	$\tilde{\theta}$	$P_0(\tilde{\theta})$	$P(x\|\tilde{\theta})$	$P(x\tilde{\theta})$	$P_1(\tilde{\theta})$	Regret a_1	Regret a_2
$x = 2$:	0.3	0.7	0.26	0.182	0.92	1500	0
	0.1	0.3	0.29	0.015	0.23	0	6000
				0.197		1380	480*
$x = 3$:	0.3	0.7	0.08	0.056	1	1500	0
	0.1	0.3	0.0	0	0	0	6000
				0.056		1500	0*
$x = 4$:	0.3	0.7	0.01	0.007	1	1500	0
	0.1	0.3	0.0	0	0	0	6000
				0.007		1500	0*

Summary of Posterior Expected Regret for $n = 4$, X

X	Decision	Marginal Probability	Regret
0	a_1	0.366	690
1	a_1	0.374	1155
2	a_2	0.197	480
3	a_2	0.056	0
4	a_2	0.007	0
		Posterior Expected Regret:	779.07

Prior EVPI	1050.00
Post EVPI	-779.07
EVSI (4)	270.93
$C(n = 4)$	-200.00
ENGS (4)	70.93

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SIMULTANEOUS TRIBONACCI REPRESENTATIONS

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1. INTRODUCTION AND DEFINITIONS

The two-sided sequence $\{t_n\}_{-\infty}^{\infty}$ of Tribonacci numbers is defined by $t_{-1} = 0$, $t_0 = 0$, $t_1 = 1$ and the recursion $t_{n+3} = t_{n+2} + t_{n+1} + t_n$. A Tribonacci representation of the integer a is an expression $a = \Sigma K_i t_i$ where $\{K_n\}_{-\infty}^{\infty}$ is a finitely nonzero sequence of integers.

This paper attempts to generalze to Tribonacci representations some of the results of Robert Silber's and my joint paper [7], "The Ring of Fibonacci Representations." I advise reading that paper before this one because, among other reasons, there one can see how much can be done in the order 2 case.

It is a pleasure to acknowledge here the extensive and essential assistance that Professor Silber gave me in working on the present paper.

Although I had originally planned to attempt to generalize all of [7], for a variety of reasons only parts of Section 3 of [7] were attempted. Some terminology must be introduced to explain these generalizations.

A finitely nonzero sequence of integers $\{K_i\}_{-\infty}^{\infty}$ will be called canonical (of order *three*) iff

A. Either (a) all the nonzero K_n are +1 or (b) all the nonzero K_n are -1;

B. No *three* consecutive k's are nonzero.

If (a) holds, we call the sequence positive canonical; if (b) holds, it is negative canonical.

Theorem 3.6 of [7] generalizes straightforwardly to: Every triple of integers (a,b,c) can be written $(\Sigma K_i t_i,\ \Sigma K_i t_{i+1},\ \Sigma K_i t_{i+2})$ for a unique canonical sequence $\{K_i\}$. These are the "simultaneous Tribonacci representations" of the title. The resolution algorithm, which (among other things) enables one to find the sequence given the triple, was altered from that of [7] not in an essential way.

For a finitely nonzero sequence $\{K_n\}$ define the upper (lower) degree to be the largest (smallest) integer p (r) such that $K_p \neq 0$ ($K_r \neq 0$). By definition, the identically zero sequence has lower degree $+\infty$ and upper degree $-\infty$. By a straightforward generalization of the order 2 case, those triples for which the associated canonical sequence has given upper degree are found.

Theorem 3.4 in [7], "Every integer r has a unique positive canonical Fibonacci representation with negative upper degree" has the not so obvious generalization "Every integer pair (a,b) can be written $(\Sigma K_i t_i,\ \Sigma K_i t_{i+1})$ for a unique positive canonical sequence $\{K_n\}$ of upper degree ≤ -2."

The above results which comprise Sections 2 and 3 of this paper, are mostly obvious enough generalizations of [7] that they are included here only because they are needed in parts of Sections 4 and 5, which attempt to answer the question of which triples have canonical sequences with given lower degree. The answer to the analogous question for the Fibonacci case is fairly easy to state (see Theorem 3.10, [7]).

Carlitz, Scoville, and Hoggatt [2] show that the solution to a problem intimately related to the Tribonacci lower degree problem is not the obvious generalization of the order 2 answer.

I have not solved the lower-degree problem. However, a computer-drawn region in the complex plane is shown to have the property that a certain algebraic expression in a, b, c, and r lies in this region if the associated canonical sequence has lower degree r.

For the above problem of Carlitz-Scoville-Hoggatt, a computer draws a diagram which divides the unit square into regions, and it is shown that this diagram is a solution to the problem in the sense that explicit formulas for this diagram would solve the problem.

In practice, however, accuracy is guaranteed only for a (probabilistic) propostion of integers. One must make some calculations with irrational numbers and plot a point on the unit square. If this point is far enough away from the curves of the diagram, one is assured of accuracy. The probability of accuracy can be increased by improving the accuracy of the calculations, of plotting points on the square, and of the diagram itself by increase computer time and improving the accuracy of the computer's sketching ability, finer tipped pens, etc.

Since there is no practical need at present for more accurate approximate solutions of this problem, I have made fiarly rough diagrams, and paid more attention to the variety of theoretical questions which appear.

Certain questions can be answered completely even with the rough sketches, though.

I had hopes before the first very rough diagram was drawn that it would turn out to be some familiar shape which would indicate the correct analytic solution. However, the complicated and unfamiliar shape that appeared indicates that any analytic solution is likely to be very complicated.

2. THE RESOLUTION ALGORITHM

Approximations to the three roots of $x^3 - x^2 - x - 1 = 0$ are:

$$\alpha = 1.839286754$$
$$\beta = -.419643377 + .606290729i$$
$$\gamma = -.419643377 - .606290729i$$

$Z[\alpha]$ forms a ring and also is a free module of dimension 3 over Z; $\{1, \alpha, \alpha^2\}$ is one basis. α is invertible in $Z[\alpha]$. In fact, $0 = \alpha^3 - \alpha^2 - \alpha - 1$ implies $\alpha^{-1} = -1 - \alpha + \alpha^2$. We take $\{\alpha^{-2}, \alpha^{-1}, 1\}$ as the standard basis of $Z[\alpha]$ over Z.

Let A be the linear transformation of IR^3 defined by $A(d,e,f) = (f,\ d+f,\ e+f)$.

<u>*Lemma 1*</u>: Given any three integers d_0, e_0, and f_0 not all zero, let $(d_n, e_n, f_n) = A\ (d_0, e_0, f_0)$. Then, for sufficiently large n, the three integers d_n, e_n, and f_n are of the same sign.

Proof: The characteristic polynomial of A is $x^3 - x^2 - x - 1$. An eigenvector associated with maximum eigenvalue α is $(1, \alpha^{-1} + 1, \alpha)$. Thus, as $n \to \infty$, either

(a) $(1, e_n/d_n, f_n/d_n) \to (1, \alpha^{-1} + 1, \alpha)$

or

(b) $(d_n, e_n, f_n) \to (0, 0, 0)$ (since $|\beta| = |\gamma| < 1$).

(This is an application of the "power method," see [5, Section 9.6].) Since d_n, e_n, and f_n are integers not all zero, (b) cannot hold. Then (a) implies that d_n, e_n, and f_n must all eventually have the same sign.

The next theorem is found generalized in [3, Theorem A] except for a slightly different version of uniqueness. Also the alternate proof of existence here is by means of a practical algorithm.

