$$= \frac{C_p(1, 0, 1, 0, 0, \ldots, 0)/2}{C_p(1, 1, 0, 0, 0, \ldots, 0)/2} = \frac{2/2}{2/2} = 1.$$

Thus, $C_{2p}(x_1, \ldots, x_{2p}) = p^2$.

Theorem 9: When $x_1 = j + 1$, $x_2 = j$, $x_3 = x_4 = \cdots = x_{p+1} = j + 1$, $x_{p+2} = \cdots = x_{2p} = j$, one has $C_{2p}(x_1, \ldots, x_{2p}) = (2j + 1)p^2$.

Proof: $C_{2p}(x_1, \ldots, x_{2p}) = p^2(2pj + p)/p = (2j + 1)p^2$, by using Lemma 5 and Theorem 1.

Theorem 10: $V_{2p} = [\{m : p \nmid m\} \cup \{m : p^2 \mid m\}] \cap [\{m : 2 \nmid m\} \cup \{m : 4 \mid m\}]$.

Proof: By Theorems 6 and 7, no other values are possible. The only possible values are the integers not divisible by 2 or p [by using $A(2p, r, j)$ with $\gcd(2p, r) = 1$], the multiples of 4 that are not divisible by p (by using Theorem 8), the multiples of p^2 that are not divisible by 2 (by using Theorem 9), and the multiples of $4p^2$ (by using Theorem 3). Thus, V_{2p} consists of the integers m satisfying either $p \nmid m$ or $p^2 \mid m$ and also satisfying either $2 \nmid m$ or $4 \mid m$.

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POWERS OF MATRICES AND RECURRENCE RELATIONS

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0. INTRODUCTION

This article arose out of the desire to demonstrate an interesting and perhaps initially surprising application of the theory of matrices to final year high school students. Thus, we consider a matrix-theoretic approach to firstly the solution of two simultaneous first-order recurrence relations and secondly to the solution of a single second-order recurrence relation, together with the proofs of a few identities.

It is well known that the solution of an mth order linear homogeneous recurrence relation can be found by means of the theory of matrices. Indeed, Rosenbaum [4] gave an approach which is based on the Jordan normal form; the reader should also see the recent article [5] of Ryavec. The technique used in Section 1 of this paper is based upon the Cayley-Hamilton theorem for 2 x 2 matrices and is particularly elementary. A novel feature of Section 2 is the use of 2 x 2 matrices to obtain generalizations of a few well-known identities which interrelate the Fibonacci and Lucas numbers.

1. POWERS OF 2 x 2 MATRICES

Let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ be a 2 x 2 matrix whose entries are real, or even complex, numbers.

The *characteristic polynomial* of A is

$$\det(\lambda I - A) = \begin{vmatrix} \lambda - a_{11} & -a_{12} \\ -a_{21} & \lambda - a_{22} \end{vmatrix} = \lambda^2 - (a_{11} + a_{22}) + (a_{11}a_{22} - a_{12}a_{21}).$$

It can be verified by direct computation that

$$A^2 - (a_{11} + a_{22})A + (a_{11}a_{22} - a_{12}a_{21})I = 0.$$

This is a special case of the famous Cayley-Hamilton theorem which says that if

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mm} \end{pmatrix} \text{ is an } m \times m \text{ matrix } (m \geq 1)$$

and $c(\lambda) = \det(\lambda I - A) = \lambda^m + c_{m-1}\lambda^{m-1} + \cdots + c_1\lambda + c_0$, then $c(A) = 0$ in the sense that $A^m + c_{m-1}A^{m-1} + \cdots + c_1 A + c_0 I = 0$. Here, $c_{m-1} = -(a_{11} + a_{22} + \cdots + a_{mm})$ and $c_0 = (-1)^m$ det A, as is consistent with the above case when $m = 2$.

