# THEORY OF EXTRA NUMERICAL INFORMATION APPLIED TO THE FIBONACCI SUM

JEROME HINES
*South Orange, New Jersey*

In both logic and mathematics the comma is used to represent the *ordered* and *unordered* concepts of *and*. This equivocation in the use of the comma is bad notation which can lead to serious problems. It is also unwise to indicate ordering by changing brackets to parentheses. To avoid these problems, we will denote the *unordered and* by the common plus sign $+$, and the *ordered and* by the symbol $\dotplus$, to be called *proto-plus*. $+$ will be employed as ordinary addition when it is used with real and complex numbers. Obviously this creates a problem regarding the use of the *unordered and* in set theory. For example, instead of the set $\{2, 3, 4\}$, we would be obliged to write $\{2 + 3 + 4\}$, which would indicate adding 2, 3, and 4, yielding 9, which was not the original intention. This problem is resolved by building an enlightening new set theory out of the properties of its own elements. The first step in this direction is to introduce *ordered multiplication*, a noncommutative operation denoted by the symbol $\circ$. This operation will enable us to differentiate between concepts such as "two" and "a two" (one *of* two, or a pair, denoted by $1 \circ 2$). $2 \circ 3$ would then be understood as "two triples." The axioms for $+$ and $\dotplus$ will be given later. Next we introduce the concept of "*any* counting number," denoted by $\omega$, where $\omega + \omega = \omega$. A set containing pencils ($p$) and erasers ($e$) would not be written as $\{p, e\}$, but as $\omega \circ p + \omega \circ e$. Naturally, $\omega \circ 2 + \omega \circ 3 + \omega \circ 4$ does not equal $\omega \circ 9$. Adding two operations of "choice" ($\subset$) and "anti-choice" ($\not\subset$) completes the list of operations necessary for the construction of this new set theory. One of the interesting consequences of this approach is that the operations of $+$ (ordinarily denoted by the comma between elements) and *union* (ordinarily denoted by $\cup$) are one and the same. Many other interesting insights arise from this approach.

The ordered collection "*a and then b and then c*" will be written as $a \dotplus b \dotplus c$, and we will introduce a sigma type notation, parallel to the common use of $\Sigma$, for iterated use of $\dotplus$, to be denoted by $\sigma$. $\overset{n}{\underset{i=1}{\sigma}} f(n)$ will then denote $f(1) \dotplus f(2) \dotplus f(3) \dotplus \cdots \dotplus f(n)$, and will be called a *proto-sum*. If "$a$" and "$b$" are real numbers, "$a \dotplus b$" will be called a *proto-number* (as well as a *proto-sum*). Obviously

(1a) $$a \dotplus b \neq b \dotplus a$$

We define proto-minus $\dotminus$ by

(1b) $$a \dotplus (-b) = a \dotminus b$$

Note that $1 \dotminus 1$ is *not* zero, but differs from it only by the *extra-numerical information* of *ordering*. We shall call such a term a *proto-null*, and, since it is not zero, we can use it as a divisor.

We will now present the axioms for $\dotplus$ and the *proto-numbers*. Given a collection of real numbers $R$, with elements "$a$" and "$b$", and a collection of proto-numbers $P$, with elements "$p$", "$q$", "$r$", and "$s_i$" ($i$ any counting number), and three operations $+$, $\dotplus$, and $\cdot$ in $R$ and $P$, then

| | | |
|---|---|---|
| (2a) | $(\forall a)(\forall b)$ | $a \dotplus b \in P$ |
| (2b) | $(\forall p)(\forall q)$ | $p + q \in P$ |
| (2c) | $(\forall p)(\forall q)$ | $p \dotplus q \in P$ |
| (2d) | $(\forall p)(\forall q)$ | $p \cdot q \in P$ |
| (2e) | $(\forall p)(\forall q)$ | $p + q = q + p$ |
| (2f) | $(\forall p)(\forall q \neq P)$ | $p \dotplus q \neq q \dotplus p$ |
| (2g) | $(\forall p)(\forall q)$ | $p \cdot q = q \cdot p$ |
| (2h) | $(\forall p)(\forall q)(\forall r)$ | $p + (q + r) = (p + q) + r$ |
| (2i) | $(\forall p)(\forall q)(\forall r)$ | $p + (q \dotplus r) = (p + q) \dotplus r = p + q \dotplus r$* |
| (2j) | $(\forall p)(\forall q)(\forall r)$ | $p \cdot (q \cdot r) = (p \cdot q) \cdot r$ |
| (2k) | $(\forall p)(\exists 0)$ | $p + 0 = 0 + p = p$ |
| (2l) | $(\forall p)(\exists 0)$ | $p \dotplus 0 = p$ |
| (2m) | $(\forall p \neq 0)$ | $0 \dotplus p \neq p$ |
| (2n) | $(\forall p)(\exists -p)$ | $-p + p = p + (-p) = 0$ |
| (2o) | $(\forall p)(\forall q)(\forall r)$ | $p \cdot (q + r) = p \cdot q + p \cdot r$ |
| (2p) | $(\forall p)(\forall s_i)$ | $p \cdot \overset{\infty}{\underset{i=1}{\sigma}} (s_i) = \overset{\infty}{\underset{i=1}{\sigma}} (p \cdot s_i)$ |