We shall call two finitely nonzero sequences of integers $\{K_n\}_{-\infty}^{\infty}$ and $\{K_n'\}_{-\infty}^{\infty}$ equivalent if $\Sigma K_n X_n = \Sigma K_n' X_n$ for every complex sequence $\{X_n\}_{-\infty}^{\infty}$ which satisfies $X_{n+3} = X_{n+2} + X_{n+1} + X_n$ for all n.

Theorem 2: For any finitely nonzero sequence of integers $\{K_n\}_{-\infty}^{\infty}$ there is a unique equivalent canonical sequence $\{K_n'\}_{-\infty}^{\infty}$. $\{K_n'\}_{-\infty}^{\infty}$ is in fact the unique canonical sequence satisfying $\Sigma K_n'\alpha^n = \Sigma K_n\alpha^n$.

Proof: *Uniqueness.*—First note that the sequence is positive or negative canonical according as $\Sigma K_n'\alpha^n$ is nonnegative or nonpositive. By factoring out a minus sign if necessary, we may now assume without loss of generality that $\{K_n'\}$ is positive.

Claim.—p is the upper degree of the sequence iff $\alpha^p \le \Sigma K_n'\alpha^n < \alpha^{p+1}$. Since $K_p' = 1$, the left-hand inequality is clear. For the right-hand inequality, note that $\{K_n'\}$ has zeros at least in one of every three consecutive terms and thus $\Sigma K_n'\alpha^n$ would be increased if the one's in the series were moved upward in position (if necessary) and then more one's added (when necessary) to form a new series $\Sigma K_n''\alpha^n$ with $K_n'' = 0$ for $n > p$, $K_p'' = 1$, $K_{p-1}'' = 1$, $K_{p-2}'' = 0$, and successive decreasing terms 1, 1, 0, 1, 1, 0, ..., ad infinitum. We obtain

$$\Sigma K_n'\alpha^n < \sum_{n=0}^{\infty} \alpha^{p-3i} + \alpha^{p-1-3i} = (\alpha^p + \alpha^{p-1})/(1 - \alpha^{-3})$$

$$= \alpha^{p+1}(\alpha^{-1} + \alpha^{-2})/(\alpha^{-1} + \alpha^{-2})$$

$$= \alpha^{p+1}.$$

Thus p is determined by the value of $\Sigma K_n'\alpha^n$. To find the next lower "one" in the sequence, merely examine the powers of a that $\Sigma K_n\alpha^n - K_p\alpha^p$ lies between. Successively subtracting off suitable powers of a will determine the positions of each of the other "ones" in the sequence.

Existence.—It is clear that the following operation replaces sequences by equivalent ones:

Choose integers n and K

Replace K_n by $K_n + K$

K_{n-1} by $K_{n-1} - K$

K_{n-2} by $K_{n-2} - K$

and K_{n-3} by $K_{n-3} - K$

Since the following *resolution algorithm* involves only repeated applications of the proper choice of this operation, existence will be proven if the algorithm is proven to terminate with a canonical sequence.

Step I: To replace $\{K_n\}_{-\infty}^{\infty}$ by an equivalent sequence in which all nonzero terms are of the same sign. If the upper degree of the sequence is p, replace K_p by 0 and add K_p to K_{p-1}, K_{p-2}, and K_{p-3}. Repeat this procedure until all nonzero terms are of like sign.

For analyzing Step I, let r denote the lower degree. If $p - r \ge 3$, p is reduced by at least 1 but r is unaltered. Thus, eventually, no more than three terms are nonzero. If $K_{p-2} = d$, $K_{p-1} = e$, $K_p = f$ and all other terms are zero, application of the procedure yields a new sequence with consecutive terms $(f, d + f, e + f)$ and all other terms zero. It now follows from Lemma 1 that eventually all nonzero terms are of the same sign.

Step II: If all nonzero terms are negative, factor out a minus sign, and treat the sequence as if all terms were nonnegative. Thus, without loss of generality, assume henceforth that the sequence is nonnegative.

Step III: (i) If any three consecutive terms are nonzero, choose three such terms K_{n-3}, K_{n-2}, and K_{n-1}, pick a positive integer $K \le \min\{K_{n-3}, K_{n-2}, K_{n-1}\}$, subtract K from each of K_{n-3}, K_{n-2}, and K_{n-1} and add K to K_n.

(ii) If no three consecutive terms are nonzero, either all nonzero terms are 1, in which case the sequence is canonical and Step III terminates, or else (a) choose any $K_n > 1$, choose positive $J < K_n$, replace K_n by $K_n - J$, and replace K_{n-1}, K_{n-2}, and K_{n-3} by $K_{n-1} + J$, $K_{n-2} + J$, and $K_{n-3} + J$, respectively; then (b) [actually applying (i) in a specific way] choose positive $K \le \min\{K_n - J, K_{n-1} + J, K_{n-2} + J\}$ and replace $K_n - J$, $K_{n-1} + J$, and $K_{n-2} + J$ by $K_n - J - K$, $K_{n-1} + J - K$, and $K_{n-2} + J - K$, respectively, and replace K_{n+1} by $K_{n+1} + K$. Repeat Step III until the sequence obtained is canonical.

In order to show that Step III terminates in a finite number of repetitions, first introduce the parameter $N = \Sigma K_n$. Note that (i) reduces N by $2K > 0$. Thus (i) cannot be repeated consecutively indefinitely. Any infinite repetition of Step III would have an infinite number of times (ii) is applied. The next thing to show is that from the position before one use of (ii) to the position before the next use of (ii) N is not increased. (ii) itself adds $2J - 2K$ to N so (ii) increases N only if $J > K$. In this case, the new consecutive nonzero terms $K_n + J - K$, $K_{n-2} + J - K$, and $K_{n-3} + J$ have minimum $\ge J - K$. Thus (i) must next be applied, and must be repeated until at least one of these three terms is reduced to zero. But if (i) reduces an individual term by K', then that application of (i) reduces N by $2K'$. Thus (i) must be repeated at least until N is brought back down to its value before the most recent use of (ii).

There still remains the possibility of an infinite sequence of Step III's, each with $N = N_0$ just before each application of (ii). To show this is impossible, order the set of all finitely nonzero nonnegative integer sequences lexicographically. Note that (i) and (ii) both strictly increase the lexicographic order. Consider only those $\{K_n\}_{-\infty}^{\infty}$ produced with $N = N_0$. These form a sequence of nonnegative sequences of given entry-sum $N = N_0$ which is increasing in lexocographic order. Such a sequence of sequences must be finite it if is bounded above. The following is a proof of this.

Consider the highest-position nonzero term in each of the sequences. This single term will be nondecreasing (in lexicographic order) and because of the existence of the upper bound and the requirement $N = N_0$ can only move through a finite number of values. Thus, the highest-position term becomes fixed after a certain point. Beyond this point, consider also the next highest-position term. This term must now be nondecreasing and so also eventually becomes fixed. Continuing on, one by one each of the successive nonzero terms becomes fixed and since there are at most N_0 nonzero terms, eventually all the terms become fixed.

It now only remains to show the existence of an upper bound for the sequences under consideration. Pick m such that $\alpha^m > \Sigma K_n \alpha^n$. If $\{K_n\}$ and $\{K_n''\}$ are equivalent nonnegative sequences and p'' is the upper degree of $\{K_n''\}$, then $p'' < m$. Reason: if $p'' \ge m$, then $\alpha^m \le K''_{p''} \alpha^{p''} \le \Sigma K_n'' \alpha^n = \Sigma K_n \alpha^n$. Thus, one can choose the sequence with 1 in the mth place and zeros elsewhere as a lexicographic upper bound to all nonnegative sequences equivalent to $\{K_n\}$. The proof is complete.

