences (the restriction being for aesthetic reasons) and have been useful in preparing this paper. These representations will be included without proof as the substance of the next section.

Let $a_1, a_2, \ldots, a_n$ be complex numbers, with $a_n \neq 0$. We define a linear recursion sequence $\{Q_i^{a(x),U}\}$ by

(1) $$Q_i^{a(x),U} = \sum_{j=1}^{n} a_j Q_{i-j}^{a(x),U} \quad \text{for } n \geq 1$$

where $U = [u_1, u_2, \ldots, u_n]$, $Q_{i-n}^{a(x),U} = u_i$ for $1 \leq i \leq n$, and $a(x) = x^n - a_1x^{n-1} - a_2x^{n-2} \ldots - a_n$. We will refer to $a(x) = 0$ as the auxiliary equation. The absence of the row vector U from the notation will imply that $U = [0, 0, \ldots, 0, 1]$, representing the normalized sequence we will be most concerned with in this paper. The order of the sequence $\{Q^{a(x),U}\}$ is n, and hence the restriction that $a_n \neq 0$ incures a unique definition of order.

2. REPRESENTATIONS

Some representations for linear recursion sequences will be helpful, and are presented here.

Noticing that the recursion relation (1) has a form similar to that of scalar multiplication of n-tuples leads to a matrix approach, presented for instance in Raphael [4]. Explicitly, we may write

(2) $$\begin{bmatrix} a_1 & a_2 & \ldots & a_n \\ 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & & & \vdots \\ 0 & \ldots & 1 & 0 \end{bmatrix}^m \cdot \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} Q_m^{a(x)} \\ Q_{m-1}^{a(x)} \\ \vdots \\ Q_{m-n+1}^{a(x)} \end{bmatrix} \quad \text{for } m \geq 0.$$

Another approach by Raphael [4] relates linear recursion sequences to power series in the following way:

(3) $$\sum_{i=0}^{\infty} Q_i^{a(x)} x^i = 1/(1 - a_1x - a_2x^2 - \cdots - a_nx^n).$$

Let $r_1, \ldots, r_n$ be the n complex roots of $a(x)$ (repeated according to their multiplicity). Then, as Jarden [3, pp. 106-107] noted,

$$(4) \qquad D^{a(x)} Q_m^{a(x)} = D_1^{a(x)} r_{1,d_1}^{(m)} + D_2^{a(x)} r_{2,d_2}^{(m)} + \cdots + D_n^{a(x)} r_{n,d_n}^{(m)}, \qquad \text{for } m \geq 0.$$

where $D^{a(x)}$ is the constant determinant

$$(5) \qquad D^{a(x)} = \begin{vmatrix} r_{1,d_1}^{(0)} & r_{2,d_2}^{(0)} & \cdots & r_{n,d_n}^{(0)} \\ r_{1,d_1}^{(1)} & r_{2,d_2}^{(1)} & \cdots & r_{n,d_n}^{(1)} \\ \vdots & \vdots & & \vdots \\ r_{1,d_1}^{(n-1)} & \cdots & & r_{n,d_n}^{(n-1)} \end{vmatrix},$$

the constant determinants $D_i^{a(x)}$ are as in (5) with ith column deleted, and replaced by $\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$, d_i is the multiplicity of r_i among $r_1, \ldots, r_{i-1}$, and

$$(6) \qquad r_{i,d_i}^{(m)} = \binom{m}{d_i} r_i^{m-d_i}.$$

Also involving the roots $r_1, \ldots, r_n$ of $a(x)$, it has been shown (see [5], for example) that

$$(7) \qquad Q_m^{a(x)} = \sum_{\sum_{i=1}^{n} m_i = m} r_1^{m_1} \cdot r_2^{m_2} \cdot \cdots \cdot r_n^{m_n} \text{ for } m \geq 0,\ m_i \geq 0.$$

3. CONVERGENCE THEOREMS

In this section we will look at the convergence of $\{Q_i^{a(x)}\}_{i=0}^{\infty}$. Convergence of $\{Q_i^{a(x)}\}_{i=0}^{\infty}$ to zero will be considered first.

