$$a(k, q)b(k - q, p - q) = \frac{1}{2}\{a(k, q)b(k - q, p - q) + a(k - q, p - q)b(k, q)\}$$

(3.4)

$$= \frac{1}{2}\binom{k}{q}\binom{p}{q}\left\{\sum_{i=0}^{M}\sum_{j=1}^{M-i} j!(j + 1)!(-1)^{i+j+1}\ell_i(S_i^q S_{i+j}^{p-q} + S_{i+j}^q S_i^{p-q})_2\right\}$$

where $M$ is the greater of $q$ and $(p - q)$ and $\ell_i$ is Lucas' sequence, given by

$$\ell_n = \ell_{n-1} + \ell_{n-2} \quad \text{and} \quad \ell_0 = 2,\ \ell_1 = 1.$$

Also, if $a$ and $b$ are roots of the equation $y^2 - y - 1 = 0$, the lefthand side of equation (3.3) should be [3]

$$-\sum_{i=1}^{q} i!F_{i+1}S(q, i)$$

from equation (3.2), $F_i$ being the Fibonacci sequence. Then

$$\sum_{i=0}^{q} i!F_{i+1}S(q, i) = -\frac{1}{2}\binom{p}{q}\sum_{i=0}^{M}\sum_{j=0}^{M-i}(-1)^{i+j+1}j!(j + 1)!\ell_i(S_j^q S_{i+j}^{p-q} + S_{i+j}^q S_j^{p-q}), \tag{3.5}$$

which is the required identity.

## CONCLUSION

We have defined $\psi_p$ difference equations as generalizations of the periodic difference equations. This is a much wider class of difference equations than the periodic ones, but does not contain all difference equations. We extended Minkowski's operational calculus to deal with a large class (but not all) $\psi_p$ difference equations. This is of interest in itself as a means of solving more difference equations than Minkowski's calculus enabled us to. It is also of interest inasmuch as it provides an independent means of solving periodic difference equations and thereby discovering new identities between combinations of various sequences. Thus, it can also be regarded as being of interest in number theory.

## ACKNOWLEDGMENT

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# SOLUTION OF PSEUDO-PERIODIC DIFFERENCE EQUATIONS

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## ABSTRACT

A method for getting the particular solution of pseudo-periodic difference equations, by using a discrete periodic function has been given. Some identities, equalities, and inequalities have been derived by using the above-mentioned discrete function.

## 1. INTRODUCTION

Periodic difference equations have been previously solved [1] by the use of Minkowski's operational calculus. The type of equations solved by this method are

$$P(E)f(x) = (a_1, a_2, \ldots, a_n)_n, \tag{1.1}$$

where $P(E)$ is a polynomial function of $E$ with constant coefficients, $n$ is the period, and the $a_i$'s are constants. An obvious extension of this would be to make the $a$ 's functions of the variable $x$. Since the resulting equations would no longer be periodic, we call them pseudo-periodic.

To solve pseudo-periodic cifference equations, we define a discrete function $h(x, m)$,

$$(1.2) \qquad \begin{cases} h(x, m) = 1 & \text{when } m/x \\ \quad\quad\quad\;\; = 0 & \text{when } m\nmid x \end{cases}$$

It can easily be seen that $h(x, m)$ satisfies the following properties:

(1) $\{h(x, m)\}^j = h(x, m)$ for all integers $j > 0$;

(2) $\sum_{j=0}^{m-1} h(x + j, m) = 1$;

(3) $h(x, m_1)h(x, m_2) = h(x, m)$, $m$ being the L.C.M. of $m_1$ and $m_2$;

(4) $h(x + mk, m) = h(x, m)$ for all integers $k \geq 0$;

(5) $h(nx, m) = h(x, m)$, $n$ and $m$ being relatively prime.