The above resolution algorithm is most conveniently done by first writing the sequence $\{K_n\}$ in usual *positional* notation (the reverse of usual sequential order) with a dot setting off position zero from position -1. Thus, the sequence with $K_{-2} = 2$, $K_{-1} = 1$, $K_0 = 3$, $K_1 = 0$, $K_2 = 4$, and all other terms 0, would be written 403.12 in this notation.

Applying the algorithm,

$$403.12 \xrightarrow{(i)} 412.01 \xrightarrow{(i)} 1301.01 \xrightarrow{(iia)} 1212.11 \xrightarrow{(iib)} 2101.11 \xrightarrow{(i)} 2110. \xrightarrow{(i)} 11000.$$

This shows that the sequence $\{K_n'\}$ with $K_3' = 1$, $K_4' = 1$, and all other terms 0, is the canonical sequence equivalent to $\{K_n\}$.

3. SIMULTANEOUS TRIBONACCI REPRESENTATIONS

The following is found greatly generalized in [8, Theorem 2.4].

Lemma 3: For all $n \in Z$,

$$\alpha_i = t_i \alpha^{-2} + (t_{i+2} - t_{i+1})\alpha^{-1} + t_{i+1}.$$

Proof: First check this formula explicitly for $i = 0, -1, -2$ (making suitable use of the relations $\alpha^{n+3} = \alpha^{n+2} + \alpha^{n+1} + \alpha^n$). The formula then follows for all i, since both left and right sides satisfy the recursion $X_{i+3} = X_{i+2} + X_{i+1} + X_i$ for all i.

Proposition 4: There are unique integers a, b, and c such that $\Sigma K_i \alpha^i = a\alpha^{-2} + (c - b)\alpha^{-1} + b$. These integers are given by

$$a = \Sigma K_i t_i, \quad b = \Sigma K_i t_{i+1}, \text{ and } c = \Sigma K_i t_{i+2}.$$

Proof: Applying Lemma 3,

$$\Sigma K_i\alpha^i = (\Sigma K_i t_i)\alpha^{-2} + (\Sigma K_i t_{i+2} - \Sigma K_i t_{i+1})\alpha^{-1} + (\Sigma K_i t_{i+1}).$$

Now use the fact that α^{-2}, α^{-1}, and 1 are a basis of $Z[\alpha]$ over Z.

Theorem 5: Existence, uniqueness, and construction of simultaneous Tribonacci representations.

(a) For every integer triple (a, b, c) there is a unique canonical sequence $\{K_i\}$ such that

$$a = \Sigma K_i t_i,\ b = \Sigma K_i t_{i+1},\ \text{and}\ c = \Sigma K_i t_{i+2}.$$

(b) The sequence $\{K_i\}$ can be found by resolving the sequence with a in the -2 position, $c - b$ in the -1 position, b in the 0 position, and zeros elsewhere, that is, $(b).(c - b)(a)$ in positional notation.

Proof: By Theorem 2, there is a unique canonical sequence $\{K_i\}$ such that

$$a\alpha^{-2} + (c - b)\alpha^{-1} + b = \Sigma K_i\alpha^i.$$

Now apply Propostion 4.

Comment: Theorem 5(a) was stated first in [4]. I believe that the use of the resolution algorithm to find the canonical sequence $\{K_n\}$ is in the majority of cases the most efficient method now available.

Example: Find the canonical sequence $\{K_i\}$ such that $a = \Sigma K_i t_i = -1$, $b = \Sigma K_i t_{i+1} = 4$, and $c = \Sigma K_i t_{i+2} = 3$.

$$(b).(c - b)(a) = 4.(-1)(-1) \xrightarrow{\text{I}} .334 \xrightarrow{(\text{i})} 2.112 \xrightarrow{(\text{i})} 11.002 \xrightarrow{(\text{iia})} 11.001111 \xrightarrow{(\text{iib})} 11.010001$$

Verification:

$$t_{-6} + t_{-2} + t_0 + t_1 = -3 + 1 + 0 + 1 = -1$$

$$t_{-5} + t_{-1} + t_1 + t_2 = 2 + 0 + 1 + 1 = 4$$

$$t_{-4} + t_0 + t_2 + t_3 = 0 + 0 + 1 + 2 = 3$$

Theorem 6(a)—First proven in [4, (5.2)]: The triple (a, b, c) has its simultaneous Tribonacci representation using a positive canonical sequence $\{K_n\}$ iff $a\alpha^{-2} + b(1 - \alpha^{-1}) + c\alpha^{-1} \geq 0$.

Theorem 6(b): In addition, the sequence $\{K_n\}$ will have upper degree p iff

$$\alpha^p \leq a\alpha^{-2} + b(1 - \alpha^{-1}) + c\alpha^{-1} < \alpha^{p+1}.$$

Proof: As was noted in the proof of uniqueness in Theorem 2, $\{K_i\}$ is positive canonical iff $\Sigma K_i\alpha^i \geq 0$, and in addition has upper degree p iff $\alpha^p \leq \Sigma K_i\alpha^i < \alpha^{p+1}$. By Propositions 4 and 5, $\Sigma K_i\alpha^i = a\alpha^{-2} + (c - b)\alpha^{-1} + b$. Substituting this into the above inequalities yields the conclusion.

Theorem 7: For each pair (a,b) of integers, there is a unique positive canonical sequence of upper degree ≤ -2 such that $a = \Sigma K_i t_i$ and $b = \Sigma K_i t_{i+1}$. The triple (a,b,c) for the given pair (a,b) is found by the formula

$$c = -[a\alpha^{-1} + b(\alpha - 1)].$$

Proof: By Theorem 6, $\{K_i\}$ is positive canonical and of upper degree ≤ -2 iff

$$0 \leq a\alpha^{-2} + b(1 - \alpha^{-1}) + c\alpha^{-1} < \alpha^{-1},$$

or, equivalently,

$$-a\alpha^{-1} - b(\alpha - 1) \leq c < 1 - a\alpha^{-1} - b(\alpha - 1).$$

This formula gives a unique integer c for each pair (a,b) and the existence of uniqueness of $\{K_i\}$ then follows by Theorem 5.

4. THE LOWER DEGREE I

In this section and in Section 5, all sequences $\{K_i\}_{-\infty}^{\infty}$ not otherwise described will be assumed canonical. This property will not be explicitly stated again.

Define $\mathcal{S} = \{\Sigma K_i\beta^i\colon \{K_i\}$ is positive and of lower degree $\geq 0\}$. (β is defined at the beginning of Section 2.) Also define $\mathcal{S}^0 = \{\Sigma K_i\beta^i\colon \{K_i\}$ is positive and of lower degree $0\}$. We wish to describe $\mathcal{S}$ and $\mathcal{S}^0$, at least approximately, as subsets of the complex plane.

Let $\mathcal{S}_{11} = \{\Sigma K_i\beta^i\colon \{K_i\}$ is positive of lower degree ≥ 0 and upper degree $\leq 11\}$. $\mathcal{S}_{11}$ is a finite set of complex numbers.

$$\text{Let } \mathcal{S}_{11}^0 = \mathcal{S}^0 \cap \mathcal{S}_{11}.$$

$$\text{Let } \mathcal{T} = \{\Sigma K_i\beta^i\colon \{K_i\} \text{ is positive}\}.$$

Proposition 8: If $\Sigma K_i\beta \in \mathcal{T}$, then $|\Sigma K_i\beta^i - \sum^{11} K_i\beta^i| < .075$. In particular, every point of $\mathcal{S}$ is within .075 of a point in $\mathcal{S}_{11}$ and every point of $\mathcal{S}^0$ is within .075 of a point in $\mathcal{S}^0_{11}$.

