AN INTRODUCTION TO FIBONACCI DISCOVERY

Brother U. Alfred

The Fibonacci Association

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INTRODUCTION

Anybody who claims to put out a book devoted to the discovery of mathematics is faced with a problem. A certain amount of explanation is necessary in order to introduce the topic. In an extreme form, once this explanation is concluded, people could then be encouraged to make discoveries. On this basis, the present book would contain one or two pages. But evidently, for most people this would not be sufficient. A certain amount of suggestion is required: suggestion of problems to be studied; indications of how to attack the problem; and in many cases, the solution of the problem, so that a person will be able to make the next step should an impasse be reached.

Evidently, preparing a book on discovering mathematics is not a simple task. This volume is based on the assumption that a person who uses it wants to discover mathematics; that, therefore, he will make a real effort to find the answer by himself before looking to the solution in the key section of the book.

Hence the following general approach is used.

(1) The topic for research is introduced by explanation. Examples are given when this seems advisable.

(2) There are suggested exercises to familiarize the researcher with the idea that has been explained. For these exercises answers are provided in another part of the book. However, the answers are not in sequence, so that the danger of looking ahead on the answers CAN be avoided.

(3) Specific points for research are then given. Once again, answers are usually provided and even the complete solution in certain instances.

These arrangements allow for independent activity while at the same time obviating the dangers of frustration.

Going through a discovery book in mathematics is not like reading a textbook, much less a novel. Ideas take TIME to develop. It is better to wait and come to an answer on one's own rather than short-circuit the process by looking too readily to the solution or answer in the book. After a while, this process will pay off in a sense of achievement and mastery.

Brother U. Alfred

1. DISCOVERING FIBONACCI FORMULAS

DEFINITION OF THE FIBONACCI SEQUENCE

A sequence is an ordered set of quantities. Thus the following are sequences:

1, 2, 3, 4, 5, 6, 7.......

2, 4, 6, 8, 10, 12,.......

2, 4, 8, 16, 32, 64, 128,.....

In a sequence, there is a first term, a second, a third, etc. and if the sequence is infinite, there is no last term. Essentially, what we are doing is setting the terms of the sequence into one-to-one correspondence with the natural numbers: 1, 2, 3,..

So when we speak of the Fibonacci sequence, we are talking about a certain set of quantities arranged in order. These are identified by the letter capital F with a subscript 1, 2, 3, 4......to indicate which term of the sequence we are talking about. Thus F_{16} would mean the sixteenth Fibonacci number.

We start with $F_1 = 1$, $F_2 = 1$. This is all that is needed along with the LAW OF THE FIBONACCI SEQUENCE, namely, that every term is the sum of the two preceding terms. So

$$F_3 = F_2 + F_1 = 1 + 1 = 2$$

$$F_4 = F_3 + F_2 = 2 + 1 = 3$$

$$F_5 = F_4 + F_3 = 3 + 2 = 5$$

$$F_6 = F_5 + F_4 = 5 + 3 = 8$$

In general, $F_{n+1} =$ ________ $+$ ________ ?

(See answer #1, p. 36))

PROBLEMS

P. 1. Find the first twenty terms of the Fibonacci sequence. (Compare your results with Table 1, p. 52)

Let us write the Fibonacci numbers out less formally: 1, 1, 2, 3, 5, 8, 13, 21,..

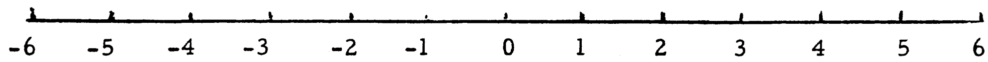
Question: Is it possible to go backward as well as forward in the sequence? If so, how can this be done?

Question: Going backward, we have 34, 21, and then 13. How is the 13 obtained from 34 and 21? Does this rule seem to work for other cases? Could you formulate in words a rule for obtaining the previous Fibonacci number from two successive Fibonacci numbers? (See answer #2, p. 35)

Using F_{n+1} and F_n as two successive Fibonacci numbers, write a formula for F_{n-1}. (See answer # 3, p. 34)

Applying this formula, or using the verbal rule, obtain the twenty Fibonacci numbers that precede 1, 1. (See answer # 4, p. 33)

How should we name these Fibonacci numbers? We continue backward with our subscripts on the number line.



The Fibonacci number before F_1 would thus be F_0 ; the Fibonacci number before F_0 would be F_{-1}; etc.

