A MATRIX GENERATION OF FIBONACCI IDENTITIES FOR F_{2nk}

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A series of identities involving even-subscripted Fibonacci numbers and binomial coefficients are derived in this paper by means of a sequence of special 2 x 2 matrices. We begin with the simplest case.

Let $R = \begin{pmatrix} 3 & 1 \\ -1 & 0 \end{pmatrix}$ and the characteristic equation, of course, is $x^2 - 3x + 1 = 0$, which is related to the recursion formula for the alternate Fibonacci numbers. By induction, one can easily establish that, for all integers n,

$$R^n = \begin{pmatrix} F_{2n+2} & F_{2n} \\ -F_{2n} & -F_{2n-2} \end{pmatrix},$$

and, if the auxiliary matrix $S = \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}$, then

$$R^nS = \begin{pmatrix} F_{2n+3} & F_{2n+1} \\ -F_{2n+1} & -F_{2n-1} \end{pmatrix},$$

where F_n is the nth Fibonacci number defined by $F_{n+1} = F_n + F_{n-1}$, $F_1 = F_2 = 1$. Since R satisfies its own characteristic equation, $R^2 - 3R + I = 0$ or $(R + I)^2 = 5R$, which leads to

$$R^m(R + I)^{2n} = 5^nR^{n+m}, \tag{1}$$

$$R^m(R + I)^{2n}S = 5^nR^{n+m}S, \tag{2}$$

$$R^m(R + I)^{2n+1} = 5^nR^{n+m}(R + I), \tag{3}$$

$$R^m(R + I)^{2n+1}S = 5^nR^{n+m}(R + I)S. \tag{4}$$

We use the binomial theorem to rewrite equation (1) and equate elements in the upper right from equations (1) and (2), which gives us

$$\sum_{k=0}^{2n}\binom{2n}{k}R^{k+m} = 5^nR^{n+m},$$

$$\sum_{k=0}^{2n}\binom{2n}{k}F_{2k+2m} = 5^nF_{2n+2m}, \tag{1'}$$

$$\sum_{k=0}^{2n}\binom{2n}{k}F_{2k+2m+1} = 5^nF_{2n+2m+1}. \tag{2'}$$

Similarly, from equations (3) and (4), we can obtain

$$\sum_{k=0}^{2n+1}\binom{2n+1}{k}F_{2k+2m} = 5^n(F_{2n+2m+2} + F_{2n+2m}) = 5^nL_{2n+2m+1}, \tag{3'}$$

$$\sum_{k=0}^{2n+1}\binom{2n+1}{k}F_{2k+2m+1} = 5^n(F_{2n+2m+3} + F_{2n+2m+1}) = 5^nL_{2n+2m+2}, \tag{4'}$$

where L_n is the nth Lucas number defined by $L_{n+1} = L_n + L_{n-1}$, $L_1 = 1$, $L_2 = 3$.

The equations above can be simplified still further. Equations (1′) and (2′) can be combined by letting $p = 2m$ in (1′) and $p = 2m + 1$ in (2′), and noting that p takes on any integral value, we write, finally,

$$\sum_{k=0}^{2n} \binom{2n}{k} F_{2k+p} = 5^n F_{2n+p}.$$

Similarly, equations (3′) and (4′) can be combined into the single identity

$$\sum_{k=0}^{2n+1} \binom{2n+1}{k} F_{2k+p} = 5^n L_{2n+1+p}.$$

As an interesting special case, let $p = -(2n+1)$ in the above equation, and use the index replacement $n - k$ for k, yielding

$$\sum_{k=0}^{2n+1} \binom{2n+1}{k} F_{2k-(2n+1)} = 2\left\{\sum_{k=0}^{n} \binom{2n+1}{n-k} F_{2k-1}\right\} = 5^n L_0$$

or

$$\sum_{k=0}^{n} \binom{2n+1}{n-k} F_{2k-1} = 5^n,$$

a result given by S. G. Guba in [2].

Returning to the characteristic polynomial of R, since $R^2 - 3R + I = 0$, $(R - I)^2 = R$, which leads to

(5) $$R^m(R - I)^{2n} = R^{n+m},$$

(6) $$R^m(R - I)^{2n}S = R^{n+m}S,$$

(7) $$R^m(R - I)^{2n+1} = R^{n+m}(R - I),$$

(8) $$R^m(R - I)^{2n+1}S = R^{n+m}(R - I)S.$$

Proceeding as before and equating elements in the upper right for the four matrix equations above, we have

(5′) $$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k} F_{2k+2m} = F_{2n+2m},$$

(6′) $$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k} F_{2k+2m+1} = F_{2n+2m+1},$$

(7′) $$\sum_{k=0}^{2n+1} (-1)^{k+1} \binom{2n+1}{k} F_{2k+2m} = (F_{2n+2m+2} - F_{2n+2m}) = F_{2n+2m+1},$$