Let μ_1 and μ_2 be the roots of the *characteristic equation*

$$\lambda^2 - (a_{11} + a_{22}) + (a_{11}a_{22} - a_{12}a_{21}) = 0.$$

Hence, $\mu_1 + \mu_2 = a_{11} + a_{22}$ and $\mu_1\mu_2 = a_{11}a_{22} - a_{12}a_{21}$. Thus,

$$(A - \mu_1 I)(A - \mu_2 I) = A^2 - (\mu_1 + \mu_2)A + \mu_1\mu_2 I = 0 = (A - \mu_2 I)(A - \mu_1 I)$$

by the Cayley-Hamilton theorem; we will use these relations in the proofs of equations 1.2 and 1.7 below.

Case 1: $\mu_1 \neq \mu_2$.

Firstly, let us assume that the roots μ_1, μ_2 of the characteristic equation are distinct. Then, we may meaningfully introduce the matrices

$$E_1 = \frac{1}{\mu_1 - \mu_2}(A - \mu_2 I), \quad E_2 = \frac{1}{\mu_2 - \mu_1}(A - \mu_1 I),$$

which have the following properties:

(1.1) $$E_1 + E_2 = I.$$

The proof is a direct computation:

(1.2) $$E_1E_2 = 0 = E_2E_1;$$

(1.3) $$E_1^2 = E_1 \text{ and } E_2^2 = E_2.$$

Proof: By (1.1) and (1.2), $E_1 = E_1 I = E_1(E_1 + E_2) = E_1^2 + E_1E_2 = E_1^2 + 0 = E_1^2$. Similarly, $E_2^2 = E_2$.

(1.4) $$A = \mu_1 E_1 + \mu_2 E_2.$$

By (1.2), E_1 and E_2 commute, hence (1.4) and the "binomial theorem" yield

$$(\mu_1 E_1 + \mu_2 E_2)^n = \mu_1^n E_1^n + \binom{n}{1}\mu_1^{n-1}\mu_2 E_1^{n-1}E_2 + \cdots + \mu_2^n E_2^n \text{ for } n \geq 1,$$

and so

$$A^n = \mu_1^n E_1 + \mu_2^n E_2 \text{ for } n \geq 1.$$

To take into account the case of $n = 0$ and the possibility that one of μ_1 and μ_2 is zero, *we adopt the definitions*

$$A^0 = I \text{ and } 0^0 = 1;$$

the latter definition is not so common, yet it will be useful in all that follows. Thus, these definitions and (1.1) allow us to assert that

$$A^n = \mu_1^n E_1 + \mu_2^n E_2 \text{ for } n \geq 0$$

and then substituting for E_1 and E_2, we obtain

(1.5) $$A^n = \frac{1}{\mu_1 - \mu_2}\begin{pmatrix} a_{11}(\mu_1^n - \mu_2^n) + (\mu_1\mu_2^n - \mu_2\mu_1^n) & a_{12}(\mu_1^n - \mu_2^n) \\ a_{21}(\mu_1^n - \mu_2^n) & a_{22}(\mu_1^n - \mu_2^n) + (\mu_1\mu_2^n - \mu_2\mu_1^n) \end{pmatrix}, \quad n \geq 0.$$

Case 2: $\mu_1 = \mu_2 = \mu$.

Secondly, we assume that the characteristic equation has a repeated root $\mu = \mu_1 = \mu_2$. Let $H = A - \mu I$. Then we have the following properties:

(1.6) $$A = \mu I + H;$$

(1.7) $$H^2 = 0.$$

Proof: $H^2 = (A - \mu I)^2 = (A - \mu_1 I)(A - \mu_2 I) = 0.$

Because μI and H commute, the "binomial theorem," (1.6) and (1.7) give

(1.8) $$A^n = \mu^n I + n\mu^{n-1}H,\ n \geq 1.$$

Substituting for H in terms of A we obtain

(1.9) $$A^n = \mu^{n-1}\begin{pmatrix} n(a_{11} - \mu) + \mu & na_{12} \\ na_{21} & n(a_{22} - \mu) + \mu \end{pmatrix},\ n \geq 1.$$

We now consider the simultaneous first-order linear recurrence relations

(1.10) $$\begin{aligned} y_{n+1} &= a_{11}y_n + a_{12}z_n \\ z_{n+1} &= a_{21}y_n + a_{22}z_n \end{aligned}$$

which hold for $n \geq 0$, and wherein the coefficients a_{ij} are independent of n.