Theorem 3.1: Let $\{Q_i^{a(x)}\}$ be a linear recursion sequence. Then $\{Q_i^{a(x)}\}_{i=0}^{\infty}$ converges to zero if and only if all the roots of $a(x)$ lie in $\{z \mid |z| < 1\}$.

Proof: Suppose all the roots of $a(x)$ lie in $\{z \mid |z| < 1\}$. Notice that $a(x)$ is the characteristic polynomial of the matrix

$$(8) \qquad A = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \cdot & & \vdots \\ 0 & & 1 & 0 \end{bmatrix}$$

Since by hypothesis, all the roots of $a(x)$, the characteristic polynomial of A, lie in $\{z \mid |z| \leq 1\}$, from Bodewig [1, p. 57], $\lim_{m \to \infty} A^m = 0$. It now follows from equation (2) that

$$\lim_{m \to \infty} Q_m^{a(x)} = 0.$$

Let $\lim_{m \to \infty} Q_m^{a(x)} = 0$, and define $E = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$. Then E, $A \cdot E$, $A^2 \cdot E$, $\ldots$, $A^{n-1} \cdot E$ form a basis for the field of all n-dimensional column vectors, since $a(x)$ is also the minimum polynomial of A. Thus an arbitrary column vector X can be written as $C_0 \cdot E + C_1(A \cdot E) + C_2(A^2 \cdot E) + \cdots + C_{n-1}(A^{n-1} \cdot E)$. We compute using (2) that

$$(9) \qquad \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \lim_{m \to \infty} \begin{bmatrix} Q_m^{a(x)} \\ Q_{m-1}^{a(x)} \\ \vdots \\ Q_{m-n+1}^{a(x)} \end{bmatrix} = \lim_{m \to \infty} \begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & \cdots & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \lim_{m \to \infty}(A^m \cdot E) = (\lim_{m \to \infty} A^m) \cdot E.$$

Similarly, $\lim_{m\to\infty}(A^{m-i} \cdot A^i \cdot E) = 0$ for $1 \le i \le n - 1$. Thus,

$$(10) \qquad 0 = C_0 \cdot \lim_{m\to\infty}(A^m \cdot E) + C_1 \cdot \lim_{m\to\infty}(A^m \cdot A \cdot E) + \cdots + C_{n-1} \cdot \lim_{m\to\infty}(A^m \cdot A^{n-1} \cdot E) = \lim_{m\to\infty}(A^m \cdot X).$$

Therefore, $\lim_{m\to\infty} A^m = 0$, since X was arbitrary, and from Bodewig [1, p. 57] all the roots of $a(x)$, the characteristic polynomial of A, must lie in $\{z \mid |z| < 1\}$.

We may use this theorem and equation (3) to prove the following corollary.

Corollary 3.2: The infinite sum $\sum_{i=0}^{\infty} Q_i^{a(x)}$ exists and equals $1/(1 - a_1 - \cdots - a_n) = \frac{1}{a(1)}$ if and only if all the roots of $a(x)$ lie in $\{z \mid |z| < 1\}$.

Proof: If $\sum_{i=0}^{\infty} Q_i^{a(x)}$ exists, then $\lim_{m\to\infty} Q_i^{a(x)} = 0$, and from Theorem 3.1, all the roots of $a(x)$ must lie in $\{z \mid |z| < 1\}$.