We shall use these properties to evaluate the expression

$$f(x) = \frac{1}{(E^{m_1 m_2} - a^{m_1 m_2})}\left[\frac{nx}{m_1}\right]^k h(x, m_2),\ a \neq 1,$$

where $\left[\frac{a}{b}\right]$ is the bracket function, being the largest integer less than or equal to $\frac{a}{b}$. This can be used to solve any pseudo-periodic difference equation, in principle. All pseudo-periodic difference equations being periodic difference equations we can, of course, solve periodic difference equations, as well, by this method.

The plan of work is as follows. In Section 2 we solve equation (1.3). As an example of this method, we have solved a previously solved difference equation

$$(E - a)(E - b)f(x) = x^k,$$

where $a$ and $b$ are the roots of the equation $y^2 - y - 1 = 0$. This yields an identity involving Fibonacci numbers and Sterling's numbers of the first and second kind.

In Section 3 we give some equalities and inequalities involving bracket functions that can be derived by using the discrete function $h(x, m)$.

## 2. SOLUTION OF PSEUDO-PERIODIC DIFFERENCE EQUATIONS

We shall first find the particular solution of the difference equation

$$(1.2) \qquad \Delta f(x) = \left(\frac{x}{m}\right)^k h(x, m).$$

Any polynomial with different periods can be constructed from terms of the type of the righthand side of (2.1) with different values of $k$ and $m$. We can thus construct artibtary functions and solve an linear, first-order pseudo-periodic difference equation.

Consider the action of the difference operator on the $k$th Bernoulli polynomial [1, 2] with the argument $\left[\frac{x-1}{m}\right] + 1$,

$$\Delta B_{k+1}\left(\left[\frac{x-1}{m}\right] + 1\right) = \sum_{i=0}^{k+1}\binom{k+1}{i}(1 + B)^{k-i+1}\left[\frac{x-1}{m}\right]^i,$$

where $B^r \equiv B_r$ is the $r$th Bernoulli number. Using property (2) of $h(x, m)$ and equation (3.2) given in the next section, we get

$$\Delta B_{k+1}\left(\left[\frac{x-1}{m}\right] + 1\right) = \sum_{i=1}^{k+1}(-1)^{k-i+1}\binom{k+1}{i}B_{k-i+1}\sum_{j=1}^{i}(-1)^{j+1}\binom{i}{j}\left(\frac{x}{m}\right)^{i-j}h(x, m),$$

putting $j = i - r$, changing the order of summation, and then putting $i = s + r$,

$$\Delta B_{k+1}\left(\left[\frac{x-1}{m}\right] + 1\right) = \sum_{r=0}^{k}\sum_{s=1}^{k-r+1}(-1)^{k-r}\binom{k+1}{s+r}\binom{s+r}{r}B_{k-s-r+1}\left(\frac{x}{m}\right)^r h(x, m).$$

Now
$$\binom{k+1}{s+r}\binom{s+r}{r} = \binom{k+1}{r}\binom{k-r+1}{s}$$

and
$$\sum_{s=1}^{k-r+1} B_{k-s-r+1}\binom{k-s+1}{s} = (1+B)^{k-r+1} - B^{k-r+1},$$

$$\therefore \quad \Delta B_{k+1}\left(\left[\frac{x-1}{m}\right]+1\right) = \sum_{r=0}^{k}(-1)^{k-r}\binom{k+1}{r}\left(\frac{x}{m}\right)^r \left\{(1+B)^{k-r+1} - B^{k-r+1}\right\}h(x, m)$$

$$= (k+1)\left(\frac{x}{m}\right)^k h(x, m)$$

(2.2)
$$f(x) = \frac{1}{k+1}B_{k+1}\left(\left[\frac{x-1}{m}\right]+1\right) + c.$$

It can be seen that if
$$\Delta^{-1}f(x) = F(x) + c$$
$$\Delta^{-1}f\left(\left[\frac{x}{m}\right]\right) = \sum_{i=0}^{m-1} F\left(\left[\frac{x+i}{m}\right]\right) + c$$

then (2.2) gives
$$\Delta^{-1}\left[\frac{x}{m}\right]^k = \frac{1}{k+1}\sum_{r=0}^{m-1} B_{k+1}\left(\left[\frac{x+r}{m}\right]\right)$$