In terms of matrices, (1.10) can be expressed in the form

(1.11) $$\begin{pmatrix} y_n & y_{n+1} \\ z_n & z_{n+1} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} y_{n-1} & y_n \\ z_{n-1} & z_n \end{pmatrix},\ n \geq 1.$$

Hence,

(1.12) $$\begin{pmatrix} y_n & y_{n+1} \\ z_n & z_{n+1} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}^n \begin{pmatrix} y_0 & y_1 \\ z_0 & z_1 \end{pmatrix},\ n \geq 0,$$

or, more briefly,

(1.13) $$Y_n = A^n Y_0,\ n \geq 0,$$

where

$$Y_n = \begin{pmatrix} y_n & y_{n+1} \\ z_n & z_{n+1} \end{pmatrix} \text{ for } n \geq 0 \text{ and } A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

Applying (1.5) and (1.9) to yield expressions for A^n and then equating the elements in the first row and column of left-hand and right-hand sides of (1.13) gives

(1.14) *Theorem*: Let μ_1 and μ_2 be the roots of the characteristic equation of the matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

and consider the recurrence relations

$$\begin{aligned} y_{n+1} &= a_{11}y_n + a_{12}z_n \\ z_{n+1} &= a_{21}y_n + a_{22}z_n,\ n \geq 0. \end{aligned}$$

If μ_1 and μ_2 are distinct, then for any $n \geq 0$,

$$y_n = \frac{(a_{11}y_0 + a_{12}z_0)(\mu_1^n - \mu_2^n) + y_0(\mu_1\mu_2^n - \mu_2\mu_1^n)}{\mu_1 - \mu_2}$$

$$z_n = \frac{(a_{21}y_0 + a_{22}z_0)(\mu_1^n - \mu_2^n) + z_0(\mu_1\mu_2^n - \mu_2\mu_1^n)}{\mu_1 - \mu_2}.$$

If μ_1 and μ_2 are both equal to μ, then for any $n \geq 1$,

$$y_n = \mu^{n-1}(n(a_{11}y_0 + a_{12}z_0 - \mu y_0) + \mu y_0)$$

$$z_n = \mu^{n-1}(n(a_{21}y_0 + a_{22}z_0 - \mu z_0) + \mu z_0).$$

We now consider the second-order linear recurrence relation

(1.15) $$u_{n+2} = au_{n+1} + bu_n,$$

which holds for $n \geq 0$, and wherein a and b are independent of n.

Of course, (1.15) can be regarded as a special case of (1.10) if we set $y_n = u_n$ and $z_n = u_{n+1}$ for all $n \geq 0$, $a_{11} = 0$, $a_{12} = 1$, $a_{21} = b$, and $a_{22} = a$. Then (1.13) gives

(1.16) $$U_n = B^n U_0, \; n \geq 0,$$

where

$$U_n = \begin{pmatrix} u_n & u_{n+1} \\ u_{n+1} & u_{n+2} \end{pmatrix} \text{ for } n \geq 0 \text{ and } B = \begin{pmatrix} 0 & 1 \\ b & a \end{pmatrix}.$$

Moreover, Theorem 1 specializes to yield the following important and well-known result.

