Again, $yq + p^2q^2 = t^2$ for some integer t. Solving for y,

(12) $$y = \frac{t^2 - p^2q^2}{q}.$$

But y is positive, hence $t = s + pq$ for some integer $s \geq 1$. Substituting t into (12) yields

(13) $$y = \frac{s(s + 2pq)}{q}.$$

Since q is a product of distinct primes, q must divide s, i.e., $s = nq$ for some integer $n \geq 1$. Substituting s into (13) yields the desired formula for y,

(14) $$y = nq(n + 2p),$$

and substituting (14) for y in (11) yields

$$x = 2pq(n + p).$$

REFERENCES

1. O. Ore. *Number Theory and Its History*. New York: McGraw-Hill, 1948.
2. J. V. Uspensky and M. A. Heaslet. *Elementary Number Theory*. New York: McGraw-Hill, 1939.

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ON PRIMITIVE WEIRD NUMBERS

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1. INTRODUCTION

Let n be a positive integer. Denote by $\sigma(n)$ the sum of divisors of n. It is called n perfect if $\sigma(n) = 2n$, abundant if $\sigma(n) \geq 2n$, and deficient if $\sigma(n) < 2n$. Further, n is defined to be pseudoperfect if it is the sum of some of its proper divisors that all are distinct (d is a proper divisor of n, if d/n and $d < n$).

An integer n is called weird if n is abundant but not pseudoperfect. It is primitive abundant if it is abundant but all its proper divisors are deficient. If n is primitive abundant but not pseudoperfect, it is called primitive weird.

It is not known [1] if there are infinitely many primitive weird numbers or any odd weird numbers. A list of weird and primitive weird numbers not exceeding 10^6 is given in [1]. However, there is a misprint in [1] on page 618: instead of 539774 one should read 539744.

In this note we let n specially be of the form

(1) $$n = 2^\alpha pq \quad (\alpha \geq 1,\ p < q,\ p \text{ and } q \text{ odd primes}),$$

and give necessary and sufficient conditions under which n is primitive weird. As far as we know this cannot be found in the literature. As an application, we list some primitive weird numbers exceeding 10^6.

Throughout this note, let p and q be odd primes and $p < q$.

We use the following notations:

$$S = \sum_{\nu=0}^{\alpha} 2^\nu = 2^{\alpha+1} - 1, \quad S' = \sum_{(\nu)} 2^\nu$$

(the sum being taken over some of the indices ν);

$$S_p = \sum_{\nu=0}^{\alpha} 2^\nu p = (2^{\alpha+1} - 1)p, \quad S_p^m = S_p - mp \quad (0 \leq m \leq 2^{\alpha+1} - 1);$$

$$S_q = \sum_{\nu=0}^{\alpha} 2^\nu q = (2^{\alpha+1} - 1)q, \quad S_q^n = S_q - nq \quad (0 \leq n \leq 2^{\alpha+1} - 1);$$

$$S_{pq} = \sum_{\nu=0}^{\alpha-1} 2^\nu pq = (2^\alpha - 1)pq, \quad S_{pq}^k = S_{pq} - kpq \quad (0 \leq k \leq 2^\alpha - 1).$$

Theorem 1: The integer n in (1) is primitive weird iff

(2) $$2^{\alpha+1} + 1 \le p \le q < \frac{(2^{\alpha+1} - 1)(p + 1)}{p - (2^{\alpha+1} - 1)}$$

is true and

(3) $$pq = S_p^m + S_q^n + S' \quad \text{for some } m,n$$

is false.

Theorem 2: Assume that the primes p and q are of the forms

$$p = 2^{\alpha+1} + x, \quad 1 \le x \le 2^{\alpha+1} - 3$$

$$q = \frac{\tau p - i}{x + 1}, \quad 1 \le i \le x \text{ and } \tau \text{ an integer}$$

such that

$$\frac{2^{\alpha+1}p - i}{x + 2} \le q \le \frac{2^{\alpha+1}p + x - 1}{x + 1}.$$

Then the integer n in (1) is primitive weird.

2. APPLICATIONS

Theorem 2 gives, e.g., the following primitive weird numbers $n = 2^\alpha pq$.