If all the roots of $a(x)$ lie in $\{z \mid |z| < 1\}$, then all of the roots of $1 - a_1x - a_2x^2 - \cdots - a\,x$ must lie outside $\{z \mid |z| \le 1\}$, since the roots of $a(x)$ are the reciprocals of the roots of $1 - a_1x - a_2x^2 - \cdots - a_nx^n$. Note that $a(x)$ has no zero roots, since $a_n \ne 0$. Hence, the power series for $1/(1 - a_1x - \cdots - a_nx^n)$ is valid at $x = 1$. Thus, from equation (3), $\sum_{i=0}^{\infty} Q_i^{a(x)}$ exists, and $\sum_{i=0}^{\infty} Q_i^{a(x)} = 1/(1 - a_1x - \cdots - a_nx^n)$.

We are now ready to prove a theorem about the convergence of linear recursion sequences to nonzero complex numbers.

Theorem 3.3: Let $\{Q_i^{a(x)}\}$ be a linear recursion sequence. Then $\lim_{m\to\infty} Q_m^{a(x)} = b \ne 0$ if and only if 1 is a root of $a(x)$, and all the roots of $a(x)/(x - 1)$ lie in $\{z \mid |z| < 1\}$. Furthermore, if $a^*(x) = a(x)/(x - 1)$, then $\lim_{m\to\infty} Q_m^{a(x)} = 1/(a^*(1))$.

Proof: Let $\lim_{m\to\infty} Q_m^{a(x)} = b \ne 0$. Using equation (1), we find

$$(11) \qquad 0 \ne b = \lim_{m\to\infty} Q_m^{a(x)} = \lim_{m\to\infty}\left(\sum_{i=1}^{n} a_i Q_{m-i}^{a(x)}\right) = \sum_{i=1}^{n} a_i \lim_{m\to\infty} Q_{m-i}^{a(x)} = \left(\sum_{i=1}^{n} a_i\right) \cdot b.$$

Thus $\sum_{i=1}^{n} a_i = 1$ and hence $a(1) = 0$. Therefore, 1 is a root of $a(x)$. Let $r_1, \ldots, r_n$ be the roots of $a(x)$ (repeated to their multiplicity) with $r_n = 1$. Then if $a^*(x) = a(x)/(x - 1)$, $r_1, \ldots, r_{n-1}$ are the $(n - 1)$ roots of $a^*(x)$. From equation (7),

$$(12) \qquad Q^{a(x)} = \sum_{\sum_{i=1}^{n} m_i = m} r_1^{m_1} \cdot \cdots \cdot r_{n-1}^{m_{n-1}} \cdot r_n^{m_n} = \sum_{m_n=0}^{m}\left(1^{m_n} \sum_{\sum_{i=1}^{n-1} m_i = m - m_n} r_1^{m_1} \cdot \cdots \cdot r_{n-1}^{m_{n-1}}\right) = \sum_{m_n=0}^{m} Q_{m_n}^{a^*(x)}.$$

Thus, since $\lim_{m\to\infty} Q_m^{a(x)}$ exists, $\sum_{m=0}^{\infty} Q_m^{a^*(x)}$ exists, both equal $1/a^*(1)$, and all the roots of $a^*(x)$ lie in $\{z \mid |z| < 1\}$ by Corollary 3.2.

If $a(x)$ has a root of 1, and all the roots of $a^*(x) = a(x)/(x - 1)$ lie in $\{z \mid |z| < 1\}$, by Corollary 3.2,

$$\sum_{m=0}^{\infty} Q_m^{a^*(x)} = 1/(a^*(1))$$

and from equation (12), $\lim_{m\to\infty} Q_m^{a(x)} = 1/(a^*(1))$.

4. RELATED THEOREMS

Theorems 3.1 and 3.3 together give necessary and sufficient conditions for convergence, in the usual sense, of a linear recursion sequence to a complex number. We will now consider some other aspects of linear recursion sequences related to convergence. The next theorem concerns the ratio of consecutive terms of a linear recursion sequence.

Theorem 4.1: Let $\{Q_i^{a(x)}\}$ be a linear recursion sequence. If among the roots of largest norm for $a(x)$ there is a unique root, r_n, of greatest multiplicity, then there exists $N > 0$ such that if $m > N$, $Q_{m+1}^{a(x)}/Q_m^{a(x)}$ exists, and $\lim Q_{m+1}^{a(x)}/Q^{a(x)} = r_n$.