(1.17) *Corollary*: Let μ_1 and μ_2 be the roots of the characteristic equation of the matrix

$$B = \begin{pmatrix} 0 & 1 \\ b & a \end{pmatrix}$$

and consider the recurrence relation

$$u_{n+2} = au_{n+1} + bu_n, \; n \geq 0.$$

If μ_1 and μ_2 are distinct, then for any $n \geq 0$,

$$u_n = \frac{u_1(\mu_1^n - \mu_2^n) + u_0(\mu_1\mu_2^n - \mu_2\mu_1^n)}{\mu_1 - \mu_2}.$$

If μ_1 and μ_2 are both equal to μ, then for any $n \geq 1$,

$$u_n = \mu^{n-1}(n(u_1 - \mu u_0) + u\mu_0).$$

We close this section with an example which occurs as Exercise 3-9 in [3, p. 92]. Beforehand, we note two consequences of (1.16) and (1.17) for the special case of the Fibonacci numbers, which are defined by the recurrence $f_{n+2} = f_{n+1} + f_n$ for $n \geq 0$, where $f_0 = 0$ and $f_1 = 1$. Then (1.16) shows that

$$\begin{pmatrix} f_n & f_{n+1} \\ f_{n+1} & f_{n+2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^{n+1},$$

and (1.17) gives Binet's formula

$$f_n = \frac{\mu_1^n - \mu_2^n}{\sqrt{5}}, \text{ where } \mu_1 = \frac{1 + \sqrt{5}}{2} \text{ and } \mu_2 = \frac{1 - \sqrt{5}}{2} = -\frac{1}{\mu_1}.$$

Define two sequences $\{y_n\}$, $\{z_n\}$ in terms of binomial coefficients by

$$y_n = \sum_{k=0}^{n-1} \binom{n+k}{2k+1} \text{ when } n \geq 1, \text{ and } y_0 = 0,$$

$$z_n = \sum_{k=0}^{n} \binom{n+k}{2k} \text{ when } n \geq 1, \text{ and } z_0 = 1.$$

Using Pascal's relation $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$, it readily follows that

$$y_{n+1} = y_n + z_n$$

$$z_{n+1} = y_{n+1} + z_n, \text{ for } n \geq 0.$$

Whence we obtain the special case

$$y_{n+1} = y_n + z_n,\ n \geq 0,\ y_0 = 0$$
$$z_{n+1} = y_n + 2z_n,\qquad z_0 = 1$$

of (1.10). Here

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} f_1 & f_2 \\ f_2 & f_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^2$$

in terms of the Fibonacci numbers. Then (1.13) yields

$$\begin{pmatrix} y_n & y_{n+1} \\ z_n & z_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^{2n} \begin{pmatrix} y_0 & y_1 \\ z_0 & z_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^{2n} \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} f_{2n-1} & f_{2n} \\ f_{2n} & f_{2n+1} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} \text{ for } n \geq 1.$$

Hence, $y_n = f_{2n}$, $z_n = f_{2n+1}$ for $n \geq 0$, and Binet's formula provides closed expressions for y_n and z_n.

2. SOME IDENTITIES

In this section we restrict ourselves to demonstrating a few identities concerning sequences $\{u_n\}$ which satisfy (1.15) and which are suggested by (1.16) and (1.17). Basin and Hoggatt [1] and Bicknell [2] have previously used matrix techniques to establish identities satisfied by sequences which are defined by specializations of (1.15) and we follow their techniques. It is worth noting that in [6] Waddill used different matrix techniques to obtain identities.

(2.1) *Proposition:* Let $\{u_n\}$ be a sequence satisfying the recursion formula $u_{n+2} = au_{n+1} + bu_n$ for $n \geq 0$. Then, for $n \geq 1$,

$$u_{n-1}u_{n+1} - u_n^2 = (-b)^{n-1}(u_0u_2 - u_1^2) \textit{ and } u_{2n} = \sum_{r=0}^{n} \binom{n}{r} a^{n-r}b^r u_r.$$

Proof: Taking determinants in (1.16) gives det U_n = (det $B)^n$det U_0, i.e.,

$$u_nu_{n+2} - u_{n+1}^2 = (-b)^n(u_0u_2 - u_1^2) \text{ for } u \geq 0.$$

Replacing n by $n - 1$, we get the first identity of (2.1).

The Cayley-Hamilton theorem implies that $B^2 = aB + bI$ and so (1.16) gives

$$U_{2n} = B^{2n}U_0 = (aB + bI)^nU_0 = \left(\sum_{r=0}^{n} \binom{n}{r} a^r b^{n-r} B^r\right)U_0$$

and the second identity follows.