(8′) $$\sum_{k=0}^{2n+1} (-1)^{k+1} \binom{2n+1}{k} F_{2k+2m+1} = (F_{2n+2m+3} - F_{2n+2m+1}) = F_{2n+2m+2}.$$

Again, equations (5′) and (6′) can be combined by taking $p = 2m$ in (5′) and $p = 2m + 1$ in (6′), and letting p be any integer in the resulting identity,

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k} F_{2k+p} = F_{2n+p}.$$

Similarly, combining (7′) and (8′) leads to

$$\sum_{k=0}^{2n+1} (-1)^{k+1} \binom{2n+1}{k} F_{2k+p} = F_{2n+1+p}.$$

The two identities above can be streamlined even more by taking $q = 2n$ in the first and $q = 2n + 1$ in the second, leading to

$$\sum_{k=0}^{q} (-1)^{k+q} \binom{q}{k} F_{2k+p} = F_{q+p},$$

which holds for all integers $q \geq 0$ and for any integer p. The special case $p = -q$ yields

$$\sum_{k=0}^{q} (-1)^{k+1} \binom{q}{k} F_{q-2k} = 0.$$

In order to distinguish between matrices in our sequence, let us call the R matrix just developed R_2. The next matrix of interest is

$$R_4 = \begin{pmatrix} 7 & 1 \\ -1 & 0 \end{pmatrix}.$$

The following matrix identities are easily established by mathematical induction. The proofs are given in the general case so are here omitted for the sake of brevity. We exhibit, for any integer n,

$$R_4^n S_0 = \begin{pmatrix} F_{4n+4} & F_{4n} \\ -F_{4n} & -F_{4n-4} \end{pmatrix} \text{ for } S_0 = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix};$$

$$R_4^n S_1 = \begin{pmatrix} F_{4n+5} & F_{4n+1} \\ -F_{4n+1} & -F_{4n-3} \end{pmatrix} \text{ for } S_1 = \begin{pmatrix} 5 & 1 \\ -1 & -2 \end{pmatrix};$$

$$R_4^n S_2 = \begin{pmatrix} F_{4n+6} & F_{4n+2} \\ -F_{4n+2} & -F_{4n-2} \end{pmatrix} \text{ for } S_2 = \begin{pmatrix} 8 & 1 \\ -1 & 1 \end{pmatrix};$$

$$R_4^n S_3 = \begin{pmatrix} F_{4n+7} & F_{4n+3} \\ -F_{4n+3} & -F_{4n-1} \end{pmatrix} \text{ for } S_3 = \begin{pmatrix} 13 & 2 \\ -2 & -1 \end{pmatrix}.$$

Since R_4 must satisfy its characteristic equation, $R^2 - 7R + I = 0$ or $(R - I)^2 = 5R$, leading to

(9) $$R^m(R - I)^{2n} = 5^n R^{m+n},$$

(10) $$R^m(R - I)^{2n+1} = 5^n R^{m+n}(R - I).$$

The binomial expansion of matrix equation (9) yields

$$\sum_{j=0}^{2n} (-1)^j \binom{2n}{j} R^{j+m} = 5^n R^{m+n}.$$

Multiplication on the right by the auxiliary matrix S_s, chosen from the four listed above, and then equating elements in the upper right yields

(9′) $$\sum_{j=0}^{2n} (-1)^j \binom{2n}{j} F_{4(j+m)+s} = 5^n F_{4(m+n)+s}, \quad s = 0, 1, 2, 3.$$

On the other hand, equation (10) can be expanded as

$$\sum_{j=0}^{2n+1} (-1)^{j+1} \binom{2n+1}{j} R^{j+m} = 5^n (R^{m+n+1} - R^{m+n}).$$

By appropriate S matrices, for $s = 0, 1, 2, 3$, we have

$$\sum_{j=0}^{2n+1} (-1)^{j+1} \binom{2n+1}{j} F_{4(j+m)+s} = 5^n (F_{4(m+n+1)+s} - F_{4(m+n)+s}).$$

But, the latter two terms can be factored, using identities given by I. D. Ruggles in [1]:

(A) $$F_{n+p} - F_{n-p} = L_n F_p \text{ if } p \text{ is even } (F_n L_p \text{ if } p \text{ is odd}),$$

(B) $$F_{n+p} + F_{n-p} = F_n L_p \text{ if } p \text{ is even } (L_n F_p \text{ if } p \text{ is odd}).$$

Here, applying identity (A), we get

$$F_{(4(m+n)+s+2)+2} - F_{(4(m+n)+s+2)-2} = L_{4(m+n)+s+2} F_2.$$

Thus, for $s = 0, 1, 2, 3$,

(10′)
$$\sum_{j=0}^{2n+1} (-1)^{j+1}\binom{2n+1}{j} F_{4(j+m)+s} = 5^n L_{4(m+n)+s+2}.$$