Proof: $|\Sigma K_i\beta^i - \sum^{11} K_i\beta^i| \leq \sum_{i=12}^{\infty} K_i|\beta^i| \leq \sum_{i=1}^{\infty} |\beta|^{12+3i} + |\beta|^{13+3i} = (|\beta|^{12} + |\beta|^{13})/(1 - |\beta|^3) < .075$

where the second inequality uses an argument similar to that in the uniqueness part of Theorem 2.

The above proposition indicates that in a certain sense $\mathcal{S}_{11}$ is a good approximation to $\mathcal{S}$ and that $\mathcal{S}^0_{11}$ is a good approximation to $\mathcal{S}^0$. This, together with the following propositions, will help explain the correctness of our "sketch" of $\mathcal{S}$ and $\mathcal{S}^0$ (Figure 1, which appears at the end of this section).

Proposition 9: If $\Sigma K_i\beta^i \in \mathcal{S}^0$, then $|\Sigma K_i\beta^i| > .425$.

Proof: We calculated the minimum of the moduli of the hundreds of points of $\mathcal{S}^0_{11}$ obtaining .50088 (attained at $1 + \beta^2 + \beta^4 + \beta^7 + \beta^{10}$) and, applying Proposition 8, subtracted .075, thus obtaining the lower bound of the theorem. Clearly, a listing of the details of this proof would be unprofitable. The skeptical reader with access to a computer can easily reproduce them for himself.

Proposition 10: If $\{K_i\}$ is positive and of lower degree r, then $|\Sigma K_i\beta^i| > .425|\beta^r|$.

Proof: The lower degree of $\{K_{i+r}\}_{i=-\infty}^{\infty}$ is zero. Hence $|\Sigma K_{i+r}\beta^i| > .425$.

$$|\Sigma K_i\beta^i| = |\Sigma K_{i+r}\beta^{i+r}| = |\beta^r||\Sigma K_{i+r}\beta^i| > .425|\beta^r|.$$

Proposition 11: If $\Sigma K_i\beta^i \in \mathcal{S}$, then $|\Sigma K_i\beta^i| < 1.69$.

Proof: The proof is analogous to that of Proposition 9. The point of $\mathcal{S}_{11}$ of maximum modulus is $1 + \beta^3 + \beta^6 + \beta^8 + \beta^9 + \beta^{11}$. Its modulus is 1.6055.

Proposition 12: If $\Sigma K_i\beta^i \in \mathcal{T}$ and $|\Sigma K_i\beta^i| < 1.69$, then lower degree $\{K_i\}$ is ≥ -4.

Proof: Note that $.425|\beta^{-5}| > 1.69$ and apply Proposition 10.

To sketch $\mathcal{S}$ and $\mathcal{S}^0$, plot $\mathcal{S}_{11}$, identifying $\mathcal{S}^{11}_0$, and on the same graph plot all points $\Sigma K_i\beta^i$ where $|\Sigma K_i\beta^i| < 1.69$, lower degree of $\{K_i\}$ is between -4 and -1 inclusive, and upper degree of $\{K_i\}$ is ≤ 11. Then sketch a simple closed curve which separates $\mathcal{S}_{11}$ from the other points plotted, and add a simple curve to separate $\mathcal{S}^0_{11}$ from the rest of $\mathcal{S}_{11}$.

By Proposition 8, every point in $\mathcal{S}$ is (approximately) inside the curve drawn, and by Propositions 8 and 10, every point in $\mathcal{T}$ but not in $\mathcal{S}$ is (approximately) outside the curve, where the approximation includes the value .075 of Proposition 10 and sketching errors which will probably be smaller than .075.

The next few propositions and theorems help justify the drawing of a simple closed curve, since such a curve indeed encloses a simply-connected domain.

The polynomial $X^3 - X^2 - X - 1$ is irreducible and hence its Galois group is transitive on the roots α, β, γ [6, Chapter 3, Section 5].

Hence,

$$a\alpha^{-2} + b\alpha^{-1} + c = \Sigma K_i\alpha^i \quad \text{iff}$$
$$a\beta^{-2} + b\beta^{-1} + c = \Sigma K_i\beta^i \quad \text{iff}$$
$$a\gamma^{-2} + b\gamma^{-1} + c = \Sigma K_i\gamma^i.$$

Thus $\mathcal{T}$ = those elements of $Z[\beta]$ which become positive when Z is held fixed and α is substituted for β.

Proposition 13: $\mathcal{T}$ is dense in the complex plane.

Proof: Consider the complex set (with polar coordinates) $0 = \{(r,\theta):\ r_1 < r < r_2,\ \theta_1 < \theta < \theta_2\}$. Such sets are a base for the complex topology. Because $|\beta| < 1$ and argument β is not a rational multiple of 2π (see the proof of Theorem 18 in [2]), therefore, for certain sufficiently large integers m, $|\beta^m| < r_2 - r_1$ and $\theta_1 <$ argument $\beta^m < \theta_2$. Thus, $n\beta^m \in 0$ for a correctly chosen positive integer r. Now $n\alpha^m > 0$, so $n\beta^m \in \mathcal{T}$.

Proposition 14: If $\Sigma K_i\beta^i \in \mathcal{T}$ and $|\Sigma K_i\beta^i| < .425|\beta^{r-1}|$, then $\{K_i\}$ has lower degree $\geq r$. In particular, if $\Sigma K_i\beta^i \in \mathcal{T}$ and $|\Sigma K_i\beta^i| < .425|\beta^{-1}|$, then $\Sigma K_i\beta^i \in \mathcal{S}$.

Proof: This is an immediate corollary of Proposition 10.

Theorem 15: The lower degree is locally constant at points of $\mathcal{T}$. If $\Sigma K_i\beta^i \in \mathcal{T}$, $\{K_i\}$ has lower degree r and upper degree p, $\Sigma K_i'\beta^i \in \mathcal{T}$ and $|\Sigma K_i'\beta^i - \Sigma K_i\beta^i| < .425|\beta^{p+1}|$, then $\{K_i\}$ has lower degree r.

Proof: Let $\Sigma K_i'\beta^i - \Sigma K_i\beta^i = \Sigma K_i''\beta^i$.

Claim.—$\Sigma K_i''\beta^i \in \mathcal{T}$. If not, $\Sigma - K_i''\beta^i \in \mathcal{T}$, and by Proposition 14, $\{-K_i''\}$ would have lower degree $\geq p + 2$ and thus also would have upper degree $\geq p + 2$. Consequently, by Theorem 6, $\Sigma K_i\alpha^i - \Sigma K_i'\alpha^i = \Sigma - K_i''\alpha^i \geq \alpha^{p+2}$. But $\Sigma K_i\alpha^i < \alpha^{p+1}$ since p is the upper degree of $\{K_i\}$. This yields $\Sigma K_i'\alpha^i = \Sigma K_i\alpha^i - (\Sigma K_i\alpha^i - \Sigma K_i'\alpha^i) < \alpha^{p+1} - \alpha^{p+2} < 0$ contradicting the fact that $\Sigma K_i'\beta^i \in \mathcal{T}$.

Thus $\Sigma K_i''\beta^i \in \mathcal{T}$. As above, the lower degree of $\{K_i''\}$ is $\geq p + 2$, while the upper degree of $\{K_i\}$ is p. Hence $\{K_i + K_i''\}$ is canonical, and since $\Sigma(K_i + K_i'')\beta^i = \Sigma K_i'\beta^i$, it follows that $\{K_i'\} = \{K_i + K_i''\}$. Thus, lower degree of $\{K_i'\}$ = lower degree of $\{K_i\}$ and the proof is finished.