Give names to the twenty Fibonacci numbers before 1, 1. (See answer #5, p. 42)

QUESTION: Do you notice any relation between the Fibonacci numbers with negative subscripts and those with positive subscripts? For example: How is F_{-4} related to F_4? F_{-7} to F_7? Could you state in words a rule for this relationship? (See answer #6, p. 40). Can you state this rule in one neat formula? (See answer #7, p. 39)

A Fibonacci Formula

Now that the Fibonacci sequence has been defined, we are ready to go to work and find formulas. Let us start with sums. Possibly the most obvious question to ask is: What is the sum of the first ten terms or the first fifty terms of the Fibonacci sequence? Now, we have to know what sort of answer to look for. We are accustomed when adding n terms of a sequence to get a formula which depends entirely on n. Thus the sum of the first n integers: 1, 2, 3, 4, ... , n is:

$$\frac{n(n+1)}{2}$$

But with the Fibonacci numbers we usually have to look for an answer in terms of Fibonacci numbers themselves. One way to proceed is to make a table. In the first column we put k which takes on values 1, 2, 3, 4, . . . numbering thereby the things we are going to add. In the second column in the present case, we put F_k, the kth Fibonacci number; in the third column we put the sum of the first k terms of the Fibonacci sequence. Here is how it looks.

k	F_k	Sum
1	1	1
2	1	2
3	2	4
4	3	7
5	5	12
6	8	20
7	13	33
8	21	54
9	34	88

Looking at the sum column, do you see any relation between the numbers we obtain and the Fibonacci numbers? (For example, what is the sum of the first six Fibonacci numbers in terms of Fibonacci numbers? What is the sum of the first eight Fibonacci numbers in terms of Fibonacci numbers? Does this suggest a rule that can be put into words? (See answer #8, p 37)

DISCOVERING FIBONACCI FORMULAS

Note on Summation Notation

The sum of the first seven Fibonacci numbers can be written:

$$F_1 + F_2 + F_3 + F_4 + F_5 + F_6 + F_7$$

This way of writing the sum is not too inconvenient for a small number like seven. But if we wanted the sum of 100 terms, or if we wanted to show the sum of the first n terms--an indefinite number of terms--we would have to write something like:

$$F_1 + F_2 + F_3 + \cdots\cdots\cdots + F_{100}$$

or $$F_1 + F_2 + F_3 + \cdots\cdots\cdots + F_n$$

where the dots indicate all the terms that we have failed to write down. Still more conveniently we can use the summation notation:

$$\sum_{k=1}^{100} F_k$$

for the sum of the first 100 terms; or $\sum_{k=1}^{n} F_k$ for the sum of the first n terms.

So, in the present investigation where we are trying to find the sum of the first n terms of the Fibonacci sequence, this sum can be expressed:

$$\sum_{k=1}^{n} F_k .$$

What would be the formula for this summation? (See answer #9, p. 38)

Another Summation

Suppose that instead of adding all the Fibonacci numbers, we add all those with odd subscripts. That is, we are taking the sum:

$$F_1 + F_3 + F_5 \cdots\cdots\cdots\cdots$$

What would be the formula for this summation?

At the outset we have a problem of notation. If we number the first column k: 1, 2, 3, 4, ..., what are we going to call the second column? (See answer #10, p. 41)

k	$F_?$	$\sum_{k=1}^{n} F_?$
1	1	1
2	2	3
3	5	8
4	13	21
5	34	55
6	89	144
7	233	377

Write down the formula for the sum of the first n Fibonacci numbers with odd subscripts. (See answer # 11, p. 33)

DISCOVERING FIBONACCI FORMULAS

PROBLEM

P.2. Find the sum of the first n Fibonacci numbers with even subscripts.
(See answer P.2, p. 34)

Other Summations

In addition to finding the Fibonacci numbers directly on the surface, sometimes they appear as part of a product, as squares, etc. For example, what is the sum of the squares of the first n Fibonacci numbers? Again, we form a table.

k	F_k^2	$\sum_{k=1}^{n} F_k^2$
1	1	1
2	1	2
3	4	6
4	9	15
5	25	40
6	64	104
7	169	273

Suppose we try factoring the sums. Do we obtain Fibonacci numbers by some factorization? For example, the sum of the squares of the first six Fibonacci numbers is the product of what Fibonacci numbers? See if this works for other cases. What, then, would be the general formula for:

$$\sum_{k=1}^{n} F_k^2 \quad ?$$

(See answer # 12, p. 39)

PROBLEMS

P.3. Find the sum of $F_1 + F_5 + F_9 + \cdots\cdots$ to n terms. (Answer, p. 37)

P.4. Find the sum of $F_2 + F_6 + F_{10} + \cdots\cdots$ to n terms. (Answer, p. 40)

P.5. Find the sum of $F_3 + F_7 + F_{11} + \cdots\cdots$ to n terms. (Answer, p. 35)

P.6. Find the sum of $F_4 + F_8 + F_{12} + \cdots\cdots$ to n terms. (Answer, p. 41)

OTHER FORMULAS

Thus far we have concentrated on summations. But this is far from being the only type of Fibonacci relation. One very interesting formula is indicated by the following. Take any Fibonacci number; square it; subtract the product of the two numbers on either side. What answer (s) do you get uniformly? If this is true in general, what would be the formula for this relation? (See answer # 13, p. 36)

Possibly this suggests further inquiry along the same line. What would happen if we were to take the square of a Fibonacci number and subtract the product of the Fibonacci numbers two removed on either side? For example:

$$F_{10}^2 - F_8 F_{12} \quad \text{is what?}$$

Try this for a number of cases. What is the general relation? (See answer #14, p. 38).