*$+$ has precedence of operation over $\dotplus$.

We would like to add the axiom

(2q) $$(\forall p \neq 0)(\exists p^{-1}) \qquad p^{-1} \cdot p = p \cdot p^{-1} = 1$$

but we need extra concepts to deal with the multiplicative inverses of $0 + 1$ and $1 - 1$.

The case of $0 + 1$ can be handled by introducing the ordered operation *retro-plus*, denoted by +, which has an axiomatic system which is the exact mirror image of the axiomatic system developed for +, with the condition that

(3a) $$0 + (1 + 0) = 1$$

Then we can deal with *retro-numbers* which are ordered from right to left instead of from left to right. It follows that

(3b) $$0 + 1 = \frac{1}{0 + 1}$$

and we find that $0 + 1$ is the multiplicative inverse of $0 + 1$.

Before discussing the multiplicative inverse of $1 - 1$, we must have more tools to work with. By the above axioms we can show that

(3c) $$\mathop{\boldsymbol{\sigma}}_{i=1}^{n} (a_i + b_i) = \mathop{\boldsymbol{\sigma}}_{i=1}^{n} a_i + \mathop{\boldsymbol{\sigma}}_{i=1}^{n} b_i$$

(3d) $$\mathop{\boldsymbol{\sigma}}_{i=1}^{n} (a_i + b_i) = \mathop{\boldsymbol{\sigma}}_{i=1}^{n} a_i + \mathop{\boldsymbol{\sigma}}_{i=1}^{n} b_i$$

(3e) $$\mathop{\boldsymbol{\sigma}}_{i=1}^{m} \sum_{j=1}^{n} {}_i a_j = \sum_{j=1}^{n} \mathop{\boldsymbol{\sigma}}_{i=1}^{m} {}_i a_j$$

(3f) $$\mathop{\boldsymbol{\sigma}}_{j=1}^{n} \mathop{\boldsymbol{\sigma}}_{i=1}^{m} {}_i a_j = \mathop{\boldsymbol{\sigma}}_{i=1}^{m} \sum_{j=1}^{n} \left( {}_{i-j+1}a_j \cdot {}^{m}\mathfrak{z}_{i-j+1} \cdot {}^{n}\mathfrak{z}_i \right), \text{ where } {}^{m}\mathfrak{z}_i = 1,\ 0 < i \leq m^{*}$$
$$= 0,\ 0 \geq i > n$$

(3g) $$\left( \mathop{\boldsymbol{\sigma}}_{i=1}^{m} a_i \right) \cdot \left( \mathop{\boldsymbol{\sigma}}_{j=1}^{n} b_i \right) = \mathop{\boldsymbol{\sigma}}_{i=1}^{m} \sum_{j=1}^{n} \left( a_{i-j+1} \cdot b_j \cdot {}^{m}\mathfrak{z}_{i-j+1} \cdot {}^{n}\mathfrak{z}_j \right)$$

(3h) $$(a + b)^r = \mathop{\boldsymbol{\sigma}}_{i=1}^{r+1} \left[ \frac{r!}{(r - i + 1)!\,(i - 1)!} a^{r-i+1} b^{i-1} \right]$$

The binomial expansion is an operation between ordered sums and equation (3h) is its only legitimate expression. In this treatise we will only consider sums generated by the binomial expansion, giving us the basis for a theory of rational proto-numbers.