(2.3)
$$= \frac{m}{k+1}B_{k+1}\left(\left[\frac{x}{m}\right]\right) - \frac{1}{k+1}\left(x - m\left[\frac{x}{m}\right]\right)\left[\frac{x}{m}\right]^k$$

Now we shall consider the difference equation (1.3). To solve this, put $f(x)$ in the form

(2.4)
$$f(x) = \sum_{i=0}^{k}\sum_{j=0}^{k} a_{ij}\left(\left[\frac{nx}{m_1}\right]\right)^{k-i} h(x+j, m_2).$$

By operating on both sides of (2.4) with $(E^{m_1m_2} - a^{m_1m_2})$ and comparing with (1.3), we see that

(2.5)
$$\begin{cases} a_{00} = \dfrac{1}{1 - a^{m_1m_2}} \\ a_{i0} = -a_{00}\displaystyle\sum_{s=0}^{i-1} a_{s0}\binom{k-s}{i-s}(m_2n)^{i-s} \\ a_{ij} = 0 \quad \text{for } j \neq 0 \end{cases}$$

Denoting $a_{i0}$ by $a_i$, we get

(2.6)
$$(E^{m_1m_2} - a^{m_1m_2})\sum_{i=0}^{k} a_i\left[\frac{nx}{m_1}\right]^{k-i} h(x, m_2) = \left[\frac{nx}{m_1}\right]^k h(x, m_2).$$

Assume that for some $i = j$

(2.7)
$$a_{ij} = a_j = \frac{(m_2n)^j k^{(j)}}{j!}\sum_{r=0}^{j} r!(a_{00})^{r+1}S_n^j.$$

Then (2.5) gives
$$\frac{a_{j+1}}{a_{00}} = -\sum_{q=0}^{j}\binom{k-q}{j-q+1}(m_2n)^{j+1}\binom{k}{q}\sum_{r=0}^{q} r!(a_{00})^{r+1}S_r^q$$

where $S_r^q$ are Sterling's numbers of the second kind [2]. Substituting from (2.7)
$$\frac{a_{j+1}}{a_{00}} = -\sum_{q=0}^{j}\binom{k-q}{j-q+1}(m_2n)^{j+1}\binom{k}{q}\sum_{r=0}^{q} r!(a_{00})^{r+1}S_r^q = -(m_2n)^{j+1}\binom{k}{j+1}\sum_{p=0}^{j}\sum_{q=0}^{j}\binom{j+1}{q}p!(a_{00})^{p+1}S_p^q$$
$$= -\binom{k}{j+1}(m_2n)^{j+1}\sum_{p=0}^{j} p!(p+1)(a_{00})^{p+1}S_{p+1}^{j+1}$$

$$\because \quad \sum_{q=0}^{j} \binom{j+1}{q} S_p^q = (j+1) S_{p+1}^{j+1}$$

$$\frac{a_{j+1}}{a_{00}} = \binom{k}{j+1} (m_2 n)^{j+1} \sum_{p=0}^{j} (p+1)!(a_{00})^{p+1} S_{p+1}^{j+1}.$$

Now, $j + 1$ being positive, $S_0^{j+1} = 0$. Hence

$$a_{j+1} = \frac{(m_2 n)^{j+1} K^{(j+1)}}{(j+1)!} \sum_{t=0}^{j+1} (t+1)!(a_{00})^{t+1} S_t^{j+1}.$$

Thus, if (2.7) is true for $j$, it is also true for $(j + 1)$. It is easily seen that it is true for $j = 1$. Hence, it is true for all $j$. Putting (2.7) into (2.6), we obtain

$$f(x) = \sum_{i=0}^{k} \sum_{r=0}^{i} (m_2 n) \binom{k}{i} \left[\frac{nx}{m_1}\right]^{k-i} \frac{(r)! S_r^i}{(1 - a^{m_1 m_2})^{r+1}} h(x, m). \tag{2.8}$$