Equations (9′) and (10′) can be written in a slightly simpler form by taking $p = 4m + s$. Since there is no restriction on m, there is no restriction on the integer p in the two resulting identities below:

$$\sum_{j=0}^{2n} (-1)^j \binom{2n}{j} F_{4j+p} = 5^n F_{4n+p},$$

$$\sum_{j=0}^{2n+1} (-1)^{j+1}\binom{2n+1}{j} F_{4j+p} = 5^n L_{4n+2+p}.$$

Returning to the characteristic equation for R_4, in a completely similar manner we can obtain

(11)
$$R^m(R + I)^{2n} = 3^{2n}R^{n+m},$$

(12)
$$R^m(R + I)^{2n+1} = 3^{2n}R^{n+m}(R + I).$$

Following the previous pattern of equating elements in the upper right in the matrix equations obtained from the binomial expansions of (11) and (12) and multiplying by auxiliary matrices S_s, we are led eventually to

(11′)
$$\sum_{j=0}^{2n} \binom{2n}{j} F_{4(j+m)+s} = 3^{2n} F_{4(n+m)+s}, \quad s = 0, 1, 2, 3;$$

(12′)
$$\sum_{j=0}^{2n+1} \binom{2n+1}{j} F_{4(j+m)+s} = 3^{2n}(F_{4(m+n+1)} + F_{4(m+n)+s})$$
$$= 3^{2n+1} F_{4(m+n)+s+2}, \quad s = 0, 1, 2, 3,$$

where in (12′) we applied identity (B). Again, let us write the equations above more compactly, taking $p = 4m + s$ and noting that no restrictions on m implies no restrictions on p, as

$$\sum_{j=0}^{2n} \binom{2n}{j} F_{4j+p} = 3^{2n} F_{4n+p},$$

$$\sum_{j=0}^{2n+1} \binom{2n+1}{j} F_{4j+p} = 3^{2n+1} F_{4n+2+p}.$$

Notice that, by taking $q = 2n$ in the first and $q = 2n + 1$ in the second, we may combine the two identities above into the more general identity,

$$\sum_{j=0}^{q} \binom{q}{j} F_{4j+p} = 3^q F_{2q+p}.$$

The special case $p = -2q - 1$ yields

$$\sum_{j=0}^{q} \binom{q}{j} F_{2q+1-4j} = 3^q,$$

and similar equations arise for the special cases $p = -2q + 1$ and $p = -2q + 2$.

In the above identities, the general elements of R_4^n were written in the form of a quotient; that is, the element in the upper left of R_4^n was $F_{4n+4}/3$. While looking for a general form using a sum of Lucas of Fibonacci numbers we are led by observation of the starting values given to the following expression for the element r_n in the upper left of R_4^n:

$$r_1 = 7 = L_4,$$
$$r_2 = 48 = L_8 + 1,$$
$$r_3 = 329 = L_{12} + L_4,$$
$$r_4 = 2255 = L_{16} + L_8 + 1,$$

$$r_5 = 15456 = L_{20} + L_{12} + L_4,$$

$$r_n = \sum_{j=0}^{[(n-1)/2]} L_{4(n-2j)} + \begin{cases} 0 \text{ if } n \text{ is odd} \\ 1 \text{ if } n \text{ is even,} \end{cases}$$

where $[x]$ is the greatest integer less than or equal to x. A proof can be made by mathematical induction. Observe that the expression for r_n holds for $n = 1, 2, 3, 4, 5$. Since R satisfies its own characteristic equation, $R^{k+1} = 7R^k - R^{k-1}$, and the elements in the upper left of these matrices must satisfy $r_{k+1} = 7r_k - r_{k-1}$. Assume that the expression for r_n holds for all n up through k. Then, if k is odd,

$$\begin{aligned} r_{k+1} &= 7\left(\sum_{j=0}^{[(k-1)/2]} L_{4(k-2j)}\right) - \sum_{j=0}^{[(k-2)/2]} L_{4(k-1-2j)} - 1 \\ &= \sum_{j=0}^{[(k-2)/2]} (7L_{4(k-2j)} - L_{4(k-1-2j)}) + 7L_4 - 1 \\ &= \sum_{j=0}^{[k/2]} L_{4(k+1-2j)} + 1, \end{aligned}$$

where we noted that $[k/2] = [(k-1)/2]$, $7L_p - L_{p-4} = L_{p+4}$, and $48 = L_8 + 1$. Similarly, if k is even, since $[(k-1)/2] = [(k-2)/2]$ and $7 = L_4$,

$$r_{k+1} = \sum_{j=0}^{[(k-2)/2]} (7L_{4(k-2j)} - L_{4(k-1-2j)}) + 7 = \sum_{j=0}^{[k/2]} L_{4(k+1-2j)}.$$