Another common operation between ordered sums is long division, and it must be considered a proto-algorithm. Henceforth we shall refer to it as proto-division. It turns out to be the operational inverse of the proto-multiplication given in equation (3g). As an example of equation (3g),

(3i) $(1 + 2 + 3) \cdot (4 + 5 + 6)$

$= 1 \cdot 4 + (1 \cdot 5 + 2 \cdot 4) + (1 \cdot 6 + 2 \cdot 5 + 3 \cdot 4) + (2 \cdot 6 + 3 \cdot 5) + 3 \cdot 6$

$= 4 + 13 + 28 + 27 + 18$

The following demonstrates that proto-division is the inverse of this operation:

(3j)
$$\begin{array}{r|rrrrr}
 & 4 + & 5 + & 6 & & \\ \hline
1 + 2 + 3 & 4 + & 13 + & 28 + & 27 + & 18 \\
 & 4 + & 8 + & 12 + & 0 + & 0 \\ \hline
 & & 5 + & 16 + & 27 + & 18 \\
 & & 5 + & 10 + & 15 + & 0 \\ \hline
 & & & 6 + & 12 + & 18 \\
 & & & 6 + & 12 + & 18 \\ \hline
 & & & & 0 + & 0
\end{array}$$

Given the proto-numbers $p$, $q$ ($\neq 0$), and $r$, we define

(4a) $$r = p \div q = \frac{p}{q}$$

(4b) $$\frac{p}{q} + \frac{r}{s} = \frac{p \cdot s + q \cdot r}{q \cdot s}$$

---

*The coefficients ${}^{m}\mathfrak{z}_i$ can be more generally developed, but space does not permit further discussion.

(4c) $$\frac{p}{q} + \frac{r}{s} = \frac{p \cdot s + q \cdot r}{q \cdot s}$$

(4d) $$\frac{p}{q} \cdot \frac{r}{s} = \frac{p \cdot r}{q \cdot s}$$

(4e) $$\frac{p}{q} - \frac{r}{s} = \frac{p \cdot s - q \cdot r}{q \cdot s}$$

(4f) $$\frac{p}{q} \dot{-} \frac{r}{s} = \frac{p \cdot s \dot{-} q \cdot r}{q \cdot s}$$

(4g) $$\frac{p}{q} \div \frac{r}{s} = \frac{p \cdot s}{q \cdot r}$$

We will adhere to the additive index law for real powers of proto-numbers where, for $m,n \varepsilon R$,

(4h) $$p^n = \overbrace{p \cdot p \cdot p \cdot \cdots \cdot p}^{n}$$

(4i) $$p^m \cdot p^n = p^{m+n}$$

(4j) $$p^m \div p^n = \frac{p^m}{p^n} = p^{m-n}$$

(4k) $$p^0 = 1$$

Probably all ordered summation processes are related to this axiomatic system. For example, we cannot consider any specific infinite sum without ordering its terms. Thus, all infinite series as commonly used are actually infinite proto-sums. Consider the following example:

(5a) $$\frac{1}{1 - a} = \sum_{i=1}^{\infty} a^{i-1}$$

which should be written as

(5b) $$\frac{1}{1 \dot{-} a} = \overset{\infty}{\underset{i=1}{\boldsymbol{\sigma}}} a^{i-1}$$

In equation (5a), $\frac{1}{1 - a}$ is the total of the infinite sum and is a real number if "$a$" is real. This sort of expression fails to differentiate between a sum and its numerical *total*. In the case of equation (5b), we must remember that $\frac{1}{1 \dot{-} a}$ is *not* a real number, but a proto-number. To find a numerical *total* for the infinite proto-sum in equation (5b), we must devise a means of relating terms such as $\frac{1}{1 \dot{-} a}$ and $\frac{1}{1 - a}$. This is not as simple as it appears, and requires a study of both the null and infinite proto-numbers.

To define infinite proto-numbers, we must consider the multiplicative inverse of the simplest proto-null $1 \dot{-} 1$. Let us begin by denoting the multiplicative inverse of $1 \dot{-} 1$ by $\underline{\text{И}}$,* which we will call proto-И. Then

(6a) $$\underline{\text{И}} = \frac{1}{1 \dot{-} 1} = (1 \dot{-} 1)^{-1}$$

(6b) $$\underline{\text{И}} \dot{-} \underline{\text{И}} = \frac{1}{1 \dot{-} 1} \dot{-} \frac{1}{1 \dot{-} 1} = \frac{1 \dot{-} 1}{1 \dot{-} 1} = 1$$

Similarly, we can show that

(6c) $$(\underline{\text{И}})^r \dot{-} (\underline{\text{И}})^r = (\underline{\text{И}})^{r-1}$$

We can also prove that, as a consequence of the axiom in equation (2q),

(6d) $$\underline{\text{И}} = 1 + 1 + 1 + \cdots .$$

*И is the letter for *yee* in the Russian alphabet.