As an example of the method given above, consider the pseudo-periodic difference equation

$$(E - a)(E - b) f(x) = x^k h(x, m), \tag{2.9}$$

writing the particular solution of (2.9) as $P_k(x)$

$$\begin{aligned} P_k(x) &= (E - a)^{-1}(E - b)^{-1} x^k h(x, m) \\ &= \left\{\sum_{i=1}^{m} \sum_{j=1}^{m} E^{m-i} a^{i-1} E^{m-j} b^{j-1}\right\} \left\{\sum_{p=0}^{k} \sum_{q=0}^{k} a(k, q) b(k-q, p-q) x^{k-q}\right\}, \end{aligned} \tag{2.10}$$

where

$$\left\{ \begin{aligned} a(k, q) &= m^q \binom{k}{q} \sum_{r=0}^{q} \frac{(q)! S_r^q}{(1 - a^m)^{r+1}} \\ b(k, q) &= m^q \binom{k}{q} \sum_{r=0}^{q} \frac{(q)! S_r^q}{(1 - b^m)^{r+1}} \end{aligned} \right. \tag{2.11}$$

Notice that (2.10) must hold with $a$ and $b$ interchanged, since (2.9) is symmetric in $a$ and $b$. Thus, we shall interchange and take the sum. For $m = 1$, we get

$$\begin{aligned} \therefore \quad a(k, q) b(k-q, p-q) &= \frac{1}{2}\left\{a(k, q) b(k-q, p-q) + b(k, q) a(k-q, p-q)\right\} \\ &= \frac{1}{2}\binom{k}{p}\binom{p}{q} \left\{\sum_{i=0}^{M} \sum_{j=0}^{M-i} (j)!(j+i)!(-1)^{j+i+1} \ell_i (S_j S_{i+j}^{p-q} + S_i S_j^{p-q}\right\}. \end{aligned} \tag{2.12}$$

Where $M$ is the greater of $P$ and $P - q$ and $\{\ell_i\}$ is Lucas' sequence [2]

$$\ell_n = \ell_{n-1} + \ell_{n-2}, \quad \ell_0 = 2, \ \ell_1 = 1.$$

Also, if $a$ and $b$ are the roots of the equation $y^2 - y - 1 = 0$, the L.H.S. of (2.12) should be [3]

$$-\sum_{i=1}^{q} (i)! F_{i+1} S(q, i)$$

from (2.10). Thus,

$$\sum_{i=0}^{q} (i)! F_{i+1} S(q, i) = -\frac{1}{2}\binom{p}{q} \sum_{i=0}^{M} \sum_{j=0}^{M-i} (-1)^{i+j+1} (j)!(j+i)! \ell_i (S_j^q S_{i+j}^{p-q} + S_{i+j}^q S_j^{p-q}). \tag{2.13}$$

## 3. SOME RESULTS OBTAINED BY USING $h(x, m)$

It is easy to see that:

$$\left[\frac{x}{m}\right] = \frac{1}{m}\left\{x - m + 1 + \sum_{j=0}^{m-1} j h(x + j + 1, m)\right\}; \tag{3.1}$$

(3.2) $$\left[\frac{x-1}{m}\right]^k = \sum_{j=1}^{m-1}(-1)^{j+1}\binom{k}{j}\left(\frac{x}{m}\right)^{k-j}h(x, m).$$

Putting $k = 1$ in (3.2) and then using (3.1), we get

(3.3) $$\Delta^{-1}h(x, m) = \frac{1}{m}\left\{x - m + \sum_{j=0}^{m-1} jh(x+j, m)\right\} + c.$$