Proof: Let $r_1, \ldots, r_n$ be the n roots of $a(x)$ (repeated according to their multiplicity). Let r_n be as described in the theorem, and the r_i's be arranged so that $|r_i| < |r_n|$ for $i = 1, \ldots, j$ (j could be 0) and $|r_i| = |r_n|$ for $i = j + 1, \ldots, n - 1$. Using (4), we may then write

$$Q_m^{a(x)} = C_1^{a(x)} r_{1,d_1}^{(m)} + C_2^{a(x)} r_{2,d_2}^{(m)} + \cdots + C_n^{a(x)} r_{n,d_n}^{(m)} \quad \text{for } m > 0 \tag{13}$$

where $C_i^{a(x)} = D_i^{a(x)}/D^{a(x)}$ are constants depending on $a(x)$. Jarden [3, p. 107] observes that $D^{a(x)} \neq 0$, so this quotient is defined, for such $i = 1, \ldots, n$. Also notice using the definition of $D_n^{a(x)}$ that $D_n^{a(x)} = D_n^{a(x)/x-r_n} \neq 0$, again by Jarden's observation [3, p. 107], thus, $C_n \neq 0$, a fact we will need shortly. From the definition of $r_{i,d_i}^{(m)}$ in (6), we may write

$$Q_m^{a(x)} = \sum_{i=1}^{n} C_i^{a(x)} r_i^{m-d_i} \text{ for } m > n. \tag{14}$$

We next form the new equation

$$Q_m^{a(x)}/r_n^{m-d_n} = \sum_{i=1}^{n} C_i^{a(x)} \binom{m}{d_i}\left(r_i^{m-d_i}/r_n^{m-d_n}\right) = \sum_{i=1}^{n} E_i^{a(x)} \binom{m}{d_i}(r_i/r_n)^{m-d_n} \tag{15}$$

where $E_i^{a(x)} = C_i^{a(x)} r_i^{d_n-d_i}$ are constants depending on $a(x)$. Thus,

$$\lim_{m\to\infty} E_i^{a(x)}\binom{m}{d_i}(r_i/r_n)^{m-d_n} = 0 \quad \text{for } i = 1, \ldots, j. \tag{16}$$

Since $|r_i| < |r_n|$ for $i = 1, \ldots, j$ and there exists an $N > 0$ such that

$$\left|E_i^{a(x)}\binom{m}{d_i}(r_i/r_n)^{m-d_n}\right| < |G/n| \text{ for each } i = 1, \ldots, j \text{ when } m > N_0;\quad G = \max\{E_n, 1\}. \tag{17}$$

We consider two cases. If r_n is a simple root, then $j = n - 1$ and

$$\begin{aligned}1 &= \lim_{m\to\infty}\left[E_n\binom{m+1}{d_n}(r_n/r_n)^{m+1-d_n}\right]\Big/\left[E_n\binom{m}{d_n}(r_n/r_n)^{m+1-d_n}\right]\\ &= \lim_{m\to\infty}\left[E_n\binom{m+1}{d_n}(r_n/r_n)^{m+1-d_n} + \sum_{i=1}^{n-1} E_i\binom{m+1}{d_i}(r_i/r_n)^{m-d_n}\right]\Big/\left[E_n\binom{m}{d_n}(r_n/r_n)^{m-d_n} + \sum_{i=1}^{n-1} E_i\binom{m}{d_i}(r_i/r_n)^{m-d_n}\right]\\ &= \lim_{m\to\infty}\left[Q_{m+1}^{a(x)}/r_n^{m+1-d_n}\right]\Big/\left[Q_m^{a(x)}/r_n^{m-d_n}\right] = \lim_{m\to\infty}(1/r_n)\cdot Q_{m+1}^{a(x)}/Q_m^{a(x)}.\end{aligned} \tag{18}$$