DISCOVERING FIBONACCI FORMULAS

Pursue this line of development, taking successively numbers three removed, then four removed, etc. Write down general formulas for these cases up to seven removed. (See answer #15, p. 42).

What general formula would this indicate if we were to take the square of any Fibonacci number and subtract the product of numbers k removed from it on either side? (See answer # 16, p. 33).

LUCAS NUMBERS

For the sake of simplicity we began our discovery work by dealing with the sequence: 1, 1, 2, 3, 5, 8, 13, 21, ... known as THE FIBONACCI SEQUENCE. We would now like to point out that a sequence of the Fibonacci type can be developed by starting with any two numbers a and b, adding them to get the next term, etc., using in general the law that each succeeding term is the sum of the two previous terms.

EXAMPLE. Starting with 3, 7, the next term is 10; then 17; then 27; then 44; etc.

PROBLEMS

P. 7. Find the first ten terms of the Fibonacci sequences beginning with:
(a) 3 and 12; (b) -7 and 4; (c) 6 and - 13.
(See answer to P. 7., p. 38)

In addition to the very special Fibonacci sequence, there is a closely related sequence of the Fibonacci type known as the LUCAS SEQUENCE. We shall indicate its terms by the capital letter L with subscripts. For this Lucas sequence,

$$L_1 = 1, \qquad L_2 = 3$$

P. 8. Find the first twenty terms of the Lucas sequence. (See table 2, p. 54)

P. 9. Find the twenty terms before L_1 in the Lucas sequence. These would be indicated by L_0, L_{-1}, etc. (See P. 9, p. 36)

P. 10. What is the relation between L_{-k} and L_k? (See P. 10, p. 39)

With the Lucas sequence evidently, we have some brand new territory for exploring all the various relations we considered in the Fibonacci sequence. A word of caution may be in order. We must not expect that when there is a simple formula in the Fibonacci sequence we will find a like situation in the Lucas sequence.

PROBLEMS

P. 11. Find the formula for the sum of the first n terms of the Lucas sequence. (See answer P. 11, p. 42)

P. 12. Find the formula for the sum of the first n Lucas numbers with odd subscripts. (See answer P. 12, p. 37)

P. 13. Determine the formula for the sum of the first n Lucas numbers with even subscripts. (See answer P. 13, p. 35)

P. 14. Find the formula for

$$\sum_{k=1}^{n} L_{4k}$$

(Answers may come out in Fibonacci numbers as well as Lucas numbers. See P. 14, p. 40)

DISCOVERING FIBONACCI FORMULAS

P. 15. Determine the formula for $\sum_{k=1}^{n} L_{4k-3}$

(Answer P. 15, p. 41)

P. 16. Determine the formula for $\sum_{k=1}^{n} L_{4k-2}$

(Answer P. 16, p. 36)

P. 17. Determine the formula for $\sum_{k=1}^{n} L_{4k-1}$

(Answer P. 17, p. 39)

P. 18. Find the formula for $L_n^2 - L_{n-1}L_{n+1}$. (Answer P. 18, p. 42)

P. 19. Determine the formula for $L_n^2 - L_{n-2}L_{n+2}$. (Answer P. 19, p. 37)

P. 20. Find the formula for $L_n^2 - L_{n-3}L_{n+3}$. (Answer P. 20, p. 33)

P. 21. Determine the formula for $L_n^2 - L_{n-4}L_{n+4}$. (Answer P. 21, p. 38)

P. 22. Determine the formula for $L_n^2 - L_{n-5}L_{n+5}$. (Answer P. 22, p. 35)

P. 23. Find the formula for $L_n^2 - L_{n-k}L_{n+k}$. (Answer P. 23, p. 40)

RELATION OF FIBONACCI AND LUCAS NUMBERS

Let us make a table of Fibonacci and Lucas numbers placing those with the same subscript next to each other in the same line.

k	F_k	L_k
1	1	1
2	1	3
3	2	4
4	3	7
5	5	11
6	8	18
7	13	29
8	21	47
9	34	76

The seventh Lucas number is 29. The sum of the 6th and 8th Fibonacci numbers is 8 + 21 or 29. Does this seem to hold in general? If so, what relation is there between Lucas numbers and Fibonacci numbers? Expressed by formula:

$L_n =$? in terms of Fibonacci numbers.