When $b = 1$, i.e., the sequence $\{u_n\}$ is given by $u_{n+2} = au_{n+1} + u_n$, another type of identity is easily derived from (1.16). For then $B = \begin{pmatrix} 0 & 1 \\ 1 & a \end{pmatrix}$ and the set of matrices which commute with $\begin{pmatrix} 0 & 1 \\ 1 & a \end{pmatrix}$ is $\left\{\begin{pmatrix} x & y \\ y & ay+x \end{pmatrix} : x, y \text{ arbitrary}\right\}$. In particular, U_0 commutes with B and so

$$U_mU_n = B^mU_0B^nU_0 = B^{m+n}U_0^2 = U_{m+n}U_0.$$

Hence, we obtain

(2.2) *Proposition:* Let $\{u_n\}$ be a sequence satisfying the recursion formula $u_{n+2} = au_{n+1} + u_n$ for $n \geq 0$. Then, for any m, $n \geq 0$,

$$u_mu_n + u_{m+1}u_{n+1} = u_{m+n}u_0 + u_{m+n+1}u_1,$$

and

$$u_mu_{n+1} + u_{m+1}u_{n+2} = u_{m+n}u_1 + u_{m+n+1}u_2.$$

(2.3) *Lemma:* Let x and y be arbitrary and $n \geq 0$. Then,

$$\begin{pmatrix} \frac{x+y}{2} & \frac{x-y}{2} \\ \frac{x-y}{2} & \frac{x+y}{2} \end{pmatrix}^n = \begin{pmatrix} \frac{x^n+y^n}{2} & \frac{x^n-y^n}{2} \\ \frac{x^n-y^n}{2} & \frac{x^n+y^n}{2} \end{pmatrix}$$

Proof: Of course (2.3) can be proved by induction. However, we will give a proof which is in the spirit of this paper. Let

$$E = \tfrac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad F = \tfrac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

Then, $E^2 = E$, $F^2 = F$, $EF = 0 = FE$, $E + F = I$. In addition,

$$\begin{pmatrix} \frac{x+y}{2} & \frac{x-y}{2} \\ \frac{x-y}{2} & \frac{x+y}{2} \end{pmatrix} = xE + yF. \quad \text{Hence,} \quad \begin{pmatrix} \frac{x+y}{2} & \frac{x-y}{2} \\ \frac{x-y}{2} & \frac{x+y}{2} \end{pmatrix}^n = x^nE + y^nF,$$

and the lemma follows.

(2.4) *Proposition:* Suppose the roots μ_1, μ_2 of the characteristic equation of the matrix

$$B = \begin{pmatrix} 0 & 1 \\ b & a \end{pmatrix}$$

are distinct. Let $\{p_n\}$ and $\{q_n\}$ be two sequences satisfying the recursion formula $u_{n+2} = au_{n+1} + bu_n$ and such that $p_0 = 0$, $p_1 = 1$, $q_0 = 2$, $q_1 = a$. Then, for $m, n \geq 0$,

$$\begin{aligned} q_n^2 - (a^2 + 4b)p_n^2 &= 4(-b)^n, \\ q_mq_n + (a^2 + 4b)p_mp_n &= 2q_{m+n}, \\ q_mp_n + q_np_m &= 2p_{m+n}. \end{aligned}$$

Proof: Applying (1.17) and simplifying, we obtain $p_n = \dfrac{\mu_1^n - \mu_2^n}{\mu_1 - \mu_2}$ and $q_n = \mu_1^n + \mu_2^n$ for any $n \geq 0$. Hence, (2.3) yields

$$\tfrac{1}{2}\begin{pmatrix} q_n & (\mu_1 - \mu_2)p_n \\ (\mu_1 - \mu_2)p_n & q_n \end{pmatrix} = \begin{pmatrix} \frac{\mu_1+\mu_2}{2} & \frac{\mu_1-\mu_2}{2} \\ \frac{\mu_1-\mu_2}{2} & \frac{\mu_1+\mu_2}{2} \end{pmatrix}^n, \; n \geq 0. \tag{2.5}$$

Since $\mu_1\mu_2 = -b$ and $\mu_1 + \mu_2 = a$, $(\mu_1 - \mu_2)^2 = (\mu_1 + \mu_2)^2 - 4\mu_1\mu_2 = a^2 + 4b$. Using this observation and taking determinants in (2.5), we obtain the first identity of (2.4).