Hence $\lim_{m\to\infty} Q_{m+1}^{a(x)}/Q_m^{a(x)} = r_n$. Notice that $Q_{m+1}^{a(x)}/Q_m^{a(x)}$ exists for $m > N_0$, since

$$\begin{aligned}\left|Q_m^{a(x)}/r_n^{m-d_n}\right| &= \left|\sum_{i=1}^{n-1} E_i\binom{m}{d_i}(r_i/r_n)^{m-d_n} + E_n\binom{m}{d_n}(r_n/r_n)^{m-d_n}\right|\\ &\geq -\sum_{i=1}^{n-1}\left|E_i\binom{m}{d_i}(r_i/r_n)^{m-d_n}\right| + \left|E_n\binom{m}{0}\right|\\ &\geq -\left|\frac{(n-1)\cdot E_n}{n}\right| + |E_n| > 0 \text{ for } m > N_0, \text{ and we are finished.}\end{aligned} \tag{19}$$

If r_n is not a simple root, $d_n \neq 0$. Let $M > 0$ and define

$$(20) \qquad N_M = \left(\max\left\{N_0,\ M\left[(n-j)\cdot\max_{i=j+1}^{n-1}\{|E_i|\} + d_n + 1\right]\right\}\right)|E_n|.$$

Then we find

$$(21) \qquad \left|E_n\binom{m}{d_n}\right| = \left||E_n|[(m-d_n)/d_n]\binom{m}{d_{n-1}}\right|$$
$$\geq \left|M\cdot\left((n-j)\left[\max_{i=j+1}^{n-1}\{|E_i|\}\right]+1\right)\binom{m}{d_{n-1}}\right|$$
$$\geq M\cdot\left|\sum_{i=j+1}^{n-1}\left|E_i\binom{m}{d_n-1}\right|+1\right|$$
$$\geq M\cdot\left|\sum_{i=j+1}^{n-1}\left|E_i\binom{m}{d_n-1}(r_i/r_n)^{m-d_n}\right|+\sum_{i=1}^{j}\left|E_i\binom{m}{d_i}(r_i/r_n)^{m-d_n}\right|\right|$$
$$\geq M\cdot\left|\sum_{i=1}^{n-1}E_i\binom{m}{d_i}(r_i/r_n)^{m-d_n}\right| \quad \text{for } m > N_M.$$

Therefore, $Q_{m+1}^{a(x)}/Q_m^{a(x)}$ exists for $m > N_1$. Furthermore, if

$$(22) \qquad h_m = \left(\sum_{i=1}^{n-1}E_i\binom{m}{d_i}(r_i/r_n)^{m-d_n}\right)\Big/E_n\binom{m}{d_n}$$

by (21), $\lim_{m\to\infty} h = 0$ and thus

$$(23) \qquad 1 = \lim_{m\to\infty}\binom{m+1}{d_n}\Big/\binom{m}{d_n} = \lim_{m\to\infty}\left[E_n\binom{m+1}{d_n}(1+h_{m+1})\right]\Big/\left[E_n\binom{m}{d_n}(1+h_m)\right]$$
$$= \lim_{m\to\infty}\left(Q_{m+1}^{a(x)}/r_n^{m+1-d_n}\right)\Big/\left(Q_m^{a(x)}/r_n^{m-d_n}\right) = \lim_{m\to\infty} 1/r_n \cdot Q_{m+1}^{a(x)}/Q_m^{a(x)}$$
$$= 1/r_n \lim_{m\to\infty} Q_{m+1}^{a(x)}/Q_m^{a(x)},$$

and hence $\lim_{m\to\infty} Q_{m+1}^{a(x)}/Q_m^{a(x)} = r_n$.

