$$= \frac{(-1)^{\frac{m_1 p^{a-1}(p-1)}{2} - 1} p^a (p^{2a-1} + 1)(p - 1)m_1}{24}, \qquad (\text{mod } p^a).$$

This means that the exponent of 3 in the canonical expansion of $B_1(3, m_1 3^a)$ $(3 \nmid m_1)$ is $a - 1$, and

$$p^a | B_1(p, m_1 p^a) \;\; (p \nmid m_1) \quad \text{for} \quad p > 3.$$

In the last case, we do not know, however, the exact exponent of $p$ in the canonical factorization of $B_1$.

VII

Returning to the difference

$$\binom{p^{n+1}}{p^n} - \binom{p^n}{p^{n-1}} = A_p(p^{n+1}, p^n) = \binom{p^n}{p^{n-1}} \left\{ \sum_{r=1}^{m(p-1)/2} \left(p^{2n+1}(p - 1)\right)^r B_r \right\} \Bigg/ \prod_{j=1,\, p \nmid j}^{p^n - 1} j,$$

we know that

$$\binom{p^n}{p^{n-1}} = p \prod_{j=1}^{p^{n-1}} \frac{p^n - j}{p^{n-j} - j}$$

is divisible by $p$ but not by $p^2$. Thus, the results concerning the divisibility of $B_1$ give immediately the results ennounced in the theorem. More generally, if

$$M = m_1 p^a,\; K = k_1 p^b,\; \min(a, b) = c,\; (p \nmid m_1, k_1),\; \text{and}\; 2/M\left(1 - \frac{1}{p}\right),$$

then

$$p^{a+b+c+d}/D_p(K, M),$$

where

$$d = \begin{cases} -2 & \text{for} \quad p = 2,\; a \geq 2, \\ -1 & \text{for} \quad p = 3, \\ \;\;0 & \text{for} \quad p > 3. \end{cases}$$

As for $A_p(K, M)$, we have to multiply this by the power of $p$ in the factorization of $\binom{K}{M}$ which can be calculated by the theorem of Lagrange.

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## FORMATION OF GENERALIZED $F$-$L$ IDENTITIES OF THE FORM $\sum_{r=1}^{n} r^{\bar{s}} F_{kr+m}$


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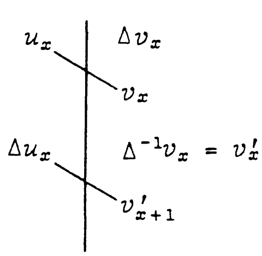
### PRELIMINARIES

$r^{\bar{s}} = r(r + 1) \cdots (r + s - 1)$. The various identities will be formed by using, if necessary, an iterated integration by parts formula for finite differences. $r^{\bar{s}}$ is convenient since $\Delta r^{\bar{s}} = s(r + 1)^{\overline{s-1}}$, $\Delta^2 r^{\bar{s}} = s(s - 1)(r + 2)^{\overline{s-2}}$, $\cdots$, $\Delta r^{\bar{s}} = s!$

$$\Delta^{-1}[u_x \Delta v_x] = u_x v_x - \Delta^{-1}[(\Delta u_x)(v_{x+1})].$$

This formula can be iterated:




*(continued)*

$$
\begin{array}{l|l}
\Delta^2 u_x & \Delta^{-1} v'_{x+1} = v''_{x+1} \\
\quad\searrow & v''_{x+2} \\
\Delta^3 u_x & \Delta^{-1} v''_{x+2} = v'''_{x+2} \\
\quad\searrow & v'''_{x+3}
\end{array}
$$

Starting with the second term, after each finite integration, the subscript $x$ is replaced by $x + 1$.

$$\Delta^{-1}[u_x v_x] = u_x v_x - (\Delta u_x)(v'_{x+1}) + (\Delta^2 u_x)(v''_{x+2}) - (\Delta^3 u_x)(v'''_{x+3}) + \cdots .$$

## 1. PURE FIBONACCI-LUCAS IDENTITIES

$$\Delta F_x = F_{x+1} - F_x = F_{x-1},\ \Delta L_x = L_{x+1} - L_x = L_{x-1},\ \Delta^{-1} F_x = F_{x+1},\ \Delta^{-1} L_x = L_{x+1}.$$

(1) $$\sum_1^n F_x = [\Delta^{-1} F_x]_1^{n+1} = [F_{x+1}]_1^{n+1} = F_{n+2} - F_2$$

(2) $$\sum_1^n L_x = [\Delta^{-1} L_x]_1^{n+1} = [L_{x+1}]_1^{n+1} = L_{n+2} - L_2$$

$$\sum_1^n x F_x = [x F_{x+1} - F_{x+3}]_1^{n+1} = (n+1)F_{n+2} - F_{n+4} - F_2 + F_4$$

(3) $$\sum_1^n x F_x = (n+1)F_{n+2} - F_{n+4} + F_3$$

(4) $$\sum_1^n x L_x = (n+1)L_{n+2} - L_{n+4} + L_3$$

$$\begin{aligned}\sum_1^n x^{\overline{2}} F_x &= [x^{\overline{2}} F_{x+1} - 2(x+1)F_{x+3} + 2F_{x+5}]_1^{n+1} \\ &= (n+1)(n+2)F_{n+2} - 2(n+2)F_{n+4} + 2F_{n+6} - 2!(F_2 - 2F_4 + F_6)\end{aligned}$$

(5) $$\sum_1^n x^{\overline{2}} F_x = (n+1)(n+2)F_{n+2} - 2(n+2)F_{n+4} + 2F_{n+6} - 2!F_4$$

