and then if

(21a) $$T_k(x) = \sum_{n=k}^{\infty} t_{n,k}x^{n-k}; \; T(x) = T_0(x),$$

(21b) $$T_k(x) = [T(x)]^{k+1}.$$

Conversely, given a triangular array satisfying (21), we may recover a sequence $\{a_n\}$ $(n \geq 0)$ via (20). What are the sequences arising in this way in the partition problems considered above [see (4, 12, 16)]?

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BREAK-UP OF INTEGERS AND BRACKET FUNCTIONS IN TERMS OF BRACKET FUNCTIONS

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ABSTRACT

We have presented a general formula for the break-up of integers into bracket functins, and some formulas for the break-up of bracket functions into other bracket functions.

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It is interesting to find break-ups of variable integers into a sum of bracket functions involving the integer we want to break up and other integers. Two well-known examples of this are

(1) $$x = \sum_{i=0}^{m-1}\left[\frac{x+i}{m}\right] \quad \text{integers } m > 0;$$

(2) $$x = \left[\frac{(p+1)x}{2p+1}\right] + \sum_{i=1}^{p}\left[\frac{x+2i}{2p+1}\right] \quad \text{integers } p > 0.$$

Here we shall find a general break-up of the variable integer into bracket functions involving two other integers (equation 12). The above-mentioned break-ups are special cases of this more general formula.

To derive the general formula, we shall need to use the h-function (defined in [1]) defined by

(3) $$\begin{cases} h(x, m) = 1 & \text{if } m/x \\ = 0 & \text{if } m\nmid x \end{cases}$$

It is easily seen that it satisfies the following properties (which we shall use later);

(4) $$\{h(x, m)\}^j = h(x, m) \quad \text{integers } j > 0;$$

(5) $$\sum_{j=1}^{m} h(x+j, m) = 1;$$

(6) $$h(x, m_1)h(x, m_2) = h(x, m) \text{ where } m = (m_1, m_2);$$

(7) $$h(x+mk, m) = h(x, m) \quad \text{integers } k;$$

(8) $$h(nx, m) = h(x, m) \text{ if } \langle n, m\rangle = 1.$$

Now, considering the difference operator, Δ, acting on the bracket function $\left[\frac{x-1}{m}\right]$:

$$\Delta\left[\frac{x-1}{m}\right] = \left[\frac{x}{m}\right] - \left[\frac{x-1}{m}\right] = \begin{cases} 1 & \text{if } m/x \\ 0 & \text{if } m\nmid x \end{cases}$$

we see that we can put

$$\Delta\left[\frac{x-1}{m}\right] = h(x, m)$$

(9) $$\Delta^{-1}h(x, m) = \left[\frac{x-1}{m}\right] + c_1$$

where c_1 is an arbitrary constant. Applying the inverse difference operator to equation (5), we obtain

$$x = \sum_{j=1}^{m} \Delta^{-1}h(x + j, m) + c_2 = \sum_{j=1}^{m}\left[\frac{x + j - 1}{m}\right] + c_3.$$

To evaluate the constant here, take $x = 1$. Clearly the lefthand side is equal to the bracket function. Thus, c_3 is zero.

$$\therefore\ x = \sum_{j=1}^{m}\left[\frac{x + j - 1}{m}\right],$$

which is the same as equation (1).

To derive the general formula, consider

$$\sum_{r=1}^{n} h(nx + y + r, m) = \left|\Delta^{-1}h(nx + y + r, m)\right|_{r=1}^{n+1}$$

$$= \left[\frac{nx + y + r - 1}{m}\right] - \left[\frac{nx + y}{m}\right]$$

$$= \Delta\left[\frac{nx + y}{m}\right]$$

(10) $$\therefore\ \left[\frac{nx + y}{m}\right] = \sum_{r=1}^{n} \Delta^{-1}h(nx + y + r, m) + c.$$

We restrict our attention to relatively prime integers n and m. There must, then, exist two integers a and b such that

$$an + bm = 1$$

$$\therefore\ \left[\frac{nx + y}{m}\right] = \sum_{r=1}^{n} \Delta^{-1}h(nx + (an + bm)(y + r)m) + c.$$

Using equation (7), we now get

$$\left[\frac{nx + y}{m}\right] = \sum_{r=1}^{n} \Delta^{-1}h(nx + na(y + r), m) + c.$$