The bracket function inequality

$$\left[\frac{x+y}{mn}\right] \geq \left[\frac{\left[\frac{x}{m}\right] + \left[\frac{y}{m}\right]}{n}\right]$$

on using (3.1), gives the inequality

(3.4) $$\left\{\begin{aligned}\sum_{r=0}^{mn-1} rh(x+y+1, mn) \geq m&\left\{\sum_{r=0}^{n} rh\left(\left[\frac{x}{m}\right] + \left[\frac{y}{m}\right] + r + 1, m\right) - 1\right\}\\ &+ \sum_{r=0}^{n-1}\left\{h(x+r+1, m) + h(y+r+1, m)\right\} + 1.\end{aligned}\right.$$

Similarly, the bracket function equality

$$\left[\frac{x}{mn}\right] = \left[\frac{\left[\frac{x}{m}\right]}{n}\right] = \left[\frac{\left[\frac{x}{n}\right]}{m}\right]$$

on using (3.1), gives the equality

(3.5) $$\left\{\begin{aligned}x + 1 + \sum_{j=0}^{mn} jh(x+j+1, mn) &= n\left\{\left[\frac{x}{n}\right] + 1 + \sum_{j=0}^{m-1} jh\left(\left[\frac{x}{n}\right] + j + 1, m\right)\right\}\\ &= m\left\{\left[\frac{x}{m}\right] + 1 + \sum_{j=0}^{n-1} jh\left(\left[\frac{x}{m}\right] + j + 1, n\right)\right\}.\end{aligned}\right.$$

Now consider

$$\sum_{r=1}^{n-k} h(nx+y+r, m) = \left|\Delta^{-1}h(nx+y+r, m)\right|_{r=1}^{nk+1} = \left|\left[\frac{nx+y+r-1}{m}\right]\right|_{r=1}^{nk+1}$$
$$= \left[\frac{n(x+k)+y}{m}\right] - \left[\frac{nx+y}{m}\right] = (E^k - 1)\left[\frac{nx+y}{m}\right]$$

(3.6) $$\therefore \quad \left[\frac{nx+y}{m}\right] = \frac{1}{E^k - 1}\sum_{r=1}^{n-k} h(mx+y+r, m) + c.$$

Putting $k = 1$, we get

$$\left[\frac{nx+y}{m}\right] = \Delta^{-1}\sum_{r=1}^{n} h(nx+y+r, m) + c.$$

If $n$ and $m$ are relatively prime, there will exist two integers, $a$ and $b$, such that $an + bm = 1$,

$$\therefore \quad \left[\frac{nx+y}{m}\right] = \Delta^{-1}\sum_{r=1}^{n} h(nx + (y+r)(an+bm), m) + c.$$

Using property (4) of $h(x, m)$,

$$\left[\frac{nx+y}{m}\right] = \Delta^{-1}\sum_{r=1}^{n} h(x + a(y+r), m) + c = \sum_{r=1}^{n}\left[\frac{x+(y+r)a-1}{m}\right] + c.$$

Determine $c$ by putting $x = 0$ in the above equation:

$$C = \left[\frac{y}{m}\right] - \sum_{r=1}^{n}\left[\frac{a(y+r)-1}{m}\right]$$

(3.7) $$\therefore \quad \left[\frac{nx + y}{m}\right] = \sum_{r=1}^{n} \left[\frac{x + (y + r)a - 1}{m}\right] - \sum_{r=1}^{n} \left[\frac{(y + r)a - 1}{m}\right] + \left[\frac{y}{m}\right].$$

Putting $y = 0$, $m = pq + p + 1$ and $n = pq + 1$ in (3.7) we obtain the equation

$$\left[\frac{(pq + 1)x}{pq + p + 1}\right] = \sum_{r=1}^{pr+1} \left[\frac{x + r(q + 1) - 1}{pq + p + 1}\right] - \sum_{r=1}^{pq+1} \left[\frac{r(q + 1) - 1}{pq + p + 1}\right],$$