Then, equating elements in the upper left for R_4^{2k} and R_4^{2k+1} gives us

$$F_{4(2k+1)} = 3\sum_{j=0}^{k-1} L_{4(2k-2j)} + 3,$$

$$F_{4(2k+2)} = 3\sum_{j=0}^{k} L_{4(2k+1-2j)}.$$

From equation (9), $(R - I)^{2n} = 5^n R^n$. Considering the cases $n = 2k$ and $n = 2k + 1$ and equating elements in the upper left, one obtains

$$\sum_{j=0}^{4k} (-1)^j \binom{4k}{j} F_{4j+4} = 3 \cdot 5^n \left(\sum_{j=0}^{k-1} L_{4(2k-2j)} + 1\right),$$

$$\sum_{j=0}^{4k+2} (-1)^j \binom{4k+2}{j} F_{4j+4} = 3 \cdot 5^n \left(\sum_{j=0}^{k} L_{4(2k+1-2j)}\right).$$

Similarly, from equation (11) with $m = 0$, we find

$$\sum_{j=0}^{4k} \binom{4k+2}{j} F_{4j+4} = 3^{2n+1}\left(\sum_{j=0}^{k-1} L_{4(2k-2j)} + 1\right),$$

$$\sum_{j=0}^{4k+2} \binom{4k+2}{j} F_{4j+4} = 3^{2n+1}\left(\sum_{j=0}^{k} L_{4(2k+1-2j)}\right).$$

A third expression for R_4^n was obtained with the element in the upper left given by

$$\sum_{j=0}^{[n/2]} (-1)^j \binom{n-j}{j} L_4^{n-2j}.$$

A proof of the general case will follow, so we will proceed only to use the above form. Equating elements in the upper left of R_4 leads to

$$F_{4n+4} = 3\sum_{j=0}^{[n/2]}(-1)^j\binom{n-j}{j}L_4^{n-2j},$$

or, for the cases $n = 2k$ and $n = 2k + 1$, in that order, to

$$\sum_{j=0}^{k}(-1)^j\binom{2k-j}{j}L_4^{2k-2j} = \sum_{j=0}^{k-1}L_{4(2k-2j)} + 1,$$

$$\sum_{j=0}^{k}(-1)^j\binom{2k+1-j}{j}L_4^{2k+1-2j} = \sum_{j=0}^{k}L_{4(2k+1-2j)}.$$

Now, to exhibit the pattern in general, if

$$R_{2k} = \begin{pmatrix} L_{2k} & 1 \\ -1 & 0 \end{pmatrix},$$

then

$$R_{2k}^n = \frac{1}{F_{2k}}\begin{pmatrix} F_{(2n+2)k} & F_{2nk} \\ -F_{2nk} & -F_{(2n-2)k} \end{pmatrix}.$$

This result has already been observed for $k = 1$ and $k = 2$, and is easily established by induction. Notice that

$$R_{2k} = \frac{1}{F_{2k}}\begin{pmatrix} F_{4k} & F_{2k} \\ -F_{2k} & 0 \end{pmatrix},$$

Since it is well known that $F_{4k} = F_{2k}L_{2k}$. We assume that R_{2k}^n has the above form; then

$$R_{2k}^nR = \frac{1}{F_{2k}}\begin{pmatrix} F_{(2n+2)k}L_{2k} - F_{2nk} & F_{(2n+2)k} \\ -F_{2nk}L_{2k} + F_{(2n-2)k} & -F_{2nk} \end{pmatrix}.$$

But, by the ubiquitous identity (B),

$$F_{2nk+2k+2} + F_{2nk+2k-2k} = F_{2nk+2k}L_{2k},$$

$$F_{2nk+2k} + F_{2nk-2k} = F_{2nk}L_{2k},$$

so that the matrix above has the desired form for R_{2k}^{n+1}. Thus, by mathematical induction, R_{2k}^n has the form prescribed above for all $n > 0$.

Observe that R^{-n} is given by

$$R^{-n} = \frac{1}{F_{2k}}\begin{pmatrix} -F_{(2n-2)k} & -F_{2nk} \\ F_{2nk} & F_{(2n+2)k} \end{pmatrix} = \frac{1}{F_{2k}}\begin{pmatrix} F_{(-2n+2)k} & F_{-2nk} \\ -F_{-2nk} & -F_{(-2n-2)k} \end{pmatrix}$$

and direct multiplication yields

$$R^nR^{-n} = \frac{1}{F_{2k}^2}\begin{pmatrix} -F_{(2n-2)k}F_{(2n+2)k} + F_{2nk}^2 & 0 \\ 0 & F_{2nk}^2 - F_{(2n-2)k}F_{(2n+2)k} \end{pmatrix}.$$

Since $\det(R_{2k}) = 1$, $\det(R_{2k}^n) = 1^n$, so that

$$F_{nk}^2 - F_{(2n+2)k}F_{(2n-2)k} = F_{2k}^2,$$

and we see that $R^nR^{-n} = I$ as well as exhibiting yet another identity arising from the prolific matrices R_{2k}. Also, since $F_{-k} = (-1)^{k+1}F_k$,

$$R_{2k}^0 = \frac{1}{F_{2k}}\begin{pmatrix} F_{2k} & F_0 \\ F_0 & -F_{-2k} \end{pmatrix} = I.$$

Hence, R_{2k}^n has the form given above for all integral exponents n.