Let $\mathcal{U}$ be the interior of the closure of $\mathcal{S}$.

Let $\mathcal{U}^0$ be the interior of the closure of $\mathcal{S}^0$.

Theorem 16: $\mathcal{S} = \mathcal{T} \cap \mathcal{U}$. $\mathcal{S}^0 = \mathcal{T} \cap \mathcal{U}^0$.

Proof: Here is the proof for $\mathcal{S}$. The proof for $\mathcal{S}^0$ is similar.

For each point $z = \Sigma K_i\beta^i \in \mathcal{T}$, consider the open disc $\mathcal{B}_z$ with center z and radius $.425|\beta^{p+1}|$ where p is the upper degree of $\{K_i\}$. By Theorem 15, if $z \in \mathcal{S}$, then $\mathcal{B}_z \cap \mathcal{T} \subseteq \mathcal{S}$. Hence $z \in \mathcal{B}_z \subseteq$ Closure $(\mathcal{B}_z \cap \mathcal{T}) \subseteq$ Closure $\mathcal{S}$. Thus, $\mathcal{S} \subseteq \mathcal{U}$.

On the other hand, if $z \in \mathcal{T}\backslash\mathcal{S}$ with lower degree $r < 0$, then $\mathcal{B}_z$ does not meet $\mathcal{S}$, hence $z \notin \mathcal{U}$.

Lemma 17: If $\Sigma K_i\beta^i \in \mathcal{T}$ has lower degree r and upper degree p, if $\Sigma K_i'\beta^i \in \mathcal{T}$ and $|\Sigma K_i'\beta^i - \Sigma K_i\beta^i| < .425|\beta^p|$, then $\Sigma K_i'\beta^i$ has lower degree $\geq r$.

Proof: The proof follows most of that of Theorem 15 word for word except that the lower degree of $\{K_i''\}$ is $\geq p + 1$, while the upper degree of $\{K_i\}$ is p, so that while $\{K_i'\}$ is equivalent to $\{K_i + K_i''\}$, the latter may not be canonical. However, every term of $\{K_i + K_i''\}$ is either 0 or 1 and thus this sequence may be resolved as follows: In choosing three consecutive 1's, choose such a triple with next higher term 0. Applying operation III(i) produces a new sequence with, again, all terms either 0 or 1. The parameter N is reduced by 2. Repeat until the equivalent canonical sequence $\{K_i\}$ is obtained. Since operation III(i) never lowers the lower degree, the conclusions is obtained.

Theorem 18: $\mathcal{U}$ is connected.

Proof: Let $z \in \mathcal{U}$. Then, since $\mathcal{U}$ is open and $\mathcal{S}$ is dense in $\mathcal{U}$, the connected component of $\mathcal{U}$ containing z also contains a point $\sum_0^\infty K_i\beta^i$ in $\mathcal{S}$.

It suffices to show that $\sum_0^{n-1} K_i\beta^i$ and $\sum_0^n K_i\beta^i$ lie in the same connected component for all $n \geq 0$ $\left(\text{where } \sum_0^{-1} K_i\beta^i = 0\right)$. This clearly holds if $K_n = 0$, so assume $K_n = 1$, $K_i = 0$ for $p < i < n$ and $K_p = 1$ $\left(\text{if } \sum_0^{n-1} K_i\beta^i = 0, \text{ set } p = -1 \text{ and succeeding statements will still hold true}\right)$.

By Lemma 17 $\left(\text{or Proposition 14 if } \sum_0^{n-1} K_i\beta^i = 0\right)$, the open disc with center $\sum_0^{n-1} K_i\beta^i$ and radius $.425|\beta^p|$ is contained in $\mathcal{U}$ as is the open disc with center $\sum_0^n K_i\beta^i$ and radius $.425|\beta^n|$. But since $.425 + .425|\beta^{-1}| > 1$, $\left|\sum_0^n K_i\beta^i - \sum_0^{n-1} K_i\beta^i\right| = |\beta^n| < .425|\beta^n| + .425|\beta^{n-1}| \leq .425|\beta^n| + .425|\beta^p|$ so the two discs overlap, and the proof is complete.

Theorem 19: $\mathcal{U}$ is simply connected.

Proof: Since $\mathcal{U}$ is the interior of a closure, if z is in a bounded component of the complement of $\mathcal{U}$ then z is in the closure of the (open) exterior of $\mathcal{U}$, thus $z \in$ Closure $(\mathcal{T}\backslash\mathcal{U})$. By Propositions 11 and 12, $z \in$ Closure $\mathcal{A}r$, where $\mathcal{A}r = \{\Sigma K_i\beta^i | \{K_i\}$ has lower degree $r\}$ where r is some number between -4 and -1 inclusive. Now $\mathcal{A}r = \mathcal{C}r \cup \mathcal{D}r$ where $\mathcal{C}r = \{\beta^r + \beta^{r+2}\Sigma K_i\beta^i: \ \Sigma K_i\beta^i \in \mathcal{S}\}$ and $\mathcal{D}r = \{\beta^r + \beta^{r+1} + \beta^{r+3}\Sigma K_i\beta^i: \ \Sigma K_i\beta^i \in \mathcal{S}\}$. Then just as it was shown in Theorem 18 that $\mathcal{S} \subseteq$ interior (Closure $\mathcal{S}$), similarly, $\mathcal{C}r \subseteq$ interior [Closure$(\mathcal{C}r)$] $= \beta^r + \beta^{r+2}$ and $\mathcal{D}r \subseteq$ interior [Closure$(\mathcal{D}r)$] $= \beta^r + \beta^{r+1} + \beta^{r+3}\mathcal{U}$, and these latter sets are open, connected, and contained in the exterior of $\mathcal{U}$, and z is in the closure of one of these sets. Hence, it suffices to show that each of these eight connected sets lies in the unbounded component of the complement of $\mathcal{U}$. Now $\mathcal{U}$ lies within the open disc of center zero and radius 1.69, but

$$\beta^{-1} + \beta^2 \in \mathcal{C}_{-1},\ \beta^{-1} + 1 + \beta^2 + \beta^5 \in \mathcal{D}_{-1},\ \beta^{-2} \in \mathcal{C}_{-2},\ \beta^{-2} + \beta^{-1} + \beta^1 \in \mathcal{D}_{-2},$$

$$\beta^{-3} \in \mathcal{C}_{-3},\ \beta^{-3} + \beta^{-2} \in \mathcal{D}_{-3},\ \beta^{-4} \in \mathcal{C}_{-4},\ \beta^{-4} + \beta^{-3} \in \mathcal{D}_{-4},$$

and each of these points has modulus > 1.69.