We could have anticipated this result by dividing out $\frac{1}{1 - 1}$ by proto-division, and it can be shown that such a division in a proto-system leaves *no remainder*. Similarly, there is no remainder in a proto-system when we divide out $\frac{1}{1 - a}$ to get the result in equation (5b). Because of this, it is easily seen that the proto-binomial expansion in equation (3h) is true for *all* real values of $a$, $b$, and $r$, in a proto-system. Surprising as this result may be, we must bear in mind that an equation such as

(6e) $$\frac{1}{1 - 2} = \overset{\infty}{\underset{i=1}{\sigma}} 2^{i-1}$$

is not a contradiction since $\frac{1}{1 - 2} \neq -1$.

We are familiar with sequences having an open end; i.e., an infinite number of terms such as $f(1) + f(2) + \cdots + f(n) + \cdots$ . Sequences having unique first and last terms, with an infinite number of terms in between, are less familiar, but have been employed, for example, by Cantor and others. We will say that sums and sequences of this sort have open middles.

Using an infinite proto-sum with an open middle, we define the *intrafinite integer* И by

(6f) $$И = \overset{И}{\underset{i=1}{\sigma}} (1) = \overbrace{1 + 1 + 1 + \cdots + 1}^{И}$$

Then we introduce a *principle* of *substitution* for И such that, if

(6g) $$g(n) = \overset{n}{\underset{i=1}{\sigma}} f(i)$$

then

(6h) $$g(И) = \overset{И}{\underset{i=1}{\sigma}} f(i)$$

For $f(i) = i$ this becomes

(6i) $$g(И) = \overset{И}{\underset{i=1}{\sigma}} i = 1 + 2 + 3 + 4 + \cdots + И$$

For every term in this proto-sum, up to and including И, we can associate its value with its rank. Obviously we are not dealing with the class of natural numbers, since the natural numbers are all finite in size, despite the fact that there are an infinite number of them. We will call our collection *the amorphous intrafinite numbers*.

Henceforth we will abbreviate $\overset{И}{\underset{i=1}{\sigma}}$ by $\underset{i}{\sigma}$ and will always use it in place of $\overset{\infty}{\underset{i=1}{\sigma}}$, which is actually meaningless due to the ambiguity of ∞. We could, in a sense, interpret И as being the number of all *counting numbers*.

Now we must construct a system of numeration of radix (base) И. Note that a system of numeration is also a form of proto-math. In a system of numeration of radix Γ, we proto-add "ones" Γ times, *and then* proto-add another Γ "ones," etc. The empty frame, with Γ positions to be filled by "ones" in such a system of numeration, will be called a *Collect** of radix Γ.

If $n < \Gamma$, then $\overset{n}{\underset{i=1}{\sigma}} (1)$ will be called a proto-digit, to be written as $\underset{\sim}{n}$; i.e.,

(6j) $$\overset{n}{\underset{i=1}{\sigma}} (1) = \underset{\sim}{n}, \; n < \Gamma$$

By choosing a system of numeration of radix И for a general approach, all infinite sums will be considered as Collects of radix И, as well as all finite sums; i.e.,

(6k) $$\overset{n}{\underset{i=1}{\sigma}} f(i) = f(1) + f(2) + \cdots + f(n) + \overbrace{0 + 0 + \cdots + 0}^{И - n}$$

where the right-hand proto-sum will have И terms.

*Pronounced kólekt.

Notationwise, we will use the following convention:

(6l)
$$\underset{i}{\boldsymbol{\sigma}} f(i) = f(1) + f(2) + f(3) + \cdots + f(\text{И})$$
$$= f(1) + f(n) + f(3) + \cdots + f(n) + \cdots$$

employing either the open end or open middle notion as desired, considering them as equivalent.