2^α	p	q	n
2	5	7	70
4	11	19	836
8	17	127	17272
	19	71	10792
		61	9272
	23	43	7912
	29	31	7192
16	37	191	113072
	41	127	83312
	43	107	73616
32	67	1021	2189024
		971	2081824
		887	1901728
	71	541	1229152
		523	1188256
	79	311	786208
	83	257	682592
	97	179	555616
	101	167	539744
	109	149	519712
64	131	4159	34869056
		4093	34315712
		3733	31297472
		3373	28279232
	137	1657	14528576
	139	1471	13086016
		1469	12979264
		1447	12872512
	149	853	8134208
		839	8000704
	151	773	7470272
	157	659	6621632
	167	521	5568448
	179	433	4960448
	191	379	4632896
	239	271	4145216
	251	257	4128448
128	257	30197	993360512
		29683	976451968

(continued)

2^α	p	q	n
		25057	824275072
		24029	790457984
	263	8317	279983488
		8087	272240768
		7561	254533504
128	269	4861	167373952
		4649	160074368
	271	4217	146279296
	277	3109	110232704
	283	2557	92624768
	307	1499	58904704
		1493	58668928
		1487	58433152
	311	1399	55691392
	317	1303	52870528
	337	1039	44818304
	409	677	35442304
	499	521	33277312
256	521	25997	3467375872
		25841	3446569216
		25633	3418827008
		24851	3314526976
		24799	3307591424
	523	22271	2981819648
		21617	2894256896
		20963	2806694144
		20789	2783397632
	547	7703	1078666496
		7673	1074465536
	557	6163	878794496
		6151	877083392
256	563	5521	795730688
	569	5003	728756992
		4993	727300352
		4973	724387072
	577	4441	655988992
		4423	653330176
	587	3931	590719232
		3923	589517056
	593	3673	557590784
		3659	555465472
	599	3457	530110208
	619	2917	462239488
	631	2687	434047232
		2671	431462656
	661	2251	380905216
	683	2029	354766592
	769	1523	299823872
	811	1381	286717696
	839	1307	280722688
	911	1163	271230208
	919	1151	270788864
	937	1123	269376256
	947	1109	268857088
	1013	1031	267367168

3. PROOF OF THEOREM 1

The divisors of n in (1) are:

$$2^\nu,\ 2^\nu p,\ 2^\nu q,\ 2^\nu pq \quad (\nu = 0, 1, \ldots, \alpha).$$

We note that divisors 2^ν are always deficient. All the divisors $2^\nu p$ and $2^\nu q$ are deficient iff

(4) $$p \geq 2^{\alpha+1} + 1.$$

For such p, n is abundant iff

(5) $$q \leq \frac{(2^{\alpha+1} - 1)(p + 1)}{p - (2^{\alpha+1} - 1)}.$$

Last we see that all the divisors $2^{\nu}pq$, where $\nu < \alpha$ and p satisfies (4), are deficient. This shows that n is primitive abundant iff (2) holds.

It is clear that [if the condition (4) holds]

$$S' \leq S < p < q - 1,$$

so

$$2pq - S' > 2pq - p = (2q - 1)p > (p + q)p > S_p + S_q \geq S_p^m + S_q^n$$

or

(6) $$S_p^m + S_q^n + S' < 2pq.$$

Since

$$S_{pq}^k \leq S_{pq} = n - pq,$$

we see from (6) that n is pseudoperfect iff (3) holds.

4. PROOF OF THEOREM 2

On the basis of our choice of p and q, the condition (2) is satisfied. Write (3) in the form

(7) $$mp + nq + (2^{\alpha+1} - 1)q + (x + 1)q = (2^{\alpha+1} - 1)(p + q) + S'.$$

This implies

(8) $$nq = (2^{\alpha+1} - 1 - \tau - m)p + i + S'.$$

Write, for brevity,

$$M = 2^{\alpha+1} - 1 - \tau.$$

If $m \geq M + 1$, the right side of (8) is $\leq -p + i + S' \leq -1 < 0$, while the left side is always ≥ 0.

In the case $m = M$, (8) is equivalent to $nq = S' + i$, which cannot hold for any n, because $0 < S' + i < q$.

Finally we have the case $m < M$. Equation (8) trivially fails for $n = 0$. If $n \geq 1$, we see that

$$nq > (M - m)p + i + S'$$

if $q \geq (M + 1)p$, and this is true for

$$q \geq \frac{2^{\alpha+1}p - i}{x + 2}.$$

5. REMARK

An integer

(10) $$n = 2^{\alpha}\prod_{i=1}^{t} p_i \quad (2^{\alpha+1} < p_1 < \cdots < p_t)$$

is abundant, and all its proper divisors are deficient, if

(11) $$\frac{2^{\alpha+1}}{2^{\alpha+1} - 1}\left(1 + \frac{1}{p_1}\right) > \prod_{i=1}^{t}\left(1 + \frac{1}{p_i}\right) \geq \frac{2^{\alpha+1}}{2^{\alpha+1} - 1}$$

or

(12) $$\frac{(2^{\alpha+1} - 1)\prod_{i=1}^{t-1}(p_i + 1)}{2^{\alpha+1}\left(1 + \frac{1}{p_1}\right)\prod_{i=1}^{t-1}p_i - (2^{\alpha+1} - 1)\prod_{i=1}^{t-1}(p_i + 1)} < p_t \leq \frac{(2^{\alpha+1} - 1)\prod_{i=1}^{t-1}(p_i + 1)}{2^{\alpha+1}\prod_{i=1}^{t-1}p_i - (2^{\alpha+1} - 1)\prod_{i=1}^{t-1}(p_i + 1)}.$$