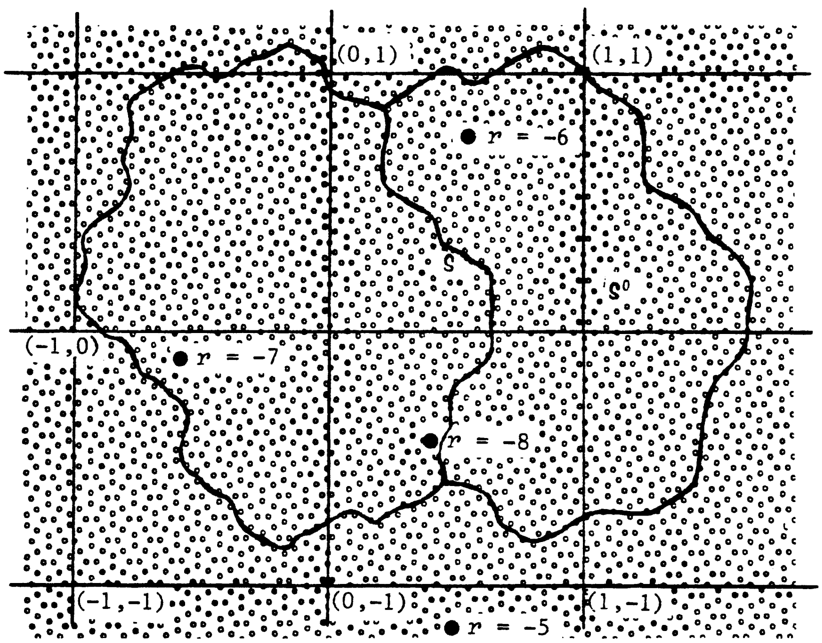


FIGURE 1. Sketches of $\mathcal{S}$ and $\mathcal{S}^0$. The dots were plotted by computer, using different colors in each of the three regions shown. Everything else was sketched by hand. The four plotted points illustrate Application 21. Only dots of modulus less than 1.69 were plotted. Thus, the right edge of the figure is a sketch of part of a circle of radius 1.69. This gives an idea of the accuracy of the rest of the sketch.

5. THE LOWER DEGREE II

If we assume that the set $\mathcal{S}$ is known, we can then solve the "lower degree problem" for simultaneous Tribonacci representations in terms of $\mathcal{S}^0$.

Theorem 20: $a = \Sigma K_i t_i$, $b = \Sigma K_i t_{i+1}$, and $c = \Sigma K_i t_{i+2}$, where $\{K_i\}$ is positive of lower degree r iff $\beta^{-r}[a\beta^{-2} + b(1 - \beta^{-1}) + c\beta^{-1}] \in \mathcal{S}^0$.

Proof: By Proposition 4 and Theorem 5, the first set of conditions is equivalent to $a\alpha^{-2} + (c - b)\alpha^{-1} + b = \Sigma K_i\alpha^i$, where lower degree of $\{K_i\} = r$. Substituting β for α and making other rearrangements yields $a\beta^{-2} + b(1 - \beta^{-1}) + c\beta^{-1} = \sum_i K_{i+r}\beta^{i+r} = \beta^r\Sigma K_{i+r}\beta^i$, which is equivalent to the second condition of the theorem.

Theorem 21: $a = \Sigma K_i t_i$, $b = \Sigma K_i t_{i+1}$, and $c = \Sigma K_i t_{i+2}$, where $\{K_i\}$ is positive of lower degree $\geq r$ iff $\beta^{-r}[a\beta^{-2} + b(1 - \beta^{-1}) + c\beta^{-1}] \in \mathcal{S}$.

Proof: The proof is similar to that of Theorem 19.

Application 22: Here is how Theorem 20 can be used in practice. Note that by Propositions 9 and 11, if $\Sigma K_i \beta^i \in \mathcal{S}^0$, then $.425 < |\Sigma K_i \beta^i| < 1.69$.

Compute $x = a\beta^{-2} + b(1 - \alpha^{-1}) + c\alpha^{-1}$. If $x \geq 0$, then $\{K_i\}$ is positive. If $x < 0$, replace (a, b, c) by $(-a, -b, -c)$.

Next compute $z = a\beta^{-2} + b(1 - \beta^{-1}) + c\beta^{-1}$. Taking absolute values, we require $.425 < |\beta|^{-r}|z| < 1.69$, or taking logs and rearranging,

$$\frac{\log|z| - \log(.425)}{\log|\beta|} < r < \frac{\log|z| - \log(1.69)}{\log|\beta|}. \tag{1}$$

This gives four or five possible values for r. Plot $\beta^{-r}z$ for these values and see which lies within $\mathcal{S}^0$.

Example: Find by this method the lower degree of the canonical sequence $\{K_n\}$ such that $a = \Sigma K_i t_i = -1$, $b = \Sigma K_i t_{i+1} = 4$, and $c = \Sigma K_i t_{i+2} = 3$.

$$-1\alpha^{-2} + 4(1 - \alpha^{-1}) + 3\alpha^{-1} = 3.16 > 0, \text{ so } \{K_n\} \text{ is positive,}$$

$$z = -1\beta^{-2} + 4(1 - \beta^{-1}) + 3\beta^{-1} = 5.42 + 3.39i, \ |z| = 6.3945131,$$

so (1) above gives

$$-8.90 < r < -4.37.$$

Plotting $\beta^{-r}z$ for $r = -8, -7, -6$, and -5 on Figure 1 shows the lower degree is -6, which agrees with the result of the Example following Theorem 5. The above method will be must more efficient than the resolution algorithm most of the time. However, accuracy will not be guaranteed if one (and therefore two) of the points plotted falls near the boundary of $\mathcal{S}^0$. If this problem comes up, r will still be known for sure to be one of two consecutive integers.

It was shown in [2, Theorem 1] that each positive integer a has a unique (Zeckendorf) representation $a = \Sigma K_i t_i$, where $\{K_i\}$ is positive and of lower degree ≥ 2.

Problem.—For a given a with this representation, find formulas for $b = \Sigma K_i t_{i+1}$ and $c = \Sigma K_i t_{i+2}$ in terms of a. Let us call such a triple (a, b, c) a Zeckendorf triple. We shall solve this problem, not with a precise formula, but rather, in terms of a picture.

Let $x + iy$ be an arbitrary point in $\mathcal{S}$. Rewrite the condition of Theorem 21 for the problem:

$$a\beta^{-4} + (c - b)\beta^{-3} + b\beta^{-2} = x + iy.$$

Equivalently,

$$a2\beta^{-2} + (c - b)2\beta^{-1} + 2b = 2\beta^2(x + iy). \tag{2}$$

We wish to break this into real and imaginary parts. To do this, we need to find the real and imaginary parts of $2\beta^n$ for various n. By the recursion relation, the values for all other n can be obtained from the values for $n = -1$, 0, and 1. Since α, β, and $\gamma = \overline{\beta}$ are the roots of $x^3 - x^2 - x - 1 = 0$, we have $\alpha + \beta + \gamma = 1$ and $\alpha\beta\gamma = 1$: Hence, $\mathrm{Re}(2\beta) = \beta + \gamma = 1 - \alpha$ and $\beta\gamma = \alpha^{-1}$. Thus, $\mathrm{Re}(2\beta^{-1}) = \beta^{-1} + \gamma^{-1} = (\beta + \gamma)/\beta\gamma = \alpha - \alpha^2 = -1 - \alpha^{-1}$. By the recursion $\mathrm{Re}(2\beta^2) = 3 - \alpha^2$ and $\mathrm{Re}(2\beta^{-2}) = -1 - \alpha^{-2}$. Let $\delta = \mathrm{Im}(2\beta) = (\beta - \gamma)/i \doteq 1.21258146$. Then $\mathrm{Im}(2\beta^{-1}) = (\beta^{-1} - \gamma^{-1})/i = -(\beta - \gamma)/\beta\gamma i = -\alpha\delta$. By the recursion $\mathrm{Im}(2\beta^2) = (1 - \alpha)\delta$ and $\mathrm{Im}(2\beta^{-2}) = (1 + \alpha)\delta$. Taking real and imaginary parts of (2) using the above data yields

$$(-1 - \alpha^{-2})a + (-1 - \alpha^{-1})(c - b) + 2b = (3 - \alpha^2)x - (1 - \alpha)\delta y,$$

and

$$(1 + \alpha)\delta a + (-\alpha\delta)(c - b) = (3 - \alpha^2)y + (1 - \alpha)\delta y. \tag{3}$$

Solving (3) for b and c yields,

$$b = a\alpha + u$$
$$c = a\alpha^2 + v,$$

where $u = (4 - \alpha^{-2} - \alpha^2)x/2 + [(\alpha - 3\alpha^{-1} + 1 - 3\alpha^{-2})\delta^{-1} - \delta(1 - \alpha)]y/2$ and $v = u + (1 - \alpha^{-1})x + (\alpha - 3\alpha^{-1})\delta^{-1}y$.