By equations (3g) and (6d), we can show that

(6m)
$$\underline{\text{И}}^{r} = \frac{1}{(1-1)^{r}} = \underset{i}{\boldsymbol{\sigma}}\left[\frac{(r+i-2)!}{(r-1)!(i-1)!}\right]$$

and

(6n)
$$\underline{\text{И}}^{-r} = (1-1)^{r} = \underset{i}{\boldsymbol{\sigma}}\left[\frac{(-1)^{i+1}r!}{(r-i+1)!(i-1)!}\right]$$

In order to obtain *totals* for infinite sums, we must relate proto-sums to their corresponding unordered sums. To do so, we must delete all information concerning ordering, introducing a certain degree of indeterminacy, so one can no longer differentiate between + and +. To accomplish this, note that

(6o)
$$\overbrace{0 + \cdots + 0}^{n} + a = a \cdot (0 + 1)^{n} = a(1 - \underline{\text{И}}^{-1})^{n}$$

Then, given $\overset{m}{\underset{i=1}{\boldsymbol{\sigma}}} a_i \underline{\text{И}}^{p}$, we see, by equation (6n) that

(6p)
$$\overset{m}{\underset{i=1}{\boldsymbol{\sigma}}} a_i \underline{\text{И}}^{p} = \sum_{i=1}^{m} a_i \underline{\text{И}}^{p} + B$$

where $B$ is a linear, unordered sum of powers of $\underline{\text{И}}$, all less than $p$. If we drop all terms of lower potency than $\underline{\text{И}}^{p}$, there is no distinction between + and + (this reasoning also holds for $m = \text{И}$, provided the proto-total of $\underset{i}{\boldsymbol{\sigma}} a_i \underline{\text{И}}^{p}$ contains no term of potency greater than $\underline{\text{И}}^{p}$). Setting $\overset{m}{\underset{i=1}{\boldsymbol{\sigma}}} a_i \underline{\text{И}}^{p}$ equal to $\sum_{i=1}^{m} a_i \underline{\text{И}}^{p}$, we have introduced a certain indeterminacy concerning all additive terms of potency less than $\underline{\text{И}}^{p}$. Such a relation will be called a *reduced equation*, or an *isonomic relation*, which we will denote by a "$p$" under the equality sign, whence

(6q)
$$\overset{m}{\underset{i=1}{\boldsymbol{\sigma}}} a_i \underline{\text{И}}^{p} \underset{p}{=} \sum_{i=1}^{m} a_i \underline{\text{И}}^{p}$$

to be read, "$\overset{m}{\underset{i=1}{\boldsymbol{\sigma}}} a_i \underline{\text{И}}^{p}$ is *isonomic to* $\sum_{i=1}^{m} a_i \underline{\text{И}}^{p}$, in a *reduced equation* of order $p$."

Now, to relate $\text{И}^{r}$ and $\underline{\text{И}}^{r}$, we must take into account the missing remainders which are peculiar to proto-math. The following example demonstrates the problem:

Applying the *principle of substitution* of И to

(6r)
$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

we have

(6s)
$$\sum_{i=1}^{\text{И}} i = \frac{\text{И}^{2}}{2} + \frac{\text{И}}{2}$$

This is the unordered sum that corresponds to the proto-sum $\underset{i}{\boldsymbol{\sigma}} i$ in equation (6i), which has as a proto-total $\underline{\text{И}}^{2}$. In a reduced equation of order 2, $\text{И}^{2}/2 + \text{И}/2$ becomes $\text{И}^{2}/2$ and we are forced to conclude that

(6t)
$$\frac{\text{И}^{2}}{2} \underset{2}{=} \underline{\text{И}}^{2}$$

More generally, we find that

(6u)
$$\text{И}^{r} \underset{r}{=} r!\,\underline{\text{И}}^{r}$$

a result that can be derived from consideration of equation (6l) and its unordered counterpart. The same result can also be derived from consideration of the missing remainders in multiplication of powers of $\underline{\aleph}$, but present space does not permit this. We would expect to find a relation between $\aleph^{-r}$ and $\underline{\aleph}^{-r}$ corresponding to equation (6u), but this has not been accomplished yet. So we will take equation (6u) as our definition for $\aleph^{r}$, for $r \geq 0$.