As $\langle n, m\rangle = 1$, using equation (8) gives

$$\left[\frac{nx + y}{m}\right] = \sum_{r=1}^{n} \Delta^{-1}h(x + a(y + r), m) + c = \sum_{r=1}^{n}\left[\frac{x + a(y + r) - 1}{m}\right] + c.$$

Putting $x = 0$ in the above equation, we obtain

$$c = \left[\frac{y}{m}\right] - \sum_{r=1}^{n}\left[\frac{a(y + r) - 1}{m}\right]$$

(11) $$\therefore\ \left[\frac{nx + y}{m}\right] = \sum_{r=1}^{n}\left[\frac{x + a(y + r) - 1}{m}\right] - \sum_{r=1}^{n}\left[\frac{a(y + r) - 1}{m}\right] + \left[\frac{y}{m}\right].$$

We now further restrict our attention to the case $n < m$. We can then write

$$n = pq + 1$$

$$m = pq + p + 1$$

as these numbers are relatively prime (as can be easily checked). Then, taking $y = 0$, we obtain

$$\left[\frac{(pq+1)x}{pq+p+1}\right] = \sum_{r=1}^{pq+1}\left[\frac{x+ra-1}{pq+p+1}\right] - \sum_{r=1}^{pq+1}\left[\frac{ra-1}{pq+p+1}\right].$$

Now a solution to the constraint on a and b with the above values of m and n is

$$a = q+1,\ b = -q.$$

Thus we get

$$\left[\frac{(pq+1)x}{pq+p+1}\right] = \sum_{r=1}^{pq+1}\left[\frac{x+r(q+1)-1}{pq+p+1}\right] - \sum_{r=1}^{pq+1}\left[\frac{r(q+1)-1}{pq+p+1}\right].$$

To obtain the required formula, we shall break up the summation into the ranges $r = 1, \ldots, p$; $r = p+1, \ldots, 2p$; $r = p(q-1)+1, \ldots, pq$, and the last term $r = pq+1$. This may be written as a double summation over i and j by writing $r = pj + i + 1$ where j goes from 0 to $q-1$ and i from 0 to $p-1$, apart from the last term. Thus we have

$$\left[\frac{(pq+1)x}{pq+p+1}\right] = \sum_{j=0}^{q-1}\sum_{i=0}^{p-1}\left\{\left[\frac{x+(pj+i+1)(q+1)-1}{pq+p+1}\right] - \left[\frac{(pj+i+1)(q+1)-1}{pq+p+1}\right]\right\} + \left[\frac{x}{pq+p+1}\right]$$

as the last term $(r = pq+1)$ is just

$$\left[\frac{x+q(pq+p+1)}{pq+p+1}\right] - \left[\frac{q(pq+p+1)}{pq+p+1}\right].$$

Now we have

$$(pj+i+1)(q+1) - 1 = j(pq+p+1) + i(q+1) + q - j.$$

Cancelling the multiples of $pq+p+1$ in both bracket functions, we obtain

$$\left[\frac{(pq+1)x}{pq+p+1}\right] = \sum_{j=0}^{q-1}\sum_{i=0}^{p-1}\left\{\left[\frac{x+i(q+1)+q-j}{pq+p+1}\right] - \left[\frac{i(q+1)+q-j}{pq+p+1}\right]\right\} + \left[\frac{x}{pq+p+1}\right].$$

Inverting the order of summation of j, we can replace $q-j$ by $j+1$.

$$\therefore\ \left[\frac{(pq+1)x}{pq+p+1}\right] = \sum_{j=0}^{q-1}\sum_{i=0}^{p-1}\left\{\left[\frac{x+i(q+1)+j+1}{pq+p+1}\right] - \left[\frac{i(q+1)+j+1}{pq+p+1}\right]\right\} - \left[\frac{x}{pq+p+1}\right].$$

Now the second bracket function on the righthand side is zero, as the maximum value of the numerator is $pq+p-1$. Changing the range of summation of j from 0 to $q-1$ to 1 to q and replacing j in the bracket function by $j-1$, we obtain

$$\left[\frac{(pq+1)x}{pq+p+1}\right] = \sum_{j=1}^{q}\sum_{i=0}^{p-1}\left[\frac{x+(q+1)i+j}{pq+p+1}\right] - \left[\frac{x}{pq+p+1}\right].$$