Equation (2.5) implies that for $m, n \geq 0$,

$$\begin{pmatrix} q_m & (\mu_1 - \mu_2)p_m \\ (\mu_1 - \mu_2)p_m & q_m \end{pmatrix}\begin{pmatrix} q_n & (\mu_1 - \mu_2)p_n \\ (\mu_1 - \mu_2)p_n & q_n \end{pmatrix} = 2\begin{pmatrix} q_{m+n} & (\mu_1 - \mu_2)p_{m+n} \\ (\mu_1 - \mu_2)p_{m+n} & q_{m+n} \end{pmatrix}. \tag{2.6}$$

Equating the elements in the first row and column of the left and right of (2.6) and also doing the same for the elements in the first row and second column gives, after simplification, the remaining two identities of (2.4).

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CONVERGENCE PROPERTIES OF LINEAR RECURSION SEQUENCES

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1. INTRODUCTION

The object of this paper is to examine convergence properties of linear recursion sequences of complex numbers. Included are several theorems providing necessary and sufficient conditions, in terms of solutions of an associated auxiliary equation, for various cases and types of convergence.

The question of convergence of linear recursion sequences was raised by Singmaster in Advanced Problem H-179 [6]. The articles of Raphael [4], Shannon [5], and Jarden [3] give representations for linear recursion sequences of integers which are valid also for complex number sequences (the restriction being for aesthetic reasons) and have been useful in preparing this paper. These representations will be included without proof as the substance of the next section.

Let $a_1, a_2, \ldots, a_n$ be complex numbers, with $a_n \neq 0$. We define a linear recursion sequence $\{Q_i^{a(x),U}\}$ by

$$Q_i^{a(x),U} = \sum_{j=1}^{n} a_j Q_{i-j}^{a(x),U} \quad \text{for } n \geq 1 \tag{1}$$

where $U = [u_1, u_2, \ldots, u_n]$, $Q_{i-n}^{a(x),U} = u_i$ for $1 \leq i \leq n$, and $a(x) = x^n - a_1x^{n-1} - a_2x^{n-2} \ldots - a_n$. We will refer to $a(x) = 0$ as the auxiliary equation. The absence of the row vector U from the notation will imply that $U = [0, 0, \ldots, 0, 1]$, representing the normalized sequence we will be most concerned with in this paper. The order of the sequence $\{Q^{a(x),U}\}$ is n, and hence the restriction that $a_n \neq 0$ incures a unique definition of order.

2. REPRESENTATIONS

Some representations for linear recursion sequences will be helpful, and are presented here.

Noticing that the recursion relation (1) has a form similar to that of scalar multiplication of n-tuples leads to a matrix approach, presented for instance in Raphael [4]. Explicitly, we may write

$$\begin{bmatrix} a_1 & a_2 & \ldots & a_n \\ 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & & & \vdots \\ 0 & \ldots & 1 & 0 \end{bmatrix}^m \cdot \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} Q_m^{a(x)} \\ Q_{m-1}^{a(x)} \\ \vdots \\ Q_{m-n+1}^{a(x)} \end{bmatrix} \quad \text{for } m \geq 0. \tag{2}$$

Another approach by Raphael [4] relates linear recursion sequences to power series in the following way:

$$\sum_{i=0}^{\infty} Q_i^{a(x)} x^i = 1/(1 - a_1x - a_2x^2 - \cdots - a_nx^n). \tag{3}$$

Let $r_1, \ldots, r_n$ be the n complex roots of $a(x)$ (repeated according to their multiplicity). Then, as Jarden [3, pp. 106-107] noted,