We now make some observations in relation to the coefficients in (3). First, as predicted by Theorem 1, the sum of the integers in row r is equal to r^3. Second, each integer in the above table is either a multiple of 3 or leaves a remainder of +1 when divided by 3. Furthermore, for any particular row, the first entry, namely 1, and every third successive entry, are exactly those integers which leave remainder +1 when divided by 3. We summarize this discussion in the following theorem.

Theorem 2: Each integer in arrangement (3) is either a multiple of 3 or leaves a remainder of +1 on division by 3. If we label the integers in any one row as $\alpha_0, \alpha_1, \ldots,$ then $\alpha_i \equiv 1 \pmod 3$ when $i \equiv 0 \pmod 3$, and $\alpha_i \equiv 0 \pmod 3$ when $i \not\equiv 0 \pmod 3$. Consequently, in row m, there are m coefficients which leave remainder +1 on division by 3, and $2(m - 1)$ which are a multiple of 3.

Proof: The form of the coefficients α_i is specified in Theorem 1. Since 3 is a prime number, the remainders after division by 3 are completely determined by the term

$$\binom{i+k-1}{k-1} = \binom{i+2}{2} = \frac{(i+1)(i+2)}{2}.$$

If $i \not\equiv 0 \pmod 3$, then i is of the form $3m - 1$ or $3m - 2$, where m is a positive integer. In either case, it is easy to see that $\frac{(i+1)(i+2)}{2}$ is divisible by 3. If, on the other hand, $i \equiv 0 \pmod 3$, then we can write $i = 3m$, and

$$\frac{(i+1)(i+2)}{2} = \frac{(3m+1)(3m+2)}{2}.$$

Consequently,

$$\frac{(3m+1)(3m+2)}{2} - 1 = \frac{9m(m+1)}{2},$$

and this is easily seen to be divisible by 3.

We can generalize the results of Theorem 2 as follows:

Theorem 3: Let k be a prime number. Then each coefficient α_i of Theorem 1 is either a multiple of k, or leaves a remainder of +1 on division by k. In any one row, $\alpha_i \equiv 1 \pmod k$ when $i \equiv 0 \pmod k$, and $\alpha_i \equiv 0 \pmod k$ when $i \not\equiv 0 \pmod k$. Consequently, in row m there are m coefficients which leave remainder +1 on division by k, and $(m - 1)(k - 1)$ which are a multiple of k.

The proof is similar to that of Theorem 2, and will not be included.

REFERENCES

1. A. M. Russell. "Functions of Bounded kth Variation." *Proc. London Math. Soc.* (3) 26 (1973):547-563.
2. D. V. Widder. *Laplace Transform.* Princeton: Princeton University Press, 1946.

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PYTHAGOREAN TRIANGLES AND MULTIPLE ANGLES

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In a paper dealing with Pythagorean triangles, Gruhn [1] asked how many pairs of primitive Pythagorean triangles exist in which the sine of one of the acute angles of the second triangle equals the sine of twice either of the acute angles of the first triangle. This question may be generalized to determining pairs of primitive Pythagorean triangles where an acute angle of the second is N times an acute angle of the first (here N can take on any positive integer value). In addition, it may be asked whether any relationship exists among the generators of such primitive Pythagorean triangles.

It is necessary to review first some known results from number theory and trigonometry. A Pythagorean triangle is a right triangle whose sides are positive integers. Such triangles will be designated by the triple (x,y,z) which satisfies the equation $x^2 + y^2 = z^2$. In the case where x and y are relatively prime, the triangle is said to be primitive. Formulas for the sides of primitive Pythagorean triangles in terms of generators m and n are (see [2]):

$$x = m^2 - n^2;\ y = 2mn;\ z = m^2 + n^2$$

where m and n are positive integers such that

$$m > n;\ (m,n) = 1;\ mn \text{ is even.}$$

For a given primitive triangle (x,y,z), the generators may be found from:

$$m = \sqrt{(z + x)/2};\ n = \sqrt{(z - x)/2}.$$

Some formulas for the expansion of sin NA and cos NA in terms of sin A and cos A are as follows (see [3, 4]):

(1) $$\sin NA = \sin A\left\{(2\cos A)^{N-1} - \binom{N-2}{1}(2\cos A)^{N-3} + \binom{N-3}{2}(2\cos A)^{N-5}\ \ldots\right\}$$

(2) $$= \binom{N}{1}\sin A\cos^{N-1}A - \binom{N}{3}\sin^3 A\cos^{N-3}A + \binom{N}{5}\sin^5 A\cos^{N-5}A\ \ldots$$

(3) $$\cos NA = \binom{N}{0}\cos^N A - \binom{N}{2}\sin^2 A\cos^{N-2}A + \binom{N}{4}\sin^4 A\cos^{N-4}A\ \ldots\ .$$