Now we must investigate the type of proto-sum generated by dividing our $1/1 \dot{-} a$ by proto-division, where $|a| > 1$. As in the case of equation (6e), we are faced, in reduced equations, with the enigma of absolutely divergent sums having finite totals. This result demonstrates that numbers of such magnitude as $2^{\aleph}$ cannot be dealt with realistically using summation, but necessitate the use of infinite products. That would be beyond the scope of this present work, but note in passing that it can be accomplished through the use of $\circ$, the operation of ordered multiplication, based upon the following additional axioms for the collection $P$ of the proto-numbers:

| | | |
|---|---|---|
| (7a) | $(\forall p)(\forall q)$ | $p \circ q \in P$ |
| (7b) | $(\forall p)(\forall q \neq p)$ | $p \circ q = q \circ p$ |
| (7c) | $(\forall p)(\forall q)(\forall r)$ | $p \cdot (q \circ r) = (p \cdot q) \circ r = p \cdot q \circ r^{*}$ |
| (7d) | $(\forall p)(\exists 0)$ | $p \circ 0 = 0 \circ p = 0$ |
| (7e) | $(\forall p)(\forall q)(\forall r)$ | $(p + q) \circ r = p \circ r + q \circ r$ |
| (7f) | $(\forall p)(\forall q)(\forall r)$ | $(p \dotplus q) \circ r = p \circ r \dotplus q \circ r$ |
| (7g) | $(\forall p)(\forall q)(\forall r)$ | $p \circ (q + r) \neq p \circ q + p \circ r$ |
| (7h) | $(\forall p)(\forall q)(\forall r)$ | $p \circ (q \dotplus r) \neq p \circ q \dotplus p \circ r$ |
| (7i) | $(\forall p)$ | $p \circ 1 = p \neq 1 \circ p$ |
| (7j) | $(\forall a \in R)(\forall b \in R)$ | $a \circ b \in P$ |
| (7k) | | $a \circ b = a \cdot b^{0 \dotplus 1}$ |

These axioms enable us to include proto-numbers as exponents.

It is also easily seen that

(7l) $$a^{b} \circ a^{c} = a^{b+c}$$

(7m) $$1 \circ a = a^{0 \dotplus 1}$$

(7n) $$\overbrace{a \circ \cdots \circ a}^{n} = a^{\underline{n}}$$

(7o) $$a \circ (b \circ c) = (a \circ b) \circ (1 \circ c)$$

(7p) $$(a \circ b) \circ (c \circ d) = [a \circ (b \cdot c)] \circ (1 \circ d)$$

(7q) $$\overbrace{a^{\underline{m}} \circ \cdots \circ a^{\underline{m}}}^{n} = a^{\underline{m} \cdot \underline{n}}$$

(7r) $$\lg_{e}(a \circ b) = \lg_{e} a \dotplus \lg_{e} b$$

In the case of proto-sums generated by proto-division such that

$$\frac{1}{1 \dot{-} a} = \underset{i}{\sigma} a^{i-1}, \quad |a| > 1$$

the results obtained by employing reduced equations are all self-consistent within the system. Take the case of equation (6e): this reduces to

(8a) $$\sum_{i=1}^{n} 2^{i-1} \underset{0}{=} -1$$

(We treat this case as a reduced equation of order 0, since all such absolutely divergent sums act as though they were convergent; i.e., having zero-order totals.) This is why we say, as in equation (8a), that this properly divergent sum is *isonomic* to -1 (*not equal to* -1).

This means that, in the proto-system, $\sum_{i=1}^{\aleph}(2^{i-1})$ has the same properties, or obeys the same laws, as -1.

As an example of this consistency within the proto-system, consider the following: Dividing a convergent sum by an absolutely divergent sum should give us a reduced total of zero, as in

---

$*\cdot$ has precedence of operation over $\circ$.

(8b) $$\frac{\underset{i}{\sigma}(2^{-i})}{\underset{j}{\sigma}(1)} = \frac{1}{2} - \underset{k}{\sigma}(2^{-k-1}) = \frac{1}{2} - \frac{1}{2}\underset{k}{\sigma}(2^{-k}) = \frac{1}{2} - \frac{1}{2}\left(\frac{1}{2 - 1}\right) = \frac{1 - 1}{2 - 1}$$

which obviously reduces to zero, as anticipated.