(6) $$\sum_1^n x^{\overline{2}} L_x = (n+1)(n+2)L_{n+2} - 2(n+2)L_{n+4} + 2L_{n+6} - 2!L_4$$

$$\begin{aligned}\sum_1^n x^{\overline{3}} F_x &= [x^{\overline{3}} F_{x+1} - 3(x+1)^{\overline{2}} F_{x+3} + 6(x+2)F_{x+5} - 6F_{x+7}]_1^{n+1} \\ &= (n+1)^{\overline{3}} F_{n+2} - 3(n+2)^{\overline{2}} F_{n+4} + 6(n+3)F_{n+6} - 6F_{n+8} - 3!(F_2 - 3F_4 + 3F_6 - F_8)\end{aligned}$$

(7) $$\sum_1^n x^{\overline{3}} F_x = (n+1)(n+2)(n+3)F_{n+2} - 3(n+2)(n+3)F_{n+4} + 6(n+3)F_{n+6} - 6F_{n+8} + 3!F_5$$

(8) $$\sum_1^n x^{\overline{3}} L_x = (n+1)^{\overline{3}} L_{n+2} - 3(n+2)^{\overline{2}} L_{n+4} + 6(n+3)L_{n+6} - 6L_{n+8} + 3!L_5$$

(9) $$\begin{aligned}\sum_1^n x^{\overline{s}} F_x = {}& (n+1)^{\overline{s}} F_{n+2} - s(n+2)^{\overline{s-1}} F_{n+4} + s(s-1)(n+3)^{\overline{s-2}} F_{n+6} \cdots \\ &+ (-1)^s s! F_{n+2s+2} + (-1)^{s+1} s! F_{s+2}\end{aligned}$$

(10) $$\begin{aligned}\sum_1^n x^{\overline{s}} L_x = {}& (n+1)^{\overline{s}} L_{n+2} - s(n+2)^{\overline{s-1}} L_{n+4} + s(s-1)(n+3)^{\overline{s-2}} L_{n+6} - \cdots \\ &+ (-1)^s s! L_{n+2s+2} + (-1)^{s+1} s! L_{s+2}\end{aligned}$$

## 2. PURE FIBONACCI-LUCAS IDENTITIES

$$\Delta F_{2x} = F_{2x+2} - F_{2x} = F_{2x+1}$$
$$\Delta^{-1} F_{2x} = F_{2x-1}, \ \Delta^{-1} L_{2x} = L_{2x-1}$$

(11) $$\sum_1^n F_{2x} = [\Delta^{-1} F_{2x}]_1^{n+1} = [F_{2x-1}]_1^{n+1} = F_{2n+1} - F_1$$

(12) $$\sum_1^n L_{2x} = L_{2n+1} - L_1$$

(13) $$\sum_1^n F_{2x+1} = F_{2n+2} - F_2$$

(14) $$\sum_1^n L_{2x+1} = L_{2n+2} - L_2$$

$$\sum_1^n xF_{2x} = [xF_{2x-1} - F_{2x}]_1^{n+1} = (n+1)F_{2n+1} - F_{2n+2} - (F_1 - F_2)$$

(15) $$\sum_1^n xF_{2x} = (n+1)F_{2n+1} - F_{2n+2} + F_0$$

(16) $$\sum_1^n xL_{2x} = (n+1)L_{2n+1} - L_{2n+2} + L_0$$

(17) $$\sum_1^n xF_{2x+1} = (n+1)F_{2n+2} - F_{2n+3} + F_1$$

(18) $$\sum_1^n xL_{2x+1} = (n+1)L_{2n+2} - L_{2n+3} + L_1$$

$$\sum_1^n x^{\bar{2}} F_{2x} = [x^{\bar{2}} F_{2x-1} - 2(x+1)F_{2x} + 2F_{2x+1}]_1^{n+1}$$
$$= (n+1)(n+2)F_{2n+1} - 2(n+2)F_{2n+2} + 2F_{2n+3} - 2!(F_1 - 2F_2 + F_3)$$

(19) $$\sum_1^n x^{\bar{2}} F_{2x} = (n+1)(n+2)F_{2n+1} - 2(n+2)F_{2n+2} + 2F_{2n+3} - 2!F_1$$

(20) $$\sum_1^n x^{\bar{2}} L_{2x} = (n+1)(n+2)L_{2n+1} - 2(n+2)L_{2n+2} + 2L_{2n+3} - 2!L_1$$

(21) $$\sum_1^n x^{\bar{2}} F_{2x-1} = (n+1)(n+2)F_{2n} - 2(n+2)F_{2n+1} + 2F_{2n+2} - 2!F_2$$

(22) $$\sum_1^n x^{\bar{2}} L_{2x-1} = (n+1)(n+2)L_{2n} - 2(n+2)L_{2n+1} + 2L_{2n+2} - 2!L_2$$

$$\sum_1^n x^{\bar{3}} F_{2x} = [x^{\bar{3}} F_{2x-1} - 3(x+1)^{\bar{2}} F_{2x} + 6(x+2)F_{2x+1} - 6F_{2x+2}]_1^{n+1}$$
$$= (n+1)^{\bar{3}} F_{2n+1} - 3(n+2)^{\bar{2}} F_{2n+2} + 6(n+3)F_{2n+3} - 6F_{2n+4} - 3!(F_1 - 3F_2 + 3F_3 - F_4)$$

(23) $$\sum_1^n x^{\bar{3}} F_{2x} = (n+1)^{\bar{3}} F_{2n+1} - 3(n+2)^{\bar{2}} F_{2n+2} + 6(n+3)F_{2n+3} - 6F_{2n+4} + 3!F_{-2}$$

(24) $$\sum_1^n x^{\bar{3}} L_{2x} = (n+1)^{\bar{3}} L_{2n+1} - 3(n+2)^{\bar{2}} L_{2n+2} + 6(n+3)L_{2n+3} - 6L_{2n+4} + 3!L_{-2}$$