The following conventions will be used throughout this paper:

1. θ: minimum of the acute angles of the original primitive Pythagorean triangle
2. N: a positive integer
3. $T_N = (x_N, y_N, z_N)$, y_N even: a primitive Pythagorean triangle where one of the acute angles is N times one of the acute angles of the original triangle
4. m_N, n_N: generators of T_N
5. $\sin\theta = \min(x_1/z_1, y_1/z_1)$

PRIMITIVENESS OF T_N

It is obvious that pairs of primitive Pythagorean triangles having an acute angle of the second N times an acute angle of the first may be obtained whenever $\theta < 90°/N$ or, equivalently, $\min(x_1/z_1, y_1/z_1) < \sin 90°/N$. In the following, therefore, when T_N is cited, it is assumed that this condition is satisfied.

Theorem 1: T_1 primitive implies T_N primitive. In order to prove this theorem, the following lemmas are needed.

Lemma 1: (i) If $x_1 < y_1$, then $\sin N\theta = \begin{cases} x_N/z_N, & N \text{ odd} \\ y_N/z_N, & N \text{ even;} \end{cases}$

(ii) If $x_1 > y_1$, then $\sin N\theta = y_N/z_N$.

Proof: Use is made of formula (1) for $\sin N\theta$. For $x_1 < y_1$, $\sin\theta = x_1/z_1$. When N is even, every term in the bracket involves 2 cos θ. Thus, the sum and also $\sin N\theta$ will be a fraction with an even numerator. The value of $\sin N\theta$ can therefore be written as y_N/z_N. When N is odd, every term in the bracket except the last term will involve 2 cos θ. The last term has value one. Thus, the bracket will be a fraction with an odd numerator and $\sin N\theta$ will be a fraction with an odd numerator, i.e., x_N/z_N. For $x_1 > y_1$, $\sin\theta = y_1/z_1$. Therefore, $\sin N\theta$ will be a fraction with an even numerator, i.e., y_N/z_N.

Lemma 2: $(z_2, x_N) = (z_2, y_N) = 1$.

Proof: It is equivalent to show that

$$(z_2,\ z_N\sin N\theta) = (z_2,\ z_N\cos N\theta) = 1$$

or

$$(x_1^2 + y_1^2,\ z_N\sin N\theta) = (x_1^2 + y_1^2,\ z_N\cos N\theta) = 1.$$

Use is made of formulas (2) and (3) for $\sin N\theta$ and $\cos N\theta$. Initially, consider the case where $x_1 < y_1$, i.e., $\sin\theta = x_1/z_1$:

$$z_N\sin N\theta = \binom{N}{1}x_1y_1^{N-1} - \binom{N}{3}x_1^3y_1^{N-3} + \binom{N}{5}x_1^5y_1^{N-5}\ \ldots$$

$$= (x_1^2 + y_1^2)Q(x_1,y_1) + x_1(2y_1)^{N-1}$$

$$= z_2Q(x_1,y_1) + x_1(2y_1)^{N-1},$$

where Q is some polynomial function of x_1 and y_1. Any divisor of z_2 and $z\sin N\theta$ must divide $x_1(2y_1)^{N-1}$. Now $(z_2,x_1) = (z_2,y_1) = 1$ since, otherwise, x_1 and y_1 would have a divisor greater than one contradicting the assumption that T_1 is primitive. Also $(z_2,2) = 1$ since z_2 is odd. Thus $(z_2, x_1(2y_1)^{N-1}) = 1$ and this implies that $(z_2, z_N\sin N\theta) = 1$.

Similarly,

$$z_N \cos N\theta = \binom{N}{0} y_1^N - \binom{N}{2} x_1^2 y_1^{N-2} + \binom{N}{4} x_1^4 y_1^{N-4} \ldots$$
$$= (x_1^2 + y_1^2) R(x_1, y_1) + y_1 (2y_1)^{N-1},$$

where R is some polynomial function of x_1 and y_1. The same reasoning as before shows that $(z_2, z_N \cos N\theta) = 1$. The case where $x_1 > y_1$, i.e., $\sin\theta = y_1/z_1$, can be handled in the same manner.