If, instead, we claim $\sum_{i=1} 2^{i-1}$ to have the properties of -1, then proto-dividing $\underset{i}{\sigma}(2^{-i})$ by $\underset{j}{\sigma}(2^{j-1})$ should produce a result that reduces to -1, since the first sum converges to 1. Indeed, by proto-division,

(8c) $$\frac{\underset{i}{\sigma}(2^{-i})}{\underset{j}{\sigma}(2^{i-1})} = \frac{1}{2} - \underset{k}{\sigma}\left(\frac{3}{2^{k+1}}\right) = \frac{1}{2} - \frac{3}{2}\underset{k}{\sigma}(2^{-k}) = \frac{1}{2} - \frac{3}{2}\left(\frac{1}{2 - 1}\right) = \frac{1 - 2}{2 - 1}$$

which reduces to -1, as anticipated.

All absolutely divergent sums of the type

(8d) $$\frac{1}{1 - a} = \underset{i}{\sigma} a^{i-1}, \quad |a| > 1$$

which behave as though they were convergent in reduced equations, will be called *co-vergent*.

The Fibonacci sum $\underset{i}{\sigma} f(i)$ is generated by proto-division thusly:

(9a) $$F = \underset{i}{\sigma} f(i) = \frac{1}{1 - 1 - 1}$$

Since

(9b) $$1 - 1 - 1 = \left(1 - \frac{1 + \sqrt{5}}{2}\right) \cdot \left(1 - \frac{1 - \sqrt{5}}{2}\right)$$

then

(9c) $$\underset{i}{\sigma} f(i) = \frac{1}{\left(1 - \frac{1 + \sqrt{5}}{2}\right) \cdot \left(1 - \frac{1 - \sqrt{5}}{2}\right)}$$

whence the proto-Fibonacci sum is the product of two proto-sums:

(9d) $$c_1 = \underset{i}{\sigma}\left[\left(\frac{1 + \sqrt{5}}{2}\right)^{i-1}\right]$$

and

(9e) $$c_2 = \underset{i}{\sigma}\left[\left(\frac{1 - \sqrt{5}}{2}\right)^{i-1}\right]$$

$c_1$ is a co-vergent proto-sum, while $c_2$ is an alternating divergent proto-sum. In reduced equations of order zero, both sums are isonomic to finite numbers:

(9f) $$c_1 \overset{=}{_0} \frac{2}{1 + \sqrt{5}}$$

and

(9g) $$c_2 \overset{=}{_0} \frac{2}{1 - \sqrt{5}}$$

As one could anticipate, the product of these two "totals" is -1, the same as the reduced "total" of the Fibonacci proto-sum.

Let us form a new proto-sum by proto-adding every other term of the Fibonacci sum, beginning with the first term, to be denoted by $F_1$. Then let us form a second one by proto-adding every other term, beginning with the second term, to be denoted by $F_2$. Then

(9h) $$F_1 = \underset{i}{\sigma} f(2i - 1)$$

and

(9i) $$F_2 = \underset{i}{\sigma} f(2i)$$

It is easily seen, by proto-division, that

(9j) $$F_1 = \frac{1 - 1}{1 - 3 + 1}$$

and

(9k) $$F_2 = \frac{1}{1 - 3 + 1}$$

whence

(9l) $$F_2 = \underset{-}{\boldsymbol{N}} \cdot F_1$$

In reduced equations of zero order, $F_1$ will be isonomic to 0 and $F_2$ will be isonomic to -1.

For notational simplicity, let us introduce $\mathfrak{z}_i$ defined by

(9m) $$\mathfrak{z}_i = 1, \text{ for an odd integer}$$
$$= 0, \text{ for an even integer*}$$

Next, let us define $F_1'$ and $F_2'$ by

(9n) $$F_1' = \underset{i}{\sigma}[\mathfrak{z}_i f(i)]$$

and

(9o) $$F_2' = \underset{i}{\sigma}[\mathfrak{z}_i f(i + 1)]$$

Obviously, $F_1'$ is the proto-sum $F_1$ with zeros inserted between all of its terms. The same is true for $F_2'$ and $F_2$. It follows that

(9p) $$F_1' + F_2' = F$$

In zero-order reduced equations, the sum of the "totals" of $F_1'$ and $F_2'$ should be -1, the "total" of $F$. In substantiation of this, it is easily seen, by proto-division, that