Theorem 4.2: Let $\{Q_m^{a(x)}\}$ be a linear recursion sequence, with all of the roots of $a(x)$ in $\{z\,|\,|z| \leq 1\}$, and let $r_{j+1}, \ldots, r_n$ be the roots of $a(x)$ in $\{z\,|\,|z| = 1\}$. If for each $i = j+1, \ldots, n$, r_i is a simple root, $r_i^m = 1$ for some integer $m > 0$, and m_i is the least positive integer with $r_i^{m_i} = 1$, then there exists $\{P_m\}_{m=1}^{\infty}$, a periodic sequence of period L.C.M. $\{m_{j+1}, \ldots, m_n\}$ such that

$$\lim_{m\to\infty}(Q_m^{a(x)} - P_m) = 0.$$

Proof: Let $r_1, \ldots, r_n$ be the roots of $a(x)$, repeated according to their multiplicity, with $r_{j+1}, \ldots, r_n$ as described in the theorem. Using (4), we may write

$$(24) \qquad Q_m^{a(x)} = \sum_{i=1}^{n} C_i^{a(x)} r_{i,d_i}^{(m)} \quad \text{for } m > 0$$

where $C_i^{a(x)} = D_i^{a(x)}/D^{a(x)}$ from equation (5). Evaluating $r_{i,d_i}^{(m)}$, we find

$$(25) \qquad Q_m^{a(x)} = \sum_{i=1}^{n} C_i^{a(x)}\binom{m}{d_i} r_i^{m-d_i} \quad \text{for } m > n.$$

Notice that

$$(26) \qquad \lim_{m\to\infty}\left(C_i^{a(x)}\binom{m}{d_i}r_i^{m-d_i}\right) = 0 \quad \text{for } i = 1, \ldots, j.$$

Hence, for each k, there exists an $N_k > 0$ such that

(27) $$\left|C_i^{a(x)}\binom{m}{d_i}r_i^{m-d_i}\right| < 1/(j\cdot k) \text{ for each } i = 1, \ldots, j.$$

Therefore,

(28) $$0 = \lim_{m\to\infty}\left[\left(\sum_{i=1}^{n}C_i^{a(x)}\binom{m}{d_i}r_i^{m-d_i}\right) - \left(\sum_{i=j+1}^{n}C_i^{a(x)}\binom{m}{d_i}r_i^{m-d_i}\right)\right]$$
$$= \lim_{m\to\infty}\left(Q_m^{a(x)} - \sum_{i=j+1}^{n}C_i^{a(x)}\binom{m}{0}r_i^{m}\right) = \lim_{m\to\infty}(Q_m^{a(x)} - P_m).$$

The powers of the r_i, $i = j+1, \ldots, n$ are periodic sequences of period m_i, and hence

$P = \sum_{i=j+1}^{n} C_i^{a(x)} r_i^m$ is a periodic sequence of period L.C.M. $\{m_{j+1}, \ldots, m_n\}$.

5. NONNORMALIZED SEQUENCES

A sequence $\{Q_m^{a(x),U}\}$ may be written as a linear combination of the terms in the sequence $\{Q_m^{a(x)}\}$ by modeling it after (2)

(29) $$\begin{bmatrix}x_n\\x_{n-1}\\\vdots\\x_1\end{bmatrix} = \begin{bmatrix}u_n\\u_{n-1}\\\vdots\\u_1\end{bmatrix}\cdot\begin{bmatrix}a_1 & a_2 & \ldots & a_n\\1 & 0 & \ldots & 0\\0 & 1 & \ldots & 0\\\vdots & \vdots & & \vdots\\0 & \ldots & 1 & 0\end{bmatrix}^{-1}$$

where the matrix has an inverse, since its characteristic polynomial $a(x)$ has no zero roots. We may then write equation (2) for the nonnormalized sequence