The proof of Theorem 1 can be accomplished by mathematical induction. The theorem is trivially true for $N = 1$. Assume that it is true for $N = k$ and try to show its validity for $N = k + 1$. Use is made of the addition formulas:

$$(4)\quad \begin{cases} \sin(k+1)\theta = \sin\theta\cos k\theta + \cos\theta\sin k\theta \\ \cos(k+1)\theta = \cos\theta\cos k\theta - \sin\theta\sin k\theta \end{cases}$$

There are three cases to consider: (i) $x_1 < y_1$, k odd; (ii) $x_1 < y_1$, k even; (iii) $x_1 > y_1$. In the first case, by use of Lemma 1, formulas (4) become

$$\frac{y_{k+1}}{z_{k+1}} = \frac{x_1 y_k}{z_1 z_k} + \frac{y_1 x_k}{z_1 z_k}$$

$$\frac{x_{k+1}}{z_{k+1}} = \frac{y_1 y_k}{z_1 z_k} - \frac{x_1 x_k}{z_1 z_k}$$

By taking $z_{k+1} = z_1 z_k$ and working only with the numerators, the equations become:

$$(5)\quad \begin{cases} y_{k+1} = x_1 y_k + y_1 x_k \\ x_{k+1} = y_1 y_k - x_1 x_k \end{cases}$$

It must be shown that $(x_{k+1}, y_{k+1}) = 1$. Now, any divisor of x_k and y_k divides both x_{k+1} and y_{k+1}. Equations (5) can be rewritten as

$$z_2 x_k = y_1 y_{k+1} - x_1 x_{k+1}$$

$$z_2 y_k = x_1 y_{k+1} + y_1 x_{k+1}$$

Since, by Lemma 2, z_2 is relatively prime to both x_{k+1} and y_{k+1}, any common divisor of x_{k+1} and y_{k+1} must divide x_k and y_k. Therefore, $(x_{k+1}, y_{k+1}) = (x_k, y_k) = 1$. The reasoning in each of the other cases is identical, appropriate substitutions being made for the various trigonometric functions.

CALCULATION OF T_N

In order to compute T_N from a given triple T_1, it is first necessary to check that $\min(x_1/z_1, y_1/z_1) < \sin 90°/N$. If this condition is satisfied, then $z_N = z_1^N$. Formulas (2) and (3) can be used to calculate $z_N \sin N\theta$ and $z_N \cos N\theta$. For x_N take the odd number of this pair while for y_N take the even number. Table 1 lists formulas for $z_N \sin N\theta$, $z_N \cos N\theta$, z_N for $N = 1, \ldots, 7$ and $x_1 < y_1$. Formulas, identical to these, for sides of $T_2, \ldots, T_5$ were cited by Vieta in 1646 [5]. He called T_2—the triangle of the double angle, T_3—the triangle of the triple angle, etc.

Examples of calculated T_N values are given in Table 2. The T_2 examples serve further to refute Gruhn's original conjecture that (3,4,5) and (7,24,25) are the only pair of primitive Pythagorean triangles in which the sine of one of the acute angles of the second triangle equals the sine of twice either of the acute angles of the first triangle. It is to be noted that both Malament [6] and Beran [7] have separately corrected Gruhn's statement.

GENERATORS OF T_N

Table 2 also lists generator values for the triangles calculated. Recursive formulas for the generators are as follows:

Theorem 2: (i) N even:

$$m_N = \max\{m_1 n_{N-1} + m_{N-1} n_1,\ m_1 m_{N-1} - n_1 n_{N-1}\}$$

$$n_N = \min\{m_1 n_{N-1} + m_{N-1} n_1,\ m_1 m_{N-1} - n_1 n_{N-1}\}$$

(ii) N odd and greater than one:

$$m_N = m_1 m_{N-1} \pm n_1 n_{N-1}$$

$$n_N = |m_1 n_{N-1} \mp n_1 m_{N-1}|$$

Note: Use upper sign for $x_1 < y_1$, otherwise use lower sign.

TABLE 1. Typical Formulas for T_N, $x_1 < y_1$

	T_N		
N	$z_N \sin N\theta$	$z_N \cos N\theta$	z_N
1	x_1	y_1	$z_1 = (x_1^2 + y_1^2)^{1/2}$
2	$2x_1y_1$	$y_1^2 - x_1^2$	z_1^2
3	$3x_1y_1^2 - x_1^3$	$y_1^3 - 3x_1^2y_1$	z_1^3
4	$4x_1y_1^3 - 4x_1^3y_1$	$y_1^4 - 6x_1^2y_1^2 + x_1^4$	z_1^4
5	$5x_1y_1^4 - 10x_1^3y_1^2 + x_1^5$	$y_1^5 - 10x_1^2y_1^3 + 5x_1^4y_1$	z_1^5
6	$6x_1y_1^5 - 20x_1^3y_1^3 + 6x_1^5y_1$	$y_1^6 - 15x_1^2y_1^4 + 15x_1^4y_1^2 - x_1^6$	z_1^6
7	$7x_1y_1^6 - 35x_1^3y_1^4 + 21x_1^5y_1^2 - x_1^7$	$y_1^7 - 21x_1^2y_1^5 + 35x_1^4y_1^3 - 7x_1^6y_1$	z_1^7