The remaining piece of machinery needed is a general expression for the auxiliary S matrices which will raise the subscripts of R_{2k}^n. The matrix

$$S_s = \begin{pmatrix} F_{2k+s} & F_s \\ -F_s & -F_{s-2} \end{pmatrix}$$

adds s to each subscript for elements of R_{2k}^n, as seen by

$$R_{2k}^n S_s = \frac{1}{F_{2k}} \begin{pmatrix} F_{2nk+2k}F_{2k+s} - F_{2nk}F_s & F_{2nk+2k}F_s - F_{2nk}F_{s-2k} \\ -F_{2nk}F_{2k+s} + F_{2nk-2k}F_s & -F_{2nk}F_s + F_{2nk-2k}F_{s-2k} \end{pmatrix}$$

$$= \begin{pmatrix} F_{2nk+2k+s} & F_{2nk+s} \\ -F_{2nk+s} & -F_{2nk-2k+s} \end{pmatrix},$$

where the two matrices can be shown equal element by element. Each case can be demonstrated by judicious use of the known formula

$$F_n F_m - F_{n-k}F_{m+k} = (-1)^{n-k}F_k F_{m+k-n}.$$

Before leaving the matrix S_s, it is interesting to notice that

$$S_1^s = F_{2k}^{s-1}S_s \quad \text{and} \quad S_1 = F_{2k}\sqrt[2k]{R_{2k}}.$$

One more bit of information will allow us to give our most general results. The even-subscripted Lucas numbers have the following curious properties:

$$\left.\begin{aligned} L_{4n} + 2 &= L_{2n}^2, \\ L_{4n} - 2 &= 5F_{2n}^2, \\ L_{4n+2} + 2 &= 5F_{2n+1}^2, \\ L_{4n+2} - 2 &= L_{2n+1}^2. \end{aligned}\right\}$$

We demonstrate the first. If $\alpha = (1 + \sqrt{5})/2$, $\beta = (1 - \sqrt{5})/2$, then $L_n = \alpha^n + \beta^n$. Thus,

$$L_{2n}^2 = (\alpha^{2n} + \beta^{2n})^2 = \alpha^{4n} + \beta^{4n} + 2\alpha^{2n}\beta^{2n} = L_{4n} + 2,$$

since $\alpha\beta = -1$. The other three can be proved just as neatly.

Now the characteristic equation of R_{2k} gives us

$$R_{2k}^2 - L_{2k}R_{2k} + I = 0 \quad \text{or} \quad (R_{2k} \pm I)^2 = (L_{2k} \pm 2)R_{2k},$$

leading to the following by considering properties of even-subscripted Lucas numbers and raising each equation to the nth power:

(15) $$R_{4q}^m(R_{4q} + I)^{2n} = L_{2q}^{2n}R_{4q}^{n+m},$$

(16) $$R_{4q}^m(R_{4q} + I)^{2n+1} = L_{2q}^{2n}R_{4q}^{n+m}(R_{4q} + I),$$

(17) $$R_{4q}^m(R_{4q} - I)^{2n} = 5^nF_{2q}^{2n}R_{4q}^{n+m},$$

(18) $$R_{4q}^m(R_{4q} - I)^{2n+1} = 5^nF_{2q}^{2n}R_{4q}^{n+m}(R_{4q} - I),$$

(19) $$R_{4q+2}^m(R_{4q+2} + I)^{2n} = 5^nF_{2q+1}^{2n}R_{4q+2}^{n+m},$$

(20) $$R_{4q+2}^m(R_{4q+2} + I)^{2n+1} = 5^nF_{2q+1}^{2n}R_{4q+2}^{n+m}(R_{4q+2} + I),$$

(21) $$R_{4q+2}^m(R_{4q+2} - I)^{2n} = L_{2q+1}^{2n}R_{4q+2}^{n+m},$$

(22) $$R_{4q+2}^m(R_{4q+2} - I)^{2n+1} = L_{2q+1}^{2n}R_{4q+2}^{n+m}(R_{4q+2} - I).$$

For each equation above, we will write the binomial expansion, multiply by the auxiliary matrix S_s, and equate elements in the upper right, leading to the correspondingly numbered equations below. For equations (15′) through (18′), $s = 0, 1, 2, \ldots, 4q - 1$; and for equations (19′) through (22′), $s = 0, 1, 2, \ldots, 4q + 1$.