(30) $$\begin{bmatrix}a_1 & a_2 & \ldots & a_n\\1 & 0 & \ldots & 0\\0 & 1 & \ldots & 0\\\vdots & \vdots & & \vdots\\0 & \ldots & 1 & 0\end{bmatrix}^{m}\cdot\begin{bmatrix}x_n\\x_{n-1}\\\vdots\\x_1\end{bmatrix} = \begin{bmatrix}Q_m^{a(x),U}\\Q_{m-1}^{a(x),U}\\\vdots\\Q_{m-n+1}^{a(x),U}\end{bmatrix}$$

or also

(31) $$Q_m^{a(x),U} = \sum_{i=1}^{n} X_{n+1-i}Q_{m+1-i}^{a(x)}, \quad m > 0.$$

Hence, we may use the normalized sequence to determine convergence for the nonnormalized sequence.

6. CONCLUSION

The theorems proved in this paper rely heavily on the relationship of the roots of the auxiliary polynomial to the region $\{z \mid |z| < 1\}$. A problem in Wall's book [7, p. 190] gives exact computations to determine this relationship. So given a linear recursion sequence and its auxiliary polynomial, it can be decided whether it converges to 0, to a nonzero complex number, or is nonconvergent in the usual sense.

Necessary and sufficient conditions for "convergence" to infinity are not given, and are not known to this author. Theorem 4.1 gives a sufficient condition that if there exists a root r_n of $a(x)$ which has norm larger than or equal to all other roots and has greatest multiplicity among the roots of its norm. The sufficiency of the conditions excluded by Theorems 3.1, 3.3, and 4.2 may be refuted by considering $a(x) = x^2 - 2$. The roots of $a(x)$ are $-\sqrt{2}$, $+\sqrt{2}$, both of which lie outside $\{z \mid |z| \le 1\}$, but the sequence begins 1, 0, 2, 0, 4, 0, 8, 0, 16, 0, ... and thus does not "converge" to infinity. This example also shows that Theorem 4.1 cannot include certain examples where there are two or more roots of largest norm and equally great multiplicity, since the quotient $Q_{m+1}^{a(x)}/Q_m^{a(x)} = 0$ for even m, and does not exist for odd m.

The original problem of Singmaster [5] asked if the conditions that the a_i all be real $a_i \geq 0$ for $i = 1, \ldots, n$, $a_1 > 0$, and $\sum_{i=1}^{n} a_i = 1$ were sufficient for $\lim_{m\to\infty} Q_m^{a(x)} = b \neq 0$. The answer to this question is affirmative. Looking at equation (2), the matrix may be viewed as a stationary Markov transition matrix, and by Doob [2, p. 256] the powers of the matrix converge. Thus, $\lim_{m\to\infty} Q_m^{a(x)}$ exists. Since $\sum_{i=1}^{n} a_i = 1$, 1 is a root of $a(x)$, and so, by Theorem 3.3, $\lim_{m\to\infty} Q_m^{a(x)} = a^*(1)$ where $a^*(x) = \dfrac{a(x)}{(x - 1)}$.

REFERENCES

1. E. Bodewig. *Matrix Calculus*. Amsterdam: North-Holland Publishing Company, 1959, p. 57.
2. J. L. Doob. *Stochastic Processes*. New York: John Wiley & Sons, Inc., 1953, p. 256.
3. D. Jarden. *Recurring Sequences*. Jerusalem: Riveon Lematematika, 1966, pp. 105-107.
4. L. Raphael. "Linearly Recursive Sequences of Integers." *The Fibonacci Quarterly* 12, No. 1 (1974):11-38.
5. A. G. Shannon. "Some Properties of a Fundamental Recursive Sequence of Arbitrary Order." *The Fibonacci Quarterly* 12, No. 4 (1974):327-334.
6. D. Singmaster. "Problem H-179." *The Fibonacci Quarterly* 9, No. 1 (1971):62.
7. H. P. Wall. *Continued Fractions*. Princeton, N.J.: D. Van Nostrand Company, Inc., 1948, p. 190.

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