and it is easily checked that $pq + 1$ and $pq + p + 1$ are relatively prime. Breaking the summation into the $q$ summations with ranges $r = 1, p$; $r = p + 1, \ldots, 2p, \ldots$; $r = p(q - 1) + 1, \ldots, pq$; and the term for $r = pq + 1$, we can write the expression as a double sum, and obtain the equation

$$\left[\frac{(pq + 1)x}{pq + p + 1}\right] = \sum_{j=0}^{q-1} \sum_{i=0}^{p-1} \left\{\left[\frac{x + (pj + i + 1)(q + 1) - 1}{pq + p + 1}\right] - \left[\frac{(pj + i + 1)(q + 1) - 1}{pq + p + 1}\right]\right\} + \left[\frac{x}{pq + p + 1}\right].$$

Taking multiples of $(pq + p + 1)$ out of the numerators of the two bracket functions, we obtain

$$\left[\frac{(pq + 1)x}{pq + p + 1}\right] = \sum_{j=0}^{q-1} \sum_{i=0}^{p-1} \left\{\left[\frac{x + (q + 1)i + q - j}{pq + p + 1}\right] - \left[\frac{(q + 1)i + q - j}{pq + p + 1}\right]\right\} + \left[\frac{x}{pq + p + 1}\right].$$

Reversing the order of summation of $j$ we notice that $q - j$ is replaced by $j + 1$. Now the maximum value of $(q + 1)i + j + 1$ is $pq + p - 1$, which is less than $pq + p + 1$. Thus, the second bracket expression is always zero. Changing the range of summation of $j$ from 0 to $q - 1$ to 1 to $q$, we obtain

$$\left[\frac{(pq + 1)x}{pq + p + 1}\right] = \sum_{j=1}^{q} \sum_{i=0}^{p-1} \left[\frac{x + (q + 1)i + j}{pq + p + 1}\right] + \left[\frac{x}{pq + p + 1}\right].$$

Now adding and subtracting the expression for $j = 0$, we get

$$\left[\frac{(pq + 1)x}{pq + p + 1}\right] = \sum_{j=0}^{q} \sum_{i=0}^{p-1} \left[\frac{x + (q + 1)i + j}{pq + p + 1}\right] - \sum_{i=0}^{p-1} \left[\frac{x + (q + 1)i}{pq + p + 1}\right] + \left[\frac{x}{pq + p + 1}\right].$$

Notice that the last bracket expression cancels the bracket expression in the second term on the L.H.S. for $i = 0$. Also, we can replace the summation over $i$ and $j$ by a summation over $t$, where the range of $t$ is 0 to $pq + p - 1$. We can thus write

(3.8) $$\left[\frac{(pq + 1)x}{pq + p + 1}\right] = \sum_{t=0}^{pq+p} \left[\frac{x + t}{pq + p + 1}\right] - \sum_{i=1}^{p-1} \left[\frac{x + (q + 1)i}{pq + p + 1}\right] - \left[\frac{x + pq + p}{pq + p + 1}\right].$$

To reduce this further, consider the inverse difference operator acting on the equation for property (2) of $h(x, m)$ and the fact that

$$\Delta^{-1}h(x + j, m) = \left[\frac{x + j - 1}{m}\right] + c$$

we obtain the equation

$$x = \sum_{j=0}^{m-1} \left[\frac{x + j - 1}{m}\right] + c.$$

Evaluating $c$ by putting $x = 0$ and absorbing into the summation, we obtain the result that

(3.9) $$x = \sum_{j=0}^{m-1} \left[\frac{x + j}{m}\right].$$

Putting (3.9) into (3.8), we obtain the identity

(3.10) $$\left[\frac{(pq + 1)x}{pq + p + 1}\right] + \sum_{i=1}^{p} \left[\frac{x + (q + 1)i}{pq + p + 1}\right] = x.$$