(25) $$\sum_1^n x^{\bar{3}} F_{2x-1} = (n+1)^{\bar{3}} F_{2n} - 3(n+2)^{\bar{2}} F_{2n+1} + 6(n+3)F_{2n+2} - 6F_{2n+3} + 3!F_{-3},$$

$$(26)\qquad \sum_{1}^{n} x^{\overline{3}} L_{2x-1} = (n+1)^{\overline{3}} L_{2n} - 3(n+2)^{\overline{2}} L_{2n+1} + 6(n+3)L_{2n+2} - 6L_{2n+3} + 3!L_{-3}$$

. . . . . . . . . . . . . . . . . . . .

$$(27)\qquad \sum_{1}^{n} x^{\overline{s}} F_{2x} = (n+1)^{\overline{3}} F_{2n+1} - s(n+2)^{\overline{s-1}} F_{2n+2} + s(s-1)(n+3)^{\overline{s-2}} F_{2n+3} - \cdots$$
$$(-1)^{s} s! F_{2n+s+1} + (-1)^{s+1} s! F_{-(s-1)}$$

$$(28)\qquad \sum_{1}^{n} x^{\overline{s}} L_{2x} = (n+1)^{\overline{s}} L_{2n+1} - s(n+2)^{\overline{s-1}} L_{2n+2} + s(s-1)(n+3)^{\overline{s-2}} L_{2n+3} \cdots$$
$$(-1)^{s} s! L_{2n+s+1} + (-1)^{s+1} s! L_{-(s-1)}$$

## 3. PURE FIBONACCI-LUCAS IDENTITIES

$$F_{3x} = F_{3x+3} - F_{3x} = \frac{1}{\sqrt{5}}[\alpha^{3x}(\alpha^{3} - 1) - \beta^{3x}(\beta^{3} - 1)]$$
$$= \frac{1}{\sqrt{5}}[\alpha^{3x}(2\alpha) - \beta^{3x}(2\beta)] = \frac{2}{\sqrt{5}}(\alpha^{3x+1} - \beta^{3x+1}) = 2F_{3x+1}$$

$$\Delta^{-1}F_{3x} = \frac{1}{2}F_{3x-1},\quad \Delta^{-1}L_{3x} = \frac{1}{2}L_{3x-1}$$

$$(29)\qquad \sum_{1}^{n} F_{3x} = [\Delta^{-1}F_{3x}]^{n+1} = \left[\frac{1}{2}F_{3x-1}\right]_{1}^{n+1} = \frac{1}{2}F_{3n+2} - \frac{1}{2}F_{2}$$

$$(30)\qquad \sum_{1}^{n} F_{3x+1} = \frac{1}{2}F_{3n+3} - \frac{1}{2}F_{3}$$

$$(31)\qquad \sum_{1}^{n} F_{3x+2} = \frac{1}{2}F_{3n+4} - \frac{1}{2}F_{4}$$

$$(32)\qquad \sum_{1}^{n} L_{3x} = \frac{1}{2}L_{3n+2} - \frac{1}{2}L_{2}$$

$$(33)\qquad \sum_{1}^{n} L_{3x+1} = \frac{1}{2}L_{3n+3} - \frac{1}{2}L_{3}$$

$$(34)\qquad \sum_{1}^{n} L_{3x+2} = \frac{1}{2}L_{3n+4} - \frac{1}{2}L_{4}$$

$$\sum_{1}^{n} xF_{3x} = \left[\frac{1}{2}xF_{3x-1} - \frac{1}{4}F_{3x+1}\right]_{1}^{n+1}$$

$$(35)\qquad \sum_{1}^{n} xF_{3x} = \frac{1}{2}(n+1)F_{3n+2} - \frac{1}{4}F_{3n+4} + \frac{F_{1}}{4}$$

$$(36)\qquad \sum_{1}^{n} xF_{3x+1} = \frac{1}{2}(n+1)F_{3n+3} - \frac{1}{4}F_{3n+5} + \frac{1}{4}F_{2}$$

$$(37)\qquad \sum_{1}^{n} xF_{3x+2} = \frac{1}{2}(n+1)F_{3n+4} - \frac{1}{4}F_{3n+6} + \frac{1}{4}F_{3}$$

$$(38)\qquad \sum_{1}^{n} xL_{3x} = \frac{1}{2}(n+1)L_{3n+2} - \frac{1}{4}L_{3n+4} + \frac{1}{4}L_{1}$$

$$(39)\qquad \sum_{1}^{n} xL_{3x+1} = \frac{1}{2}(n+1)L_{3n+3} - \frac{1}{4}L_{3n+5} + \frac{1}{4}L_{2}$$

$$(40)\qquad \sum_{1}^{n} xL_{3x+2} = \frac{1}{2}(n+1)L_{3n+4} - \frac{1}{4}L_{3n+6} + \frac{1}{4}L_{3}$$

$$\sum_{1}^{n} x^{\overline{2}}F_{3x} = \left[\frac{1}{2}x^{\overline{2}}F_{3x-1} - \frac{1}{4}2(x+1)F_{3x+1} + \frac{2}{8}F_{3x+3}\right]_{1}^{n+1}$$

(41) $$\sum_1^n x^{\overline{2}}F_{3x} = \frac{1}{2}(n+1)(n+2)F_{3n+2} - \frac{2}{4}(n+2)F_{3n+4} + \frac{2}{8}F_{3n+6} - \frac{2!}{8}F_0$$

(42) $$\sum_1^n x^{\overline{2}}F_{3x+1} = \frac{1}{2}(n+1)(n+2)F_{3n+3} - \frac{2}{4}(n+2)F_{3n+5} + \frac{2}{8}F_{3n+7} - \frac{2!}{8}F_1$$