(See answer #17, p. 34)

PROBLEMS.

P. 24. If we add the 6th and 8th Lucas numbers does this sum have any relation to the seventh Fibonacci number? Examine this for different cases. Do you arrive at a law? (See P.24, p. 41)

P.25. Take the product of two corresponding Fibonacci and Lucas numbers, such as F_4 and L_4. What do you get? Try this for different cases. What formula does this suggest? (See P.25, p. 42)

P.26. Take the sum of the squares of two consecutive terms of the Fibonacci sequence. Is there a formula that seems to fit this case? (See P.26, p. 41)

P.27. Try the same thing for the Lucas sequence. Do you find a formula? (See P.27, p. 34)

P 28. Take the product of two successive terms of the Fibonacci sequence. Compare this with the product of the terms on either side of these two terms. (For example, compare the product of F_5F_6 with the product F_4F_7). D you find a law? (See P.28, p. 40)

P.29. Do likewise for the Lucas sequence. (See P.29, p. 37)

P.30. Is there an answer for the following difference:

$$F_nL_{n+1} - F_{n+1}L_n\ ?$$ (See P.30, p. 33)

P.31. Likewise for $F_nL_{n+2} - F_{n+2}L_n$. (See P.31, p.38)

P.32. Again $F_nL_{n+3} - F_{n+3}L_n$ is what? (See P.32, p. 35)

P.33. What is $F_nL_{n+4} - F_{n+4}L_n$? (See P.33, p. 36)

P.34. Find a formula for $F_nL_{n+5} - F_{n+5}L_n$. (See P.34, p. 39)

P.35 To what general formula $F_nL_{n+k} - m\,F_{n+k}L_n$ does this lead? (See P.35, p. 37)

RESEARCH

Possibly the preceding work with Fibonacci and Lucas numbers has given you some leads. But there are many, many more possibilities for formulas. Just try anything!! Start making a collection of formulas for yourself. See how many you can find. It's an interesting game.

PROOF OF FORMULAS BY MATHEMATICAL INDUCTION

To this point we have been engaged in a freewheeling operation of arriving at plausible results without any attempt to demonstrate by proof that they are logically sound. For it is quite clear that no matter how many individual cases show that our formula is correct, this does not enable us to conclude that it holds in general for all similar cases. Psychologically, of course, we might be willing to bet one hundred to one that our result is true. But no matter how strong the probability is in favor of our formula, there is still a need for arriving at certainty.

Proof will put the results we have found on a solid basis. Our house of mathematics will then have a good foundation and we can proceed with confidence to draw other deductions from those which have already been shown to be true.

There is another value of proof which is not emphasized enough. The effort to show that our results are correct can oftentimes lead to the need of developing other formulas. In a word, PROOF IS A MAJOR ROAD TO DISCOVERY!

PROOF BY MATHEMATICAL INDUCTION

For the moment we are going to consider mathematical induction as a mode of proof while noting that there are other means of arriving at conclusions. For example, once we have one or more formulas established, we can then use them as means of deriving additional formulas.

We shall first illustrate the process of mathematical induction for a non-Fibonacci situation. Suppose we wish to know the sum of the first n integers:

$$1 + 2 + 3 + \dots + n = \sum_{k=1}^{n} k.$$

We have already mentioned a formula: $\dfrac{n(n+1)}{2}$

When n is 1, the value of $n(n+1)/2$ is 1; when n is 2, its value is 3; when n is 3, its value is 6. These results agree with the first three cases of adding n integers, when n is 1, 2, or 3 respectively.

How are we going to show that the formula is true for any value of n? The next step is to prove that <u>if</u> the formula is true up to a given integer n, then it follows <u>as a consequence</u> that it is true for $n+1$.

Given: $\sum_{k=1}^{n} k = n(n+1)/2$ for all integers $\le n$ (assumption)

To prove: $\sum_{k=1}^{n+1} k = (n+1)(n+2)/2.$

This latter relation is obtained by replacing n by $n+1$ throughout, both on the lefthand and righthand sides.

Proof. $\sum_{k=1}^{n} k = n(n+1)/2$ (assumed)

$n+1 = n+1$ (identity)

Add: $\sum_{k=1}^{n+1} k = n(n+1)/2 + n + 1 = (n+1)(n/2+1)$

$= (n+1)(n+2)/2$

Thus if the formula is true for n, it is true for $n+1$.

Now we go back to our originally verified cases. We have found that the formula holds for 1, 2, and 3. By what we have just