(15′) $$\sum_{j=0}^{2n} \binom{2n}{j} F_{4q(j+m)+s} = L_{2q}^{2n} F_{4q(n+m)+s}$$

(16′) $$\sum_{j=0}^{2n+1} \binom{2n+1}{j} F_{4q(j+m)+s} = L_{2q}^{2n}(F_{4q(n+m+1)+s} + F_{4q(n+m)+s}) = L_{2q}^{2n+1} F_{4q(n+m)+2q+s}$$

(17′) $$\sum_{j=0}^{2n} (-1)^j \binom{2n}{j} F_{4q(j+m)+s} = 5^n F_{2q}^{2n} F_{4q(n+m)+s}$$

(18′) $$\sum_{j=0}^{2n+1} (-1)^{j+1} \binom{2n+1}{j} F_{4q(j+m)+s} = 5^n F_{2q}^{2n}(F_{4q(n+m+1)+s} - F_{4q(n+m)+s}) = 5^n F_{2q}^{2n+1} L_{4q(n+m)+2q+s}$$

(19′) $$\sum_{j=0}^{2n} \binom{2n}{j} F_{(4q+2)(j+m)+s} = 5^n F_{2q+1}^{2n} F_{(4q+2)(n+m)+s}$$

(20′) $$\sum_{j=0}^{2n+1} \binom{2n+1}{j} F_{(4q+2)(j+m)+s} = 5^n F_{2q+1}^{2n}(F_{(4q+2)(n+m+1)+s} + F_{(4q+2)(n+m)+s}) = 5^n F_{2q+1}^{2n+1} L_{(4q+2)(n+m)+2q+1+s}$$

(21′) $$\sum_{j=0}^{2n} (-1)^j \binom{2n}{j} F_{(4q+2)(j+m)+s} = L_{2q+1}^{2n} F_{(4q+2)(n+m)+s}$$

(22′) $$\sum_{j=0}^{2n+1} (-1)^{j+1} \binom{2n+1}{j} F_{(4q+2)(j+m)+s} = L_{2q+1}^{2n}(F_{(4q+2)(n+m+1)+s} - F_{(4q+2)(n+m)+s}) = L_{2q+1}^{2n+1} F_{(4q+2)(n+m)+2q+1+s}$$

In each case, the proper Ruggles' identity (A) or (B) was applied.

Equations (15′) through (22′) can be rewritten in more compact forms which better display their properties. In equations (15′) through (18′) take $p = 4qm + s$ and in equations (19′) through (22′) take $p = (4q + 2)m + s$. Notice that since there are no restrictions on m and since s takes on any value from 0 through $4p - 1$ or $4q + 1$, respectively, p can be any integer. In the combined identity below, notice that equation (15′) is the case $r = 2n$ and (16′) the case $r = 2n + 1$:

(23) $$\sum_{j=0}^{r} \binom{r}{j} F_{(2q)(2j)+p} = L_{2q}^{r} F_{2qr+p}.$$

Equations (17′) and (19′), respectively, lead to

$$\sum_{j=0}^{2n} (-1)^j \binom{2n}{j} F_{(2q)(2j)+p} = 5^n F_{2q}^{2n} F_{(2q)(2n)+p},$$

$$\sum_{j=0}^{2n} \binom{2n}{j} F_{(2q+1)(2j)+p} = 5^n F_{2q+1}^{2n} F_{(2q+1)(2n)+p},$$

which can be combined into the more general identity

$$\sum_{j=0}^{2n} (-1)^{j(t+1)} \binom{2n}{j} F_{2jt+p} = 5^n F_t^{2n} F_{2nt+p}.$$

Similarly, equations (18′) and (20′) can be condensed to the identity

$$\sum_{j=0}^{2n+1}(-1)^{(j+1)(t+1)}\binom{2n+1}{j}F_{2jt+p} = 5^nF^{2n+1}L_{(2n+1)t+p},$$

which becomes (18′) when $t = 2q$ and (20′) when $t = 2q + 1$.

Equations (21′) and (22′) lead to

$$\sum_{j=0}^{r}(-1)^{r+j}\binom{r}{j}F_{(2q+1)(2j)+p} = L^r_{2q+1}F_{(2q+1)r+p}, \tag{24}$$

which is (21′) when $r = 2n$ and (22′) when $r = 2n + 1$.

Finally, equations (23) and (24) taken together provide

$$\sum_{j=0}^{r}(-1)^{(r+j)t}\binom{r}{j}F_{2jt+p} = L^r_tF_{tr+p},$$

which is (23) when $t = 2q$ and (24) when $t = 2q + 1$.

Returning to the matrix R^n_{2k}, the element in its upper left can be shown to be

$$r_n = \sum_{j=0}^{[n/2]}(-1)^j\binom{n-j}{j}L^{n-2j}_{2k},$$

which form readily becomes apparent by computing the first few powers of R_{2k}. Notice that the binomial coefficients used appear on rising diagonals of Pascal's triangle. A proof by mathematical induction is outlined below. First, if $n = 1$, the expression becomes L_{2k}, the element in the upper left of R_{2k}, and if $n = 0$, we find $r_0 = 1$, the element in the upper left of $R^0_{2k} = I$. From the characteristic equation of R_{2k}, the elements r_{p+1}, r_p, and r_{p-1} must satisfy $r_{p+1} = L_{2k}r_p - r_{p-1}$. Assume that r_p and r_{p-1} have the form given above. Then,