(43) $$\sum_1^n x^{\overline{2}}F_{3x+2} = \frac{1}{2}(n+1)(n+2)F_{3n+4} - \frac{2}{4}(n+2)F_{3n+6} + \frac{2}{8}F_{3n+8} - \frac{2!}{8}F_2$$

(44) $$\sum_1^n x^{\overline{2}}L_{3x} = \frac{1}{2}(n+1)(n+2)L_{3n+2} - \frac{2}{4}(n+2)L_{3n+4} + \frac{2}{8}L_{3n+6} - \frac{2!}{8}L_0$$

(45) $$\sum_1^n x^{\overline{2}}L_{3x+1} = \frac{1}{2}(n+1)(n+2)L_{3n+3} - \frac{2}{4}(n+2)L_{3n+5} + \frac{2}{8}L_{3n+7} - \frac{2!}{8}L_1$$

(46) $$\sum_1^n x^{\overline{2}}L_{3x+2} = \frac{1}{2}(n+1)(n+2)L_{3n+4} - \frac{2}{4}(n+2)L_{3n+6} + \frac{2}{8}L_{3n+8} - \frac{2!}{8}L_2$$

It is evident that the formulas for $\sum_1^n x^{\overline{2}}F_{3x+a}$, $a = 1, 2, \ldots$, follow directly from $\sum_1^n x^{\overline{2}}F_{3x}$. However, they are very useful for determining lower limit algorithms, if they exist.

$$\sum_1^n x^{\overline{3}}F_{3x} = \left[\frac{1}{2}x^{\overline{3}}F_{3x-1} - \frac{3}{4}(x+1)^{\overline{2}}F_{3x+1} + \frac{6}{8}(x+2)F_{3x+3} - \frac{6}{16}F_{3x+5}\right]_1^{n+1}$$

(47) $$\sum_1^n x^{\overline{3}}F_{3x} = \frac{1}{2}(n+1)^{\overline{3}}F_{3n+2} - \frac{3}{4}(n+2)^{\overline{2}}F_{3n+4} + \frac{6}{8}(n+3)F_{3n+6} - \frac{6}{16}F_{3n+8} + \frac{3!}{16}F_{-1}$$

(48) $$\sum_1^n x^{\overline{3}}L_{3x} = \frac{1}{2}(n+1)^{\overline{3}}L_{3n+2} - \frac{3}{4}(n+2)^{\overline{2}}L_{3n+4} + \frac{6}{8}(n+3)L_{3n+6} - \frac{6}{16}L_{3n+8} + \frac{3!}{16}L_{-1}$$

$$\sum_1^n x^{\overline{s}}F_{3x} = \left[\frac{1}{2}x^{\overline{s}}F_{3x-1} - \frac{s}{4}(x+1)^{\overline{s-1}}F_{3x+1} + \frac{s(s-1)}{8}(x+2)^{\overline{s-2}}F_{3x+3} - \cdots + (-1)^s s! F_{3x+2s-1}\right]_1^{n+1}$$

(49) $$\sum_1^n x^{\overline{s}}F_{3x} = \frac{1}{2}(n+1)^{\overline{s}}F_{3n+2} - \frac{s}{4}(n+2)^{\overline{s-1}}F_{3n+4} + \frac{s(s-1)}{8}(n+3)^{\overline{s-2}}F_{3n+6} - \cdots + (-1)^s\frac{s!}{2^{s+1}}F_{3n+2s+2} + (-1)^{s+1}\frac{s!}{2^{s+1}}F_{-s+2}$$

(50) $$\sum_1^n x^{\overline{s}}L_{3x} = \frac{(n+1)^{\overline{s}}}{2}L_{3n+2} - \frac{s}{4}(n+2)^{s-1}L_{3n+4} + \frac{s(s-1)}{8}(n+s)^{s-2}L_{3n+6} - \cdots + \frac{(-1)^s s!}{2^{s+1}}L_{3n+2s+2} + (-1)^{s+1}\frac{s!}{2^{s+1}}L_{-s+2}$$

*Further Remarks:* The 50 identities in Sections 1, 2, and 3 involved Fibonacci sequence properties. The following identities involve Type 1 primitive unit properties. Let $(a + b\sqrt{D})/2$ be a primitive unit in the real quadratic field $(\sqrt{D})$, $D \equiv 5 \pmod 8$, $a^2 - b^2D = -4$. Let

$$\alpha = \frac{a + b\sqrt{D}}{2},\ \beta = \frac{a - b\sqrt{D}}{2},\ \left(\frac{a + b\sqrt{D}}{2}\right)^n = \frac{L_n + F_n\sqrt{D}}{2},\ F_n = \frac{1}{\sqrt{D}}(\alpha^n - \beta^n),\ L_n = \alpha^n + \beta^n,\ \alpha\beta = -1.$$

$F_n$ and $L_n$ are also given by the finite difference equations

(*) $$F_{n+2} = aF_{n+1} + F_n,\ F_1 = b,\ F_2 = ab$$
$$L_{n+2} = aL_{n+1} + L_n,\ L_1 = a,\ L_2 = a^2 + 2$$