(9q) $$F_1' = \frac{1 + 0 - 1}{1 + 0 - 3 + 0 + 1}$$

and

(9r) $$F_2' = \frac{1}{1 + 0 - 3 + 0 + 1}$$

Note that

(9s) $$1 + 0 - 1 = (1 + 1)(1 - 1)$$

and

(9t) $$1 + 0 - 3 + 0 + 1 = (1 - 1 - 1)(1 + 1 - 1)$$

The above equations substantiate equation (9p), since

(9u) $$F_1' + F_2' = \frac{1 + 0 - 1}{1 + 0 - 3 + 0 - 1} + \frac{1}{1 + 0 - 3 + 0 - 1} = \frac{(1 + 0 - 1) + 1}{1 + 0 - 3 + 0 + 1}$$
$$= \frac{1 + 1 - 1}{(1 - 1 - 1)(1 + 1 - 1)} = \frac{1}{1 - 1 - 1} = F \quad \text{[by (9a)].}$$

If we define the alternating Fibonacci proto-sum $\overline{F}$ by

(9v) $$\overline{F} = \underset{i}{\sigma}[(-1)^{i+1} f(i)] = \frac{1}{1 + 1 - 1}$$

it follows, by equations (9q) and (9r), that

(9w) $$F_1' = (1 + 0 - 1)F_2'$$

whence, by equations (9a), (9r), (9s), (9t), (9v), and (9w),

(9x) $$F_1' = (1 + 1)(1 - 1) \cdot F \cdot \overline{F}$$

Similarly,

(9y) $$F_2' = F \cdot \overline{F}.$$

Equation (9a) reduces to

(9z) $$F \underset{0}{=} -1$$

and equation (iv) reduces to

(10a) $$\overline{F} \underset{0}{=} 1.$$

Then, from equations (9x) and (9z) and (10a), we see that

(10b) $$F_1' \underset{0}{=} 0.$$

---

*The concept of $\mathfrak{z}$ can be generalized for some complex series.

Similarly,

(10c) $$F_2' \underset{0}{=} -1$$

whence

(10d) $$F_1' + F_2' \underset{0}{=} F_1' + F_2' \underset{0}{=} -1.$$

Comparing (9z) and (10d), we find another substantiation of (9p).

These operations on the Fibonacci proto-sum show how proto-math opens new vistas of research on infinite sums. It gives us the beginning of a nonconvergency approach to the summation of infinite series, and some of the classical methods of summing divergent series will be special cases of proto-math, one example of which is Cesaro's method. This is true since the sum of the partial sums of an infinite series is simply the operation of multiplying that proto-series by

(11a) $$\underline{\text{И}} = \underset{i}{\sigma}(1).$$

In proto-math, the laws for regrouping terms in divergent series are easily found, and they vary according to the orders of the reduced equations. Given proto-sums, whose "totals" are linear sums of positive powers of $\underline{\text{И}}$, we can find these totals by inspection of the $n$th terms. We can develop the differential calculus using $\bar{1} - 1$ instead of infinitesimals, freeing us from the need for limiting processes. Indeed, it may even be possible to develop most of our present-day mathematics without recourse to limiting processes.

There seems to be a vague similarity between the approaches of proto-math and Non-Standard Analysis, but instead of using hyperreal numbers and infinitesimals (which lie on the real line, we use proto-numbers and proto-nulls (which do not lie on the real line). There is no Standard part for the infinite numbers in Non-Standard Analysis, but with our isonomic relations we seem to have achieved a generalization which enables us to enter the infinite range and deal with it in a realistic fashion.

There also seems to be a vague similarity to Cantor's work on the infinite, but there are many important differences. For example, in a proto-system of numeration of radix (base) $\text{И}$, the number of digits is not the same as the number of rational numbers constructed from these digits, in distinct contrast to the Cantorial system, where the number of natural numbers cannot be distinguished from the number of rational numbers. Also, in the proto-system, rearranging the terms in a divergent sum gives us the same "total" as in the original sum. The need for *order types* and *ordinal numbers* does not arise.

Eventually, expressions such as

(11b) $$\lg_e(0) = \infty$$

and

(11c) $$\Gamma(0) = \infty$$

can surely be rendered obsolete, and expressions such as

(11d) $$e^{1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\cdots} \underset{1}{=} \text{И}$$

can be given a rigorous foundation.

Since we have not basically employed set theory in the development of our infinite concepts, and this is a non-Boolean approach, we should expect major departure from the classical approach.

Granted that this work is still in an embryonic form, there is much yet to be done in firming up its foundations, but the promise in its unusual results and self-consistency make it worthy of further investigation.

*****