$$\begin{aligned} r_{p+1} &= \sum_{j=0}^{[p/2]}(-1)^j\binom{p-j}{j}L^{p-2j}_{2k} - \sum_{j=0}^{[(p-1)/2]}(-1)^j\binom{p-1-j}{j}L^{p-1-2j}_{2k} \\ &= \sum_{j=0}^{[p/2]}(-1)^j\binom{p-j}{j}L^{p+1-2j}_{2k} - \sum_{j=1}^{[(p+1)/2]}(-1)^{j-1}\binom{p-j}{j-1}L^{p+1-2j}_{2k} \\ &= \sum_{j=0}^{[(p+1)/2]}(-1)^j\binom{p+1-j}{j}L^{p+1-2j}_{2k} \end{aligned}$$

by the recursion relation for binomial coefficients and by carefully considering the end terms. Since r_{p+1} has the prescribed form whenever r_p and r_{p-1} do, r_n has the form given above for all integers $n \geq 0$.

Equating elements in the upper left for the matrix R^n_{2k} yields

$$F_{2k}\sum_{j=0}^{[n/2]}(-1)^j\binom{n-j}{j}L^{n-2j}_{2k} = F_{(n+1)2k}. \tag{25}$$

Using equations (15), (17), (19), and (21) with $m = 0$ and equating elements in the upper left,

$$\sum_{k=0}^{2n}\sum_{j=0}^{[k/2]}(-1)^j\binom{2n}{k}\binom{k-j}{j}L^{k-2j}_{4p} = L^{2n}_{2p}\sum_{j=0}^{[n/2]}(-1)^j\binom{n-j}{j}L^{n-2j}_{4p}, \tag{15″}$$

$$\sum_{k=0}^{2n}\sum_{j=0}^{[k/2]}(-1)^{j+k}\binom{2n}{k}\binom{k-j}{j}L^{k-2j}_{4p} = 5^nF^{2n}_{2p}\sum_{j=0}^{[n/2]}(-1)^j\binom{n-j}{j}L^{n-2j}_{4p}, \tag{17″}$$

$$(19'') \qquad \sum_{k=0}^{2n}\sum_{j=0}^{[k/2]}(-1)^j\binom{2n}{k}\binom{k-j}{j}L_{4p+2}^{k-2j} = 5^n F_{2p+1}^{2n}\sum_{j=0}^{[n/2]}(-1)^j\binom{n-j}{j}L_{4p+2}^{n-2j},$$

$$(21'') \qquad \sum_{k=0}^{2n}\sum_{j=0}^{[k/2]}(-1)^{j+k}\binom{2n}{k}\binom{k-j}{j}L_{4p+2}^{k-2j} = L_{2p+1}^{2n}\sum_{j=0}^{[n/2]}(-1)^j\binom{n-j}{j}L_{4p+2}^{n-2j}.$$

Returning to the first expression given for R_{2k}^n, in which the element in the upper right is F_{2nk}/F_{2k}, a second proof can be given which utilizes Chebyshev polynomials. A special group of Chebyshev polynomials of the second kind are defined here by $u_0(\lambda) = 0$, $u_1(\lambda) = 1$, $u_{n+1}(\lambda) = 2\lambda u_n(\lambda) - u_{n-1}(\lambda)$. [Commonly, the starting values of the same series are taken as $u_0(\lambda) = 1$, $u_2(\lambda) = 2\lambda$.] Consider the known relationship:

$$\frac{x}{1 - 2\lambda x + x^2} = \sum_{n=0}^{\infty} u_n(\lambda)x^n.$$

However, as with H. W. Gould [3], for $\alpha = (1 + \sqrt{5})/2$, $\beta = (1 - \sqrt{5})/2$, we have, by summing the geometric series,

$$\frac{1}{1 - \alpha^{2k}x} - \frac{1}{1 - \beta^{2k}x} = \sum_{n=0}^{\infty}\alpha^{2kn}x^n - \sum_{n=0}^{\infty}\beta^{2kn}x^n,$$

which can be rewritten on both sides to yield

$$\frac{(\alpha^{2k} - \beta^{2k})x}{1 - (\alpha^{2k} + \beta^{2k})x + (\alpha\beta)^{2k}x^2} = \sum_{n=0}^{\infty}(\alpha^{2kn} - \beta^{2kn})x^n.$$

Since $\alpha\beta = -1$, $\alpha^n + \beta^n = L_n$, and $(\alpha^n - \beta^n)/(\alpha - \beta) = F_n$,

$$\frac{x}{1 - L_{2k}x + x^2} = \sum_{n=0}^{\infty}\frac{(\alpha^{2nk} - \beta^{2nk})/(\alpha - \beta)x^n}{(\alpha^{2k} - \beta^{2k})/(\alpha - \beta)} = \sum_{n=0}^{\infty}\frac{F_{2nk}}{F_{2k}}x^n$$

for $k \neq 0$. But, we also have, when $\lambda = L_{2k}/2$,

$$\frac{x}{1 - L_{2k}x + x^2} = \sum_{n=0}^{\infty}u_n(L_{2k}/2)x^n,$$

which implies that $u_n(L_{2k}/2) = F_{2nk}/F_{2k}$, $k \neq 0$.