*Examples:*

$$D = 5, \quad \text{with primitive unit } \frac{1+\sqrt{5}}{2}$$

$$D = 13, \quad \text{with primitive unit } \frac{3+\sqrt{5}}{2}$$

$$D = 61, \quad \text{with primitive unity } \frac{39+5\sqrt{61}}{2}$$

## 4. TYPE 1 PRIMITIVE UNIT IDENTITIES

$$\Delta F_{4rx} = F_{4r(x+1)} - F_{4rx} = \frac{1}{\sqrt{D}}[\alpha^{4rx}(\alpha^{4r} - 1) - \beta^{4rx}(\beta^{4r} - 1)]$$

$$= \frac{1}{\sqrt{D}}[\alpha^{4rx}\alpha^{2r}(\alpha^{2r} - \beta^{2r}) + \beta^{4rx}\beta^{2r}(\alpha^{2r} - \beta^{2r})]$$

$$\Delta F_{4rx} = F_{2r}L_{2r(2x+1)}$$

$$\Delta^{-1}L_{4rx} = \frac{1}{F_{2r}}F_{2r(2x-1)}$$

$$\Delta L_{4rx} = \alpha^{4rx}\alpha^{2r}(\alpha^{2r} - \beta^{2r}) - \beta^{4rx}\beta^{2r}(\alpha^{2r} - \beta^{2r})$$

$$\Delta L_{4rx} = DF_{2r}F_{2r(2x+1)}$$

$$\Delta^{-1}F_{4rx} = \frac{1}{DF_{2r}}L_{2r(2x-1)}$$

(51) $$\sum_1^n F_{4rx} = [\Delta^{-1}F_{4rx}]_1^{n+1} = \frac{1}{DF_{2r}}L_{2r(2n+1)} - \frac{1}{DF_{2r}}L_{2r}$$

(52) $$\sum_1^n L_{4rx} = \left[\frac{1}{F_{2r}}F_{2r(2x-1)}\right]_1^{n+1} = \frac{F_{2r(2n+1)}}{F_{2r}} - \frac{F_{2r}}{F_{2r}}$$

$$\sum_1^n xF_{4rx} = \left[\frac{x}{DF_{2r}}L_{2r(2x-1)} - \frac{1}{DF_{2r}^2}F_{2r(2x)}\right]_1^{n+1}$$

(53) $$\sum_1^n xF_{4rx} = \frac{(n+1)}{DF_{2r}}L_{2r(2n+1)} - \frac{1}{DF_{2r}^2}F_{2r(2n+2)} + \frac{1}{DF_{2r}^2}F_0$$

(54) $$\sum_1^n xL_{4rx} = \frac{(n+1)}{F_{2r}}F_{2r(2n+1)} - \frac{1}{DF_{2r}^2}L_{2r(2n+2)} + \frac{1}{DF_{2r}^2}L_0$$

$$\sum_1^n x^{\overline{2}}F_{4rx} = \left[\frac{x^{\overline{2}}}{DF_{2r}}L_{2r(2x-1)} - \frac{2(x+1)}{DF_{2r}^2}F_{2r(2x)} + \frac{2}{D^2F_{2r}^3}L_{2r(2x+1)}\right]_1^{n+1}$$

$$\sum_1^n x^{\overline{2}}L_{4rx} = \left[\frac{x^{\overline{2}}}{F_{2r}}F_{2r(2x-1)} - \frac{2(x+1)}{DF_{2r}^2}L_{2r(2x)} + \frac{2}{DF_{2r}^3}F_{2r(2x+1)}\right]_1^{n+1}$$

(55) $$\sum_1^n x^{\overline{2}}F_{4rx} = \frac{(n+1)(n+2)}{DF_{2r}}L_{2r(2n+1)} - \frac{2(n+2)}{DF_{2r}^2}F_{2r(2n+2)} + \frac{2}{D^2F_{2r}^3}F_{2r(2n+3)} - \frac{2!}{D^2F_{2r}^3}L_{-2}$$

(56) $$\sum_1^n x^{\overline{2}}L_{4rx} = \frac{(n+1)(n+2)}{F_{2r}}F_{2r(2n+1)} - \frac{2(n+2)}{DF_{2r}^2}L_{2r(2n+2)} + \frac{2}{DF_{2r}^3}F_{2r(2n+3)} - \frac{2!}{DF_{2r}^3}F_{-2}$$

$$\sum_1^n x^{\overline{3}}F_{4rx} = \left[\frac{x^{\overline{3}}}{DF_{2r}}L_{2r(2x-1)} - \frac{3(x+1)^{\overline{2}}}{DF_{2r}^2}F_{2r(2x)} + \frac{6(x+2)}{D^2F_{2r}^3}L_{2r(2x+1)} - \frac{6}{D^2F_{2r}^4}F_{2r(2x+2)}\right]_1^{n+1}$$

$$\sum_1^n x^{\overline{3}}L_{4rx} = \left[\frac{x^{\overline{3}}}{F_{2r}}F_{2r(2x-1)} - \frac{3(x+1)^{\overline{2}}}{DF_{2r}^2}L_{2r(2x)} + \frac{6(x+2)}{D^2F_{2r}^3}F_{2r(2x+1)} - \frac{6}{D^2F_{2r}^4}L_{2r(2x+2)}\right]_1^{n+1}$$

(57)
$$\begin{aligned}\sum_{1} x^{\overline{3}} F_{4rx} = {} & \frac{(n+1)(n+2)(n+3)}{DF_{2r}} L_{2r(2n+1)} - \frac{3(n+2)(n+3)}{DF_{2r}^{2}} F_{2r(2n+2)} \\ & + \frac{6(n+3)}{D^{2}F_{2r}^{3}} L_{2r(2n+3)} - \frac{6}{D^{2}F_{2r}^{4}} F_{2r(2n+4)} + \frac{3!}{D^{2}F_{2r}^{4}} F_{-4r}\end{aligned}$$

(58)
$$\begin{aligned}\sum_{1}^{n} x^{\overline{3}} L_{4rx} = {} & \frac{(n+1)(n+2)(n+3)}{F_{2r}} F_{2r(2n+1)} - \frac{3(n+2)(n+3)}{DF_{2r}^{2}} L_{2r(2n+2)} \\ & + \frac{6(n+3)}{DF_{2r}^{3}} F_{2r(2n+3)} - \frac{6}{D^{2}F_{2r}^{4}} L_{2r(2n+4)} + \frac{3!}{D^{2}F_{2r}^{4}} L_{-4r}\end{aligned}$$