We see that n is not pseudoperfect if

(13) $$\sigma(n) - 2n = 2^{\alpha+1},$$

because

$$\sigma(n) - n - \sum_{\nu=0}^{\alpha} 2^{\nu} = n + 1 > n$$

and

$$\sigma(n) - n - p_1 = n - (p_1 - 2^{\alpha+1}) < n.$$

Write (13) into the form

(14) $$p_t = \frac{(2^{\alpha+1} - 1)\prod_{i=1}^{t-1}(p_i + 1) - 2^{\alpha+1}}{2^{\alpha+1}\prod_{i=1}^{t-1}p_i - (2^{\alpha+1} - 1)\prod_{i=1}^{t-1}(p_i + 1)}.$$

We see that p_t from (14) also satisfies (12) and this remains valid if we replace $2^{\alpha+1}$ in (13) and (14), e.g., by any constant $A \geq 2^{\alpha+1}$ provided that $p_1 > A$.

We can now present an algorithm for computing arbitrary long (great) primitive weird numbers n satisfying (10) and (14) if they exist.

For given α choose first the prime $p_1 > (A \geq) 2^{\alpha+1}$ and then p_2 from (14). If this is not a prime, choose p_2 an arbitrary prime $> p_1$ and calculate p_3 from (14). If this is not a prime, choose p_3 an arbitrary prime $> p_2$, and so on. The algorithm ends when we obtain a prime p_t from (14).

REFERENCE

1. S. J. Benkoski and P. Erdös. "On Weird and Pseudoperfect Numbers." *Math. of Comp.* 126 (1974):617-623.

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FIBONACCI CONCEPT: EXTENSION TO REAL ROOTS OF POLYNOMIAL EQUATIONS

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It was in November 1973 when Professor T. A. Davis was conducting a biocensus that he introduced me to the well-known Fibonacci numbers. He told me that certain limbs of a normal human body are in the Golden Ratio, viz. 1.618... . I observed that the reciprocal of the Golden Ratio (0.618...) is nothing but a root of the quadratic equation

(1) $$x^2 + x = 1$$

or

(2) $$x^2 + x - 1 = 0$$

which is formed by equating the three ratios of human limbs (each ratio, in fact, is equal to the Golden Ratio).

As is well known, this root 0.618 of (1) is the fixed ratio of the successive terms (ignoring some of the initial terms) of the Fibonacci sequence. I considered the sequence $\{U_r\}$ defined as follows:

(3) $$U_r = 1,\ \forall_r = 1, 2, 3;\ U_r = U_{r-1} + U_{r-2} + U_{r-3},\ \forall_r \geq 4.$$

Using a computer program, I found that after 21 terms of the sequence, the ratios $\left\{\frac{U_{r-1}}{U_r}\right\}$ become constant up to the 9th decimal place and is 0.543689013, which is found to be a root of the polynomial equation (cubic),

(4) $$x^3 + x^2 + x = 1.$$

Now, consider the sequences defined, analogously, as follows:

Sequence (Definitions):

(i) $U_r = 1,\ \forall_r = 1, 2, 3, 4;\ U_r = U_{r-1} + U_{r-2} + U_{r-3} + U_{r-4},\ \forall_r \geq 5;$

(ii) $U_r = 1,\ \forall_r = 1, 2, 3, 4, 5;\ U_r = U_{r-1} + U_{r-2} + U_{r-3} + U_{r-4} + U_{r-5},\ \forall_r \geq 6;$

⋮

(vii) $U_r = 1,\ \forall_r = 1, 2, 3, \ldots, 10;\ U_r = U_{r-1} + U_{r-2} + \cdots + U_{r-10},\ \forall_r \geq 11.$

The approximate limit points (which do exist) of sequences of ratios $\left\{\frac{U_{r-1}}{U_r}\right\}$ as obtained by computer, are 0.518790064, 0.508660392, 0.504138258, 0.502017055, 0.500994178, 0.500493118, and