Similar results are obtainable for the Fibonacci polynomials defined by $f_0(\lambda) = 0$, $f_1(\lambda) = 1$, $f_{n+1}(\lambda) = f_n(\lambda) + f_{n-1}(\lambda)$, which lead to

$$\frac{x}{1 - \lambda x - x^2} = \sum_{n=0}^{\infty} f_n(\lambda)x^n$$

and

$$f_n(L_{2k+1}) = F_{(2k+1)n}/F_{2k+1}.$$

A matrix having a Chebyshev polynomial as its characteristic polynomial is

$$R = \begin{pmatrix} 2\lambda & 1 \\ -1 & 0 \end{pmatrix}, \quad R^n = \begin{pmatrix} u_{n+1}(\lambda) & u_n(\lambda) \\ -u_n(\lambda) & -u_{n-1}(\lambda) \end{pmatrix},$$

while for the Fibonacci polynomials such a matrix is

$$F = \begin{pmatrix} \lambda & 1 \\ 1 & 0 \end{pmatrix}, \quad F^n = \begin{pmatrix} f_{n+1}(\lambda) & f_n(\lambda) \\ f_n(\lambda) & f_{n-1}(\lambda) \end{pmatrix}.$$

[Notice that, when $\lambda = 1$, $f_n(\lambda) = F_n$.]

By substituting $\lambda = L_{2k}/2$ in the above matrix R, we obtain

$$R_{2k} = \frac{1}{F_{2k}}\begin{pmatrix} F_{(2n+2)k} & F_{2nk} \\ -F_{2nk} & -F_{(2n-2)k} \end{pmatrix}.$$

Also, substituting $\lambda = L_{2k}/2$ into $u_{n+1}(\lambda) = 2\lambda u_n(\lambda) - u_{n-1}(\lambda)$ yields the expression for the general element in the upper left of R_{2k}^n as given in equation (25).

Since we could also show that

$$f_{n+1}(L_{2k+1}) = \sum_{j=0}^{[n/2]} \binom{n-j}{j} L_{2k+1}^{n-2j}$$

by substituting $\lambda = L_{2k+1}$ into the recursion formula for the Fibonacci polynomials, and since also $f_{n+1}(L_{2k+1}) = F_{(2k+1)(n+1)}/F_{2k+1}$, we can generalize equation (25) to the following:

$$F_{(n+1)p}/F_p = \sum_{j=0}^{[n/2]} (-1)^{j(p+1)}\binom{n-j}{j} L_p^{n-2j}, \quad p \neq 0,$$

which was a problem posed by H. H. Fern [4].

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ANTIMAGIC PENTAGRAMS WITH LINE SUMS IN ARITHMETIC PROGRESSION, $\Delta = 3$

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A pentagram or five-pointed star can be formed by extending the sides of a regular pentagon until they meet. This figure consists of five equal line segments that form a closed path. Each line intersects every other line, so that there are four intersections or vertices on each line, and two lines at each vertex.

A magic pentagram is formed by distributing ten elements on the vertices of a pentagram in such a way that the sum of the four elements (quartet) on one line equals each of the other four line sums. It has been shown [1, 2, 3, 4, 5] that no magic pentagram can be formed with the first ten positive integers.

An antimagic pentagram is one with five different line sums. Those formable with the first ten positive integers are formidably numerous. We restrict our search to those with five line sums in arithmetic progression and a common difference, $\Delta = 3$. In the sum of the five line sums, each element appears twice, so $5[2a + 4(3)]/2 = 2(55)$. Hence, the progression must be 16, 19, 22, 25, and 28.

The partitions of the five terms of this progression into four elements each < 11 are exhibited in Table 1. To make the table compact, 10 is recorded as X. Designate any quartet with a sum of x as an x-quartet. For the purposes of this discussion, two integers are said to be complementary if their sum is 11. Two quartets are complementary and two pentagrams are complementary if their corresponding elements are complementary.

To construct an antimagic pentagram, we start with the 16-quartet ($\underline{1}$, $\underline{2}$, $\underline{3}$, X) and seek a 19-quartet with which it has exactly one element in common, such as ($\underline{3}$, $\underline{7}$, 4, $\underline{5}$). A 22-quartet with exactly one element in common with each of these is ($\underline{2}$, $\underline{5}$, 6, $\underline{9}$). A 25-quartet with exactly one element in common with each of these three quartets is ($\underline{1}$, $\underline{7}$, 8, $\underline{9}$). The unduplicated elements, which are not underscored, in these four quartets form the 28-quartet (4, 6, 8, X). These five quartets can be distributed on the vertices of a pentagram with