(59)
$$\begin{aligned}\sum_{1}^{n} x^{\overline{4}} F_{4rx} = {} & \frac{(n+1)^{\overline{4}}}{DF_{2r}} L_{2r(2n+1)} - \frac{4(n+2)^{\overline{3}}}{DF_{2r}^{2}} F_{2r(2n+2)} + \frac{12(n+3)^{\overline{2}}}{D^{2}F_{2r}^{3}} L_{2r(2n+3)} \\ & - \frac{24(n+4)F_{2r(2n+4)}}{D^{2}F_{2r}^{4}} + \frac{24}{D^{3}F_{2r}^{5}} L_{2r(2n+5)} - \frac{4!}{D^{3}F_{2r}^{5}} L_{-6r}\end{aligned}$$

(60)
$$\begin{aligned}\sum_{1}^{n} x^{\overline{4}} L_{4rx} = {} & \frac{(n+1)^{\overline{4}}}{F_{2r}} F_{2r(2n+1)} - \frac{4(n+2)^{\overline{3}}}{DF_{2r}^{2}} L_{2r(2n+2)} + \frac{12(n+3)^{\overline{2}}}{DF_{2r}^{3}} F_{2r(2n+3)} \\ & - \frac{24(n+4)}{D^{2}F_{2r}^{4}} L_{2r(2n+4)} + \frac{24}{D^{2}F_{2r}^{5}} F_{2r(2n+5)} - \frac{4!}{D^{2}F_{2r}^{5}} F_{-6r}\end{aligned}$$

(61)
$$\begin{aligned}\sum_{1}^{n} x^{\overline{2s}} F_{4rx} = {} & \frac{(n+1)^{\overline{2s}}}{DF_{2r}} L_{2r(2n+1)} - \frac{2s(n+2)^{\overline{2s-1}}}{DF_{2r}^{2}} F_{2r(2n+2)} \\ & + \frac{2s(2s-1)(n+3)^{\overline{2s-2}}}{D^{2}F_{2r}^{3}} L_{2r(2n+3)} - \frac{2s(2s-1)(2s-2)(n+4)^{2s-3}}{D^{2}F_{2r}^{4}} F_{2r(2n+4)} \\ & + \cdots + \frac{(-1)^{2s}(2s)!L_{2r(2n+2s+1)}}{D^{s+1}F_{2r}^{2s+1}} + \frac{(-1)^{2s+1}L_{2r(2s-1)}(2s)!}{D^{s+1}F_{2r}^{2s+1}}\end{aligned}$$

(62)
$$\begin{aligned}\sum_{1}^{n} x^{\overline{2s}} L_{4rx} = {} & \frac{(n+1)^{\overline{2s}}F_{2r(2n+1)}}{F_{2r}} - \frac{2s(n+2)^{\overline{2s-1}}}{DF_{2r}^{2}} L_{2r(2n+2)} + \frac{2s(2s-1)(n+3)^{\overline{2s-2}}}{D^{2}F_{2r}^{3}} \\ & F_{2r(2n+3)} - \frac{(2s)(2s-1)(2s-2)(n+4)^{\overline{2s-3}}}{D^{2}F_{2r}^{4}} L_{2r(2n+4)} + \cdots \\ & + \frac{(-1)^{2s}(2s)!}{D^{2}F_{2r}^{2s+1}} F_{2r(2n+2s+1)} + \frac{(-1)^{2s\;\;1}(2s)!F_{-2r(2s-1)}}{D^{2}F_{2}^{2s+1}}\end{aligned}$$

(63)
$$\begin{aligned}\sum_{1}^{n} x^{\overline{2s+1}} F_{4rx} = {} & \frac{(n+1)^{\overline{2s+1}}}{DF_{2r}} L_{2r(2n+1)} - \frac{(2s+1)(n+2)^{\overline{2s}}F_{2r(2n+2)}}{DF_{2r}^{2}} \\ & + \frac{(2s+1)(2s)(n+3)^{\overline{2s-1}}}{D^{2}F_{2r}^{3}} L_{2r(2n+3)} - \frac{(2s+1)(2s)(2s-1)(n+4)^{\overline{2s-2}}}{D^{4}F_{2r}^{4}} \\ & F_{2r(2n+4)} + \cdots + \frac{(-1)^{2s+1}(2s+1)}{D^{s+1}F_{2r}^{2s\;\;2}} F_{2r(2n+2s+2)} + \frac{(-1)^{2s}(2s+1)!F_{-2r(2s)}}{D^{s+1}F_{2r}^{2s+2}}\end{aligned}$$

$$(64)\quad \sum_{1}^{n} x^{2s+1}L_{4rx} = \frac{(n+1)^{2s+1}F_{2r(2n+1)}}{F_{2r}} - \frac{(2s+1)(n+2)^{2s}L_{2r(2n+2)}}{DF_{2r}^{2}} + \frac{(2s+1)(2s)(n+3)^{2s-1}}{DF_{2r}^{3}}F_{2r(2n+3)} - \frac{(2s+1)(2s)(2s-1)(n+4)^{2s-2}}{D^{2}F_{2r}^{4}}L_{2r(2n+4)} + \cdots + \frac{(-1)^{2s+1}(2s+1)!}{D^{s}F_{2r}^{2s+2}}L_{2r(2n+2s+2)} + \frac{(-1)^{2s}(2s+1)!L_{-2r(2s)}}{D^{s+1}F_{2r}^{2s+2}}$$