TABLE 2. Some Examples of T_N

Example		T_1	T_2	T_3	T_4	T_5
A	x:	5	119	2035	-	-
	y:	12	120	828		
	z:	13	169	2197		
	m:	3	12	46		
	n:	2	5	9		
B	x:	7	527	11753	354144	9653287
	y:	24	336	10296	164833	1476984
	z:	25	625	15625	390625	9765625
	m:	4	24	117	527	3116
	n:	3	7	44	336	237
C	x:	35	1081	27755	462961	
	y:	12	840	42372	1816080	-
	z:	37	1396	50653	1874161	
	m:	6	35	198	1081	
	n:	1	12	107	840	
D	x:	3	7			
	y:	4	24	-	-	-
	z:	5	25			
	m:	2	4			
	n:	1	3			
E	x:	15	161	495		
	y:	8	240	4888	-	-
	z:	17	289	4913		
	m:	4	15	52		
	n:	1	8	47		

Proof of Theorem 2: Initially, consider the case where N is odd and $x_1 < y_1$. The remaining cases are proved in a similar manner. Using the addition formulas (4) for $\sin N\theta$ and $\cos N\theta$ and Lemma 1, the following values are obtained for the sides of T_N in terms of the generators of T_1 and T_{N-1}:

$$x_N = 4m_{N-1}n_{N-1}m_1n_1 + m_{N-1}^2m_1^2 - m_{N-1}^2n_1^2 - n_{N-1}^2m_1^2 + n_{N-1}^2n_1^2$$

$$y_N = 2[m_1n_1(m_{N-1}^2 - n_{N-1}^2) - m_{N-1}n_{N-1}(m_1^2 - n_1^2)]$$

$$z_N = m_{N-1}^2m_1^2 + m_{N-1}^2n_1^2 + n_{N-1}^2m_1^2 + n_{N-1}^2n_1^2$$

Consequently:

$$m_N = \sqrt{(z_N + x_N)/2} = m_1m_{N-1} + n_1n_{N-1}$$

$$n_N = \sqrt{(z_N - x_N)/2} = m_1n_{N-1} - n_1m_{N-1}$$

It is also to be noted that the sides of T_N serve as generators for T_{2N} where these exist. Thus, for instance, for $T_1 = (5,12,13)$, the sides 5 and 12 serve as generators for $T_2 = (119,120,169)$. Similarly, for $T_2 = (1081,840,1369)$, the sides serve as generators for $T_4 = (462961,1816080,1874161)$.

REFERENCES

1. E. W. Gruhn. "Parabolas and Pythagorean Triples." *Math. Teacher* (Dec. 1959):614-615.
2. Waclaw Sierpinski. *Pythagorean Triangles*. The Scripta Mathematica Studies, #9. New York: Graduate School of Science, Yeshiva University, 1962, pp. 6-7.
3. Murray R. Spiegel. *Mathematical Handbook*. Schaum's Outline Series. New York: McGraw-Hill Book Company, 1968 (Formula 1, p. 17).
4. E. W. Hobson. *A Treatise on Plane and Advanced Trigonometry*. 7th ed. New York: Dover Publications, Inc., 1928 (Formulas 2 and 3, p. 52).
5. F. Vietae. *Ad Logisticen specioseam, notae priores*. Prop. 46-51. In *Opera Mathematica*, 1646, pp. 34-37.
6. D. Malament. "Letters to Editor." *Math. Teacher* (May 1960):380.
7. R. Beran. "Letters to Editor." *Math. Teacher* (Oct. 1960):466.

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PROOF THAT THE AREA OF A PYTHAGOREAN TRIANGLE IS NEVER A SQUARE

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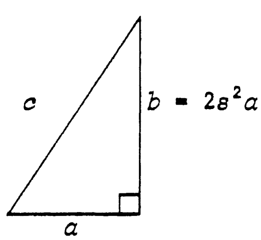
Prove that the area of an integral-sided (Pythagorean) triangle is never a square integer. In the diagrams provided below, the two triangles are equivalent. Thus, $a = a$, $b = n$, and $c = (n + k)$, where a, b, n, and k as well as s are integers. A = the area of the triangles.

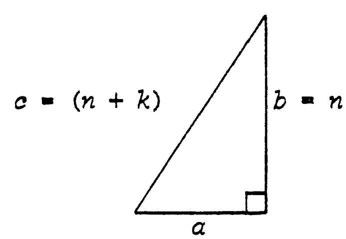


$A = \frac{1}{2}(2s^2a)a = s^2a^2$, which is a square

$a^2 + b^2 = c^2$

$a^2 + (2s^2a)^2 = c^2$

$a^2 + 4s^4a^2 = c^2$



$a^2 + b^2 = c^2$

$a^2 + n^2 = (n + k)^2; \quad a^2 = 2kn + k^2$

$(2kn + k^2) + n^2 = (n + k)^2$