Thus, there is a well-defined real matrix $\boldsymbol{T}$ such that $(u,v) = \boldsymbol{T}(x,y)$.

Now define $\mathcal{W} = \{(u,v): \ (u,v) = \boldsymbol{T}(x,y), \ x + iy \in \mathcal{S}\}$. Let $\mathcal{X}$ = interior (Closure $\mathcal{W}$). Clearly, $\mathcal{X} = \{(u,v): \ (u,v) = (x,y), \ x + iy \in \mathcal{U}\}$. A sketch of $\mathcal{W}$ (or $\mathcal{X}$) appears in Figure 2(b).

We have proven

Theorem 23: (a, b, c) is a Zeckendorf triple if there is a point $(u,v) \in \mathcal{W}$ such that $b = a\alpha + u$ and $c = a\alpha^2 + v$. Every point in $\mathcal{W}$ corresponds in this way to a unique Zeckendorf triple.

Corollary 24: The pair (b,c) in a Zeckendorf triple (a, b, c) can and does take on only the values $([\![a\alpha]\!] + j, [\![a\alpha^2]\!] + k)$ where $(j,k) = (0,0), (0,1), (1,0), (1,1)$, or $(2,1)$.

Proof: $\mathcal{W}_{11}$, $\mathcal{W}$ computed exactly with $\mathcal{S}$ replaced by $\mathcal{S}_{11}$, gives an approximation to $\mathcal{W}$. The "error" of .075 in replacing $\mathcal{S}$ by $\mathcal{S}_{11}$ is propagated into an error of no more than .098 in replacing $\mathcal{W}$ by $\mathcal{W}_{11}$.

Note that (b,c) can assume the value $([\![a\alpha]\!] + j, [\![a\alpha^2]\!] + k)$ iff there is a $(u,v) \in \mathcal{W}$ with $j - 1 < u \leq j$ and $k - 1 < v \leq k$. An examination of the thousands of points in $\mathcal{W}_{11}$ yields the results of the corollary.

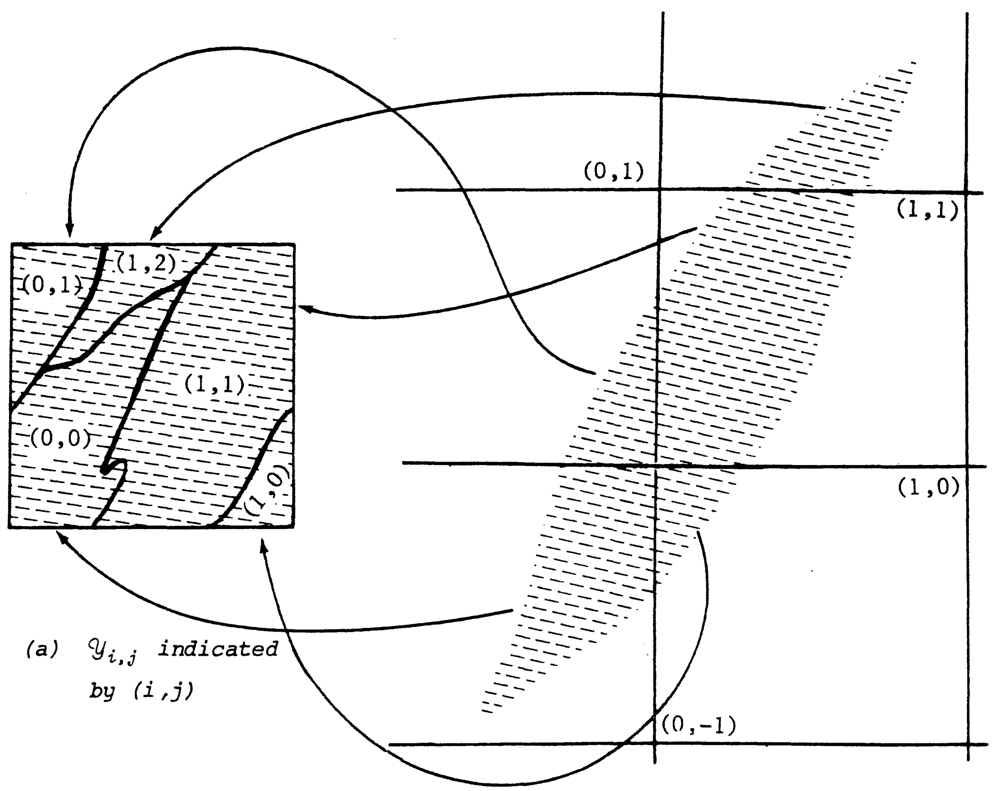
Let $\mathcal{Q}$ be the open unit square $\{(s,t): 0 < s < 1, 0 < t < 1\}$.

For each of the pairs (j,k) of Corollary 24, define $\mathcal{Y}_{j,k} = \{(s,t) \in \mathcal{Q}: (j - s, k - t) \in \mathcal{W}\}$.

Theorem 25: If (a, b, c) is a Zeckendorf triple, then $b = [\![a\alpha]\!] + j$ and $c = [\![a\alpha^2]\!] + k$ iff $(a\alpha - [\![a\alpha]\!], a\alpha^2 - [\![a\alpha^2]\!]) \in \mathcal{Y}_{j,k}$.

Proof: By Theorem 24, $b = [\![a\alpha]\!] + j$ and $c = [\![a\alpha^2]\!] + k$ iff $[\![a\alpha]\!] + j = a\alpha + u$ and $[\![a\alpha^2]\!] + k = a\alpha^2 + v$ with $(u,v) \in \mathcal{W}$ if $j - (a\alpha - [\![a\alpha]\!]), k - (a\alpha^2 - [\![a\alpha^2]\!]) \in \mathcal{W}$. Also $(a\alpha - [\![a\alpha]\!], a\alpha^2 - [\![a\alpha^2]\!])$ is always in $\mathcal{Q}$ for positive integer a.

The sets $\mathcal{Y}_{j,k}$ can be drawn approximately by rotating $\mathcal{W}$ through 180°, cutting $\mathcal{W}$ up along lines of integer value and translating the pieces by integer distances vertically + horizontally into $\mathcal{Q}$. The result is shown in Figure 2(a). It is seen that these sets appear to disjointly cover $\mathcal{Q}$. This is worth proving. Let $\mathcal{Z}_{j,k}$ = interior(Closure $\mathcal{Y}_{j,k}$). Clearly $\mathcal{Y}_{j,k} \subseteq \mathcal{Z}_{j,k} = [(j,k) - \mathcal{X}] \cap \mathcal{Q}$.



(a) $\mathcal{Y}_{i,j}$ *indicated by (i,j)*

(b) $\mathcal{W}$

FIGURE 2

Theorem 26(a): The union of the $\mathcal{Y}_{j,k}$ is dense in $\mathcal{Q}$.

Theorem 26(b): The sets $\mathcal{Z}_{j,k}$ are disjoint.

Proof (a): By Theorems 23 and 25, it is seen that the union of the $\mathcal{Y}_{j,k}$ consists of the set $\{(a\alpha - [\![a\alpha]\!], a\alpha^2 - [\![a\alpha^2]\!]): a$ positive integer$\}$. Since 1, α, and α^2 are linearly independent over $\mathcal{Z}$, this set is dense in the unit square (see [1, Chapter IV]).