Similarly, we get the equation

(3.11) $$\left[\frac{nx + y}{m}\right] = \sum_{i=1}^{p} \left[\frac{x + P_r}{m}\right] - \sum_{r=1}^{n} \left[\frac{P_r}{m}\right],$$

where

$$P_r = \frac{(n - y - r)(m^{\phi(n)} - i)}{n},$$

$\phi(n)$ being the number of natural numbers less than or equal to $n$ which are relatively prime with respect to $n$ (i.e., Euler's $\phi$-function). Then putting $y = 0$ and replacing $x$ by $n^x$, we get

$$\Delta\left[\frac{n^x}{m}\right] = \sum_{r=1}^{n-1}\left[\frac{n^x + P_r}{m}\right] - \sum_{r=1}^{n-1}\left[\frac{P_r}{m}\right]$$

(3.12) $$\therefore \quad \Delta^{-1}\sum_{r=1}^{n-1}\left[\frac{n^x + P_r}{m}\right] = \left[\frac{n^x}{m}\right] + x\sum_{r=1}^{n-1}\left[\frac{P_r}{m}\right] + c.$$

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## APPENDIX

Let

$$P_k(x) = \frac{1}{E^2 - E - 1}x^k,$$

then

$$P_k(x) = -\sum_{i=0}^{k}(F_{i+1}\Delta^i)x^k = -\sum_{i=0}^{k}F_{i+1}\Delta^i\sum_{j=0}^{k}S_{(k,j)}\,x^{(j)},$$

where $x^{(j)} = x(x-1)(x-2)\ \ldots\ (x-j+1)$, the falling factorial, where $S_j^k$ are Sterling's numbers of the second kind, and $F_i$ are Fibonacci numbers, $F_0 = 1 = F_1$, $F_n = F_{n-1} + F_{n-2}$.

$$\therefore \quad P_k(x) = -\sum_{i=0}^{k}\sum_{j=i}^{k}(j)^{(i)}\,F_{i+1}S_j^k x^{(j-i)}$$

$$= -\sum_{i=0}^{k}\sum_{j=0}^{k-i}(k-j)^{(i)}\,F_{i+1}S_{(k-j)}^k\,x^{(k-i-j)}.$$

Put $i + j = \ell$. Then $\min(\ell) = 0$ and $\max(\ell) = k$.

$$\therefore \quad P_k(x) = -\sum_{i=0}^{k}\sum_{\ell=0}^{k}(k-\ell+i)^{(i)}\,F_{i+1}S_{(k-\ell+i)}^k x^{(k-\ell)}$$

$$= -\sum_{i=0}^{k}\sum_{\ell=0}^{k}\sum_{j=0}^{k}(k-\ell+i)^{(i)}\,F_{i+1}S_{(k-\ell,\,j)}S_{(k-\ell+i)}^k x^j,$$

where $S_{(k-1,\,j)}$ are Sterling's numbers of the first kind.

Put $j + 1 = m$. Then $\min(m) = 0$ and $\max(m) = k$,

$$\therefore \quad P_k(x) = -\sum_{i=0}^{k}\sum_{\ell=0}^{k}\sum_{m=0}^{k}(k-\ell+i)^{(i)}\,F_{i+1}S_{(k-\ell+i)}^k S_{(k-1,\,k-m)}x^{k-m}.$$

Now consider the coefficients of $x^{k-m}$. By reversing the order of summation of 1, we can replace $k - \ell$ by $\ell$. Also note that

$$(\ell + i)^{(i)} = \binom{\ell + i}{i},$$

and also that

$$\sum_{\ell=k}^{k-i}S_{(i+\ell)}^k\,S_{(\ell,\,k-m)}\binom{\ell + i}{i} = \binom{k}{k-m}S_{(m,\,i)}.$$

Since the expression is zero for $\ell < k - s$ and for $1 > k - i$,

$$\therefore \quad P_k(x) = -\sum_{i=0}^{k} \sum_{m=0}^{k} i!\binom{k}{k-m} S_{(m,i)} F_{i+1} x^{k-m}$$

$$= \sum_{i=0}^{k} \sum_{m=0}^{k} \binom{k}{m} S_{(m,i)}\, i! F_{i+1} x^{k-m}.$$

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## A CLASS OF DIOPHANTINE EQUATIONS*