Adding and subtracting the term for $j = 0$,

$$\left[\frac{(pq+1)x}{pq+p+1}\right] = \sum_{j=0}^{q}\sum_{i=0}^{p-1}\left[\frac{x+(q+1)i+j}{pq+p+1}\right] - \sum_{i=0}^{p-1}\left[\frac{x+(q+1)i}{pq+p+1}\right] - \left[\frac{x}{pq+p+1}\right].$$

Now the $i = 0$ term in the second bracket on the righthand side cancels the last term. We can now again replace the double summation over i and j by a summation over t from 0 to $pq+p-1$. Adding and subtracting the term for $pq+p$, we obtain

$$\left[\frac{(pq+1)x}{pq+p+1}\right] = \sum_{t=0}^{pq+p}\left[\frac{x+t}{pq+p+1}\right] - \sum_{i=1}^{p-1}\left[\frac{x+(q+1)i}{pq+p+1}\right] - \left[\frac{x+pq+1}{pq+p+1}\right].$$

Using equation (1) for the first bracket function on the righthand side and transposing, we finally obtain

$$x = \left[\frac{(pq+1)x}{pq+p+1}\right] + \sum_{i=1}^{p-1}\left[\frac{x+(q+1)i}{pq+p+1}\right] + \left[\frac{x+(q+1)p}{pq+p+1}\right]$$

$$\text{(12)} \qquad \therefore \quad x = \left[\frac{(pq+1)x}{pq+p+1}\right] + \sum_{i=1}^{p}\left[\frac{x+(q+1)i}{pq+p+1}\right].$$

This is the general formula which we were searching for.

The special case $q = 0$ in equation (12) gives equation (1). The case $q = 1$ in equation (12) gives equation (2). Similarly, $q = 2$ gives us

$$\text{(13)} \qquad x = \left[\frac{(2p+1)x}{3p+1}\right] + \sum_{i=1}^{p}\left[\frac{x+3i}{3p+1}\right].$$

which is a new break-up of the type in equation (2). We can generate any number of such series. Separately, by choosing the special values of p we generate other break-ups. Thus, for $p = 1$

$$\text{(14)} \qquad x = \left[\frac{rx}{r+1}\right] + \left[\frac{x+r}{r+1}\right]$$

(where r is $q + 1$). We can in fact take $r \geq 0$. The next break-up in the series is, for $= 2$,

$$\text{(15)} \qquad x = \left[\frac{(2q+1)}{2q+3}\right] + \left[\frac{x+q+1}{2q+3}\right] + \left[\frac{x+2q+2}{2q+3}\right].$$

Again we can generate any number of such break-ups. It is obvious that equation (12) provides a considerable generalization of equations (1) and (2).

We are able to obtain an identity involving bracket functions by using euqation (11). It is clearly going to be equivalent to take $y = x$ and to take $y = 0$ and replace n by $n + 1$. Thus,

$$\sum_{r=1}^{n}\left[\frac{x+a(x+r)-1}{m}\right] - \sum_{r=1}^{n}\left[\frac{a(x+r)-1}{m}\right] + \left[\frac{x}{m}\right] = \sum_{r=1}^{n+1}\left[\frac{x+ar-1}{m}\right] - \sum_{r=1}^{n+1}\left[\frac{ar-1}{m}\right]$$

$$\text{(16)} \quad \therefore \left[\frac{x}{m}\right] = \sum_{r=1}^{n+1}\left[\frac{x+ar-1}{m}\right] - \sum_{r=1}^{n}\left[\frac{x+a(x+r)-1}{m}\right] + \sum_{r=1}^{n}\left[\frac{ax+ar-1}{m}\right] - \sum_{r=1}^{n+1}\left[\frac{ar-1}{m}\right].$$

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NOTE

We can also derive the result using Euler's ϕ-function, by using

$$\left[\frac{nx+y}{m}\right] = \sum_{r=1}^{n}\left[\frac{x+P_r}{m}\right] - \sum_{r=1}^{n}\left[\frac{P_r}{m}\right], \text{ where } P_r = \frac{(n-y-r)(m^{\phi(r)}-1)}{n}.$$

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