## 5. TYPE 1 PRIMITIVE UNIT IDENTITIES

$$\Delta F_{2x(2r+1)} = F_{(2x+2)(2r+1)} - F_{2x(2r+1)}$$

$$= \frac{1}{\sqrt{D}}[\alpha^{x(4r+2)}\alpha^{2r+1}(\alpha^{2r+1}+\beta^{2r+1}) - \beta^{x(4r+2)}\beta^{2r+1}(\alpha^{2r+1}+\beta^{2r+1})]$$

$$= L_{2r+1}F_{(2r+1)(2x+1)}$$

$$\Delta^{-1}F_{(2r+1)2x} = \frac{1}{L_{2r+1}}F_{(2r+1)(2x-1)},\quad \Delta^{-1}L_{(2r+1)2x} = \frac{1}{L_{2r+1}}L_{(2r+1)(2x-1)}$$

$$(65)\quad \sum_{1}^{n} F_{2x(2r+1)} = \left[\frac{1}{L_{2r+1}}F_{(2r+1)(2x-1)}\right]_{1}^{n+1} = \frac{1}{L_{2r+1}}F_{(2r+1)(2n+1)} - \frac{F_{2r+1}}{L_{2r+1}}$$

$$(66)\quad \sum_{1}^{n} L_{2x(2r+1)} = \left[\frac{1}{L_{2r+1}}L_{(2r+1)(2x-1)}\right] = \frac{1}{L_{2r+1}}L_{(2r+1)(2n+1)} - 1$$

$$\sum_{1}^{n} xF_{2x(2r+1)} = \left[\frac{x}{L_{2r+1}}F_{(2r+1)(2x-1)} - \frac{F_{(2r+1)2x}}{L_{2r+1}^{2}}\right]_{1}^{n+1}$$

$$\sum_{1}^{n} xL_{2x(2r+1)} = \left[\frac{x}{L_{2n+1}}L_{(2r+1)(2x-1)} - \frac{1}{L_{2r+1}^{2}}L_{(2r+1)(2x)}\right]_{1}^{n+1}$$

$$(67)\quad \sum_{1}^{n} xF_{2x(2r+1)} = \frac{(n+1)}{L_{2r+1}}F_{(2r+1)(2n+1)} - \frac{1}{L_{2r+1}^{2}}F_{(2r+1)(2n+2)} + \frac{F_0}{L_{2r+1}^{2}}$$

$$(68)\quad \sum_{1}^{n} xL_{2x(2r+1)} = \frac{(n+1)}{L_{2r+1}}L_{(2r+1)(2n+1)} - \frac{1}{L_{2r+1}^{2}}L_{(2r+1)(2n+2)} + \frac{L_0}{L_{2r+1}^{2}}$$

$$\sum_{1}^{n} x^{2}F_{2x(2r+1)} = \left[\frac{x^{2}}{L_{2r+1}}F_{(2r+1)(2x-1)} - \frac{2(x+1)}{L_{2r+1}^{2}}F_{(2r+1)2x} + \frac{2}{L_{2r+1}^{3}}F_{(2r+1)(2x+1)}\right]_{1}^{n+1}$$

$$\sum_{1}^{n} x^{2}L_{2x(2r+1)} = \left[\frac{x^{2}}{L_{2r+1}}L_{(2r+1)(2x-1)} - \frac{2(x+1)}{L_{2r+1}^{2}}L_{(2r+1)2x} + \frac{2}{L_{2r+1}^{3}}L_{(2r+1)(2x+1)}\right]_{1}^{n+1}$$

$$(69)\quad \sum_{1}^{n} x^{2}F_{2x(2r+1)} = \frac{(n+1)(n+2)}{L_{2r+1}}F_{(2r+1)(2n+1)} - \frac{2(n+2)}{L_{2r+1}^{2}}F_{(2r+1)(2n+2)} + \frac{2}{L_{2r+1}^{3}}F_{(2r+1)(2n+3)} - \frac{2!}{L_{2r+1}^{3}}L_{-(2r+1)}$$

(70) $$\sum_1^n x^{\overline{2}} L_{2x(2r+1)} = \frac{(n+1)(n+2)}{L_{2r+1}} L_{(2r+1)(2n+1)} - \frac{2(n+2)}{L_{2r+1}^2} L_{(2r+1)(2n+2)} + \frac{2}{L_{2r+1}^3} L_{(2r+1)(2n+3)} - \frac{2!}{L_{2r+1}^3} L_{-(2r+1)}$$

(71) $$\sum_1^n x^{\overline{s}} F_{(2r+1)2x} = \frac{(n+1)^{\overline{s}}}{L_{2r+1}} F_{(2r+1)(2n+1)} - \frac{s(n+2)^{\overline{s-1}}}{L_{2r+1}^2} F_{(2r+1)(2n+2)} + \frac{s(s-1)(n+3)^{\overline{s-2}}}{L_{2r+1}^3} F_{(2r+1)(2n+3)} + \cdots + \frac{(-1)^s s!}{L_{2r+1}^{s+1}} F_{(2r+1)(2n+s+1)} + \frac{(-1)^{s+1} s!}{L_{2r+1}^{s+1}} F_{-(s-1)(2r+1)}$$

(72) $$\sum_1^n x^{\overline{s}} L_{(2r+1)2x} = \frac{(n+1)^{\overline{s}}}{L_{2r+1}} L_{(2r+1)(2n+1)} - \frac{s(n+2)^{\overline{s-1}}}{L_{2r+1}^2} L_{(2r+1)(2n+2)} + \frac{s(s-1)(n+3)^{\overline{s-2}}}{L_{2r+1}^3} L_{(2r+1)(2n+3)} - \cdots + \frac{(-1)^s s!}{L_{2r+1}^{s+1}} L_{(2r+1)(2n+s+1)} + \frac{(-1)^{s+1} s!}{L_{2r+1}^{s+1}} L_{-(s-1)(2r+1)}$$