Proof (b): It suffices to prove that if j', j'', k', and k'' are integers, if (u',v') and $(u'',v'') \in \mathcal{X}$ and if $(j' - u', k' - v') = (j'' - u'', k'' - v'')$, then $u' = u''$ and $v' = v''$. These hypotheses imply that $u'' = u' + j$ and $v'' = v' + k$ where j and k are integers.

Let $\{(u_n', v_n')\}$ be a sequence in $\mathcal{W}$ converging to (u',v'). Corresponding to each (u_n', v_n') there is a unique Zeckendorf triple (a_n, b_n, c_n) with

$$(4)\qquad \begin{aligned} b_n &= a_n\alpha + u_n' \\ c_n &= a_n\alpha^2 + v_n'. \end{aligned}$$

By deleting from the sequence a finite number of terms, we can assume without loss of generality that $a_n\alpha^2 + ([\![a_n\alpha]\!] + j)(1 - \alpha^{-1}) + ([\![a_n\alpha^2]\!] + k)\alpha^{-1} > 0$ for all n. Since $b_n \geq [\![a_n\alpha]\!]$ and $c_n \geq [\![a_n\alpha^2]\!]$, this guarantees by Theorem 6 that a positive canonical sequence is obtained for simultaneous Tribonacci representation of the triples $(a_n, b_n + j, c_n + k)$. If we suppose that $(j,k) \neq (0,0)$, these are not Zeckendorf triples and thus they are simultaneously represented by a positive sequence $\{K_i\}$ of lower degree ≤ 1. Going through precisely the same calculations as precede the proof of Theorem 24, it is seen that $b_n + j = a_n\alpha + u_n''$ and $c_n + k = a_n\alpha^2 + v_n''$ where $(u_n'', v_n'') = \boldsymbol{T}(x_n'', y_n'')$ where $x_n'' + iy_n'' \mathcal{T}\backslash\mathcal{S}$. Hence, $(u_n'', v_n'') \in$ exterior $\mathcal{X}$. In light of (4), we have $(u_n'', v_n'') = (u_n', v_n') + (j,k)$. Hence (u_n'', v_n'') converges to (u'', v''), which thus cannot lie in $\mathcal{X}$, contradicting our hypothesis. The theorem is proven.

Example: Use Figure 2(a) to find the Zeckendorf triple with $a = 650$.

First we compute $a\alpha = 1195.5364$ and $a\alpha^2 = 2198.9343$. Then observe from Figure 2 that $(.5364, .9343) \in \mathcal{Y}_{(1,2)}$. Hence, $b = 1195 + 1 = 1196$, and $c = 2198 + 2 = 2200$, which results can be verified by direct calculation.

It was shown in [2, Theorem 12] that each integer has a unique (2nd canonical) representation $a = \Sigma K_i t_i$ where $\{K_i\}$ is positive and of lower degree positive and congruent to 1 modulo 3. For such a representation, we call (a, b, c) a 2nd canonical triple if $b = \Sigma K_i t_{i+1}$ and $c = \Sigma K_i t_{i+2}$. The following facts about 2nd canonical triples are proved similarly to their analogues for Zeckendorf triples.

Corollary 27: The pair (b,c) in a 2nd canonical triple (a, b, c) can and does take on only the values $([\![a\alpha]\!] + j, [\![a\alpha^2]\!] + k)$ where $(j,k) = (-1,-1), (-1,0), (0,-1), (0,0), (0,1), (0,2), (1,0), (1,1)$, or $(1,2)$. In [2] it was shown that j takes on the values -1, 0, and 1. Figure 3 is the analogue for 2nd canonical triples of Figure 2. A region marked (j,k) in Figure 3(a) denotes the region $\mathcal{Y}_{j,k}^2$.

Theorem 28: If (a, b, c) is a 2nd canonical triple when $b = [\![a\alpha]\!] + j$ and $c = [\![a\alpha^2]\!] + k$ iff $(a\alpha - [\![a\alpha]\!], b\alpha^2 - [\![b\alpha^2]\!]) \in \mathcal{Y}_{j,k}^2$.

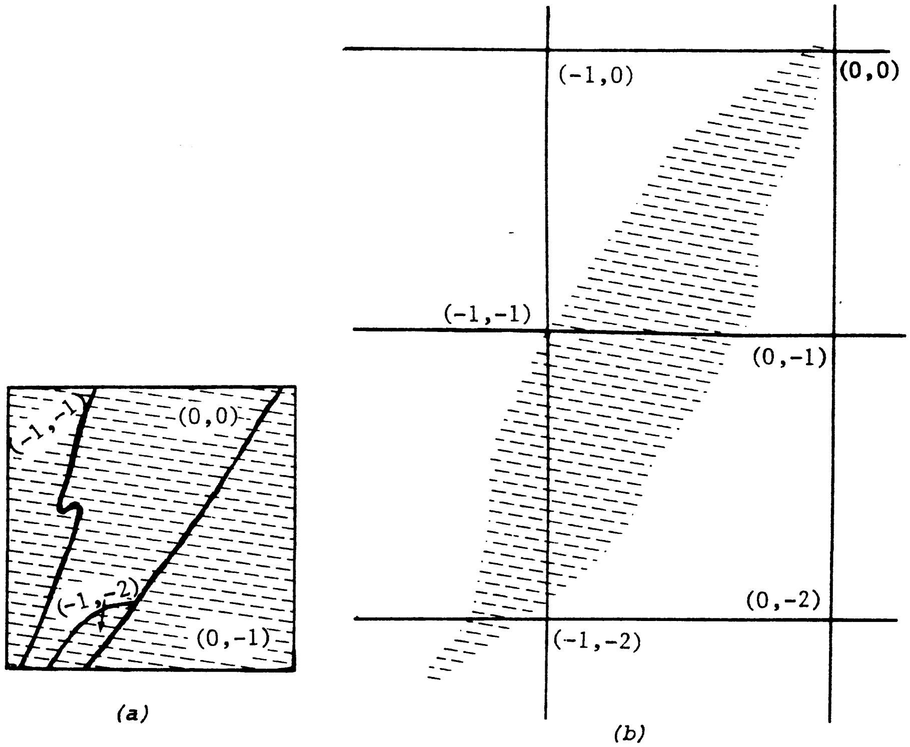


FIGURE 3. *(a) shows the unit square divided into the regions* $\mathcal{Y}^2_{i,j}$. *(b) shows the region* $\mathcal{W}^2$.

It is seen that with the given limits of accuracy, the computer sketch of $\mathcal{W}^2$ *does not indicate for sure whether* $\mathcal{Y}^2_{i,j}$ *is nonempty for (j,k) = (1,1), (1,0), (0,-2), (0,1) and (-1,0). However, accurate calculations of carefully chosen points of* $\mathcal{W}^2$ *corresponding to points of* $\mathcal{S}$ *of high upper degree show that these sets are indeed nonempty. Further theoretical considerations show that the areas near all four corners of* $\mathcal{Q}$ *are covered by infinitely many "strips" periodically alternating between those three sets* $\mathcal{Y}^2_{i,j}$ *which the sketch "allows" into the corner (for example, only* $\mathcal{Y}^2_{0,0}$, $\mathcal{Y}^2_{-1,-1}$, *and* $\mathcal{Y}^2_{-1,-2}$ *fit near (0,0) in* $\mathcal{Q}$*). Since all these strips are below accuracy level size, the corners of* $\mathcal{Q}$ *have been blacked out.*

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