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### ABSTRACT

In this paper, we prove a theorem: If $k \equiv 5 \pmod 8$ and $f \not\equiv 2^{3t-1} \pmod{2^{3t}}$ for all positive integers $t$, then $c(3a^2b + kb^3) + d(a^3 + 3kab^2) = 16f$ has no solutions in integers $ab \neq 0$ if $c$ and $d$ are both odd integers. Then it is shown how this theorem enables us to solve the diophantine equations $y^2 - k = x^3$, $k \equiv 5 \pmod 8$. In the end, we give solutions for $k = 109, 116, 125, 133, 149, 157, 165, 173, 180$, and $181$.

The Mordell equation $y^2 - k = x^3$, the simplest of all nontrivial diophantine equations of degree greater than 2, has interested mathematicians for more than three centuries, and has played an important role in the development of Number Theory.

We already know the complete solutions for $y^2 - k = x^3$, $|k| \leq 100$. The author in his doctroal dissertation (UCLA, 1971) has treated the range $100 < k \leq 200$. The present paper treats 10 particular cases in the above range.

First we prove two lemmas to prove the theorem.

*Theorem 1:* If $k \equiv 5 \pmod 8$, $f \not\equiv 2^{3t-1} \pmod{2^{3t}}$ for all positive integers $t$, then $c(3a^2b + kb^3) + d(a^3 + 3kab^2) = 16f$ has no solution in integers $ab \neq 0$ if $c$ and $d$ are both odd integers.

*Lemma 1:* Let $k \equiv 5 \pmod 8$ and $c$ and $d$ be odd integers. Then $c(3a^2b + kb^3) + d(a^3 + 3kab^2) = 0$ has only solution $a = 0$ and $b = 0$ in integers.

*Proof:* Suppose $a \neq 0$, $b \neq 0$ is a solution of

(1) $$c(3a^2b + kb^3) + d(a^3 + 3kab^2) = 0$$

in integers. ($a = 0$ implies $b = 0$, and conversely.) We see from (1) that $a \neq b$ and $a \equiv b \pmod 2$. Then $3a^2b + kb^3 = b(3a^2 + kb^2) \equiv 0 \pmod 8$ and $a^3 + 3kab^2 = a(a^2 + 3kb^2) \equiv 0 \pmod 8$, since $k \equiv 5 \pmod 8$.

Hence, $c(3a^2b + kb^3) + d(a^3 + 3kab^2) \equiv (3a^2b + kb^3) + (a^3 + 3kab^2) \pmod{16}$ as both $c$ and $d$ are odd integers. Then, from (1), we deduce that

(2) $$(3a^2b + kb^3) + (a^3 + 3kab^2) \equiv 0 \pmod{16}.$$

But

(3) $$a^3 + 3a^2b + kb^3 + 3kab^2 = (a + b)^3 + (k - 1)b^2(a + b) + 2(k - 1)ab^2.$$

Inserting $a + b = 2r$ and $k = 8\ell + 5$ in (3), we obtain

$$\begin{aligned} a^3 + 3a^2b + kb^3 + 3kab^2 &\equiv 8r(r^2 + b^2) + 8ab^2 \pmod{16} \\ &\equiv 8 \pmod{16} \text{ when both } a \text{ and } b \text{ are odd;} \\ &\equiv 0 \text{ or } 8 \pmod{16} \text{ when both } a \text{ and } b \text{ are even.} \end{aligned}$$

Then (2) implies that $a$ and $b$ are both even. Since $a \neq b$, suppose $a = 2\, m^p$ and $b = 2^q n$ where $m$ and $n$ are odd integers.

Now $(a,b)$ is a solution of (1) implies that $(a_1,b_1)$ is a solution of

$$c(3a_1^2b_1 + kb_1^3) + d(a_1^3 + 3ka_1b_1^2) = 0,$$

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