## SECTION 6

$$t = 2r+1. \quad \sum_1^n F_{xt} = \frac{1}{\sqrt{D}}[\alpha^t + \cdots + \alpha^{nt} - (\beta^t + \cdots + \beta^{nt})]$$

$$= \frac{1}{\sqrt{D}}\left[\alpha^t \frac{(\alpha^{nt}-1)}{(\alpha^t-1)} \frac{(\alpha^t+1)}{(\alpha^t+1)} - \frac{\beta^t(\beta^{nt}-1)}{(\beta^t-1)} \frac{(\beta^t+1)}{(\beta^t+1)}\right]$$

$$= \frac{1}{\sqrt{D}}\left[\frac{(\alpha^{nt}-1)(\alpha^t+1)}{\alpha^t+\beta^t} - \frac{(\beta^{nt}-1)(\beta^t+1)}{\alpha^t+\beta^t}\right]$$

$$= \frac{1}{\sqrt{D}L_t}[\alpha^{t(n+1)} + \alpha^{nt} - \alpha^t - \beta^{t(n+1)} - \beta^{nt} + \beta^t]$$

$$\sum_1^n F_{xt} = [\Delta^{-1}F_{xt}]_1^{n+1} = \frac{1}{L_t}(F_{t(n+1)} + F_{nt} - F_t)$$

$$\Delta^{-1}F_{xt} = \frac{1}{L_t}(F_{tx} + F_{t(x-1)}), \quad \Delta^{-1}L_{xt} = \frac{1}{L_t}(L_{tx} + L_{t(x-1)})$$

(73) $$\sum_1^n F_{xt} = \frac{1}{L_t}(F_{t(n+1)} + F_{tn} - F_t)$$

(74) $$\sum_1^n L_{xt} = \frac{1}{L_t}(L_{t(n+1)} + L_{tn} - L_t - L_0)$$

(75) $$\sum_1^n xF_{xt} = \left[\frac{x}{L_t}(F_{tx} + F_{t(x-1)}) - \frac{1}{L_t^2}(F_{t(x+1)} + 2F_{tx} + F_{t(x-1)})\right]_1^{n+1}$$

(76) $$\sum_{1}^{n} xL_{xt} = \left[\frac{x}{L_t}(L_{tx} + L_{t(x-1)}) - \frac{1}{L_t^2}(L_{t(x+1)} + 2L_{tx} + L_{t(x-1)})\right]_1^{n+1}$$

(77) $$\sum_{1}^{n} x^{\overline{2}}F_{tx} = \left[\frac{x^{\overline{2}}}{L_t}(F_{tx} + F_{t(x-1)}) - \frac{2(x+1)}{L_t^2}(F_{t(x+1)} + 2F_{tx} + F_{t(x-1)}) + \frac{1}{L_t^3}(F_{t(x+2)} + 3F_{t(x+1)} + 3F_{tx} + F_{t(x-1)})\right]_1^{n+1}$$

(78) $$\sum_{1}^{n} x^{\overline{2}}L_{tx} = \left[\frac{x^{\overline{2}}}{L_t}(L_{tx} + L_{t(x-1)}) - \frac{2(x+1)}{L_t^2}(L_{t(x+1)} + 2L_{tx} + L_{t(x-1)}) + \frac{1}{L_t^3}(L_{t(x+2)} + 3L_{t(x+1)} + 3L_{tx} + L_{t(x-1)})\right]_1^{n+1}$$

(79) $$\sum_{1}^{n} x^{\overline{s}}L_{tx} = \left[\frac{x^{\overline{s}}}{L_t}(L_{tx} + L_{t(x-1)}) - \frac{s(x+1)^{\overline{s-1}}}{L_t^2}(F_{(t+1)x} + 2F_{tx} + F_{t(x-1)}) + \frac{s(s-1)(x+2)^{\overline{s-2}}}{L_t^3}(F_{(t+2)x} + 3F_{(t+1)x} + 3F_{(t+1)x} + 3F_{tx} + F_{t(x-1)}) - \cdots + \frac{(-1)^s s!}{L_t^{s+1}}\left\{F_{t(x+s)} + \binom{s+1}{1}F_{t(x+s-1)} + \binom{s+1}{2}F_{t(x+s-2)} + \cdots + F_{t(x-1)}\right\}\right]_1^{n+1}$$

(80) $$\sum_{1}^{n} x^{\overline{s}}L_{tx} = \left[\frac{x^{\overline{s}}}{L_t}(L_{tx} + L_{t(x-1)}) - \frac{s(x+1)^{\overline{s-1}}}{L_t^2}(L_{t(x+1)} + 2L_{tx} + L_{t(x-1)} + \frac{s(s-1)(x+2)^{\overline{s-2}}}{L_t^3}(L_{t(x+2)} + 3L_{t(x+1)} + 3L_{tx} + L_{t(x-1)}) + \cdots + \frac{(-1)^s s!}{L_t^{s+1}}\left\{L_{t(x+s)} + \binom{s+1}{1}L_{t(x+s-1)} + \binom{s+1}{2}L_{t(x+s-2)} + \cdots + L_{t(x-1)}\right\}\right]_1^{n+1}$$

*Note 1:* The author has a slightly larger collection of corresponding formulas for

$$\sum_{1}^{2m \text{ or } 2m+1} (-1)^{x+1}x^{\overline{s}}F_{rx}.$$

The union of these formula sets makes possible the formation of identities for $\sum_{1}^{n} P(x)F_{rx}^{m}$.

*Note 2:* The author has a table of Type 1 and Type 2 primitive units for quadratic domains 5 to 9997 and a second table that includes primitive units for quadratic domains from 2 to 9999. Current on file computer programs can extend these tables to 999999. A true tested but unused program can be used for integers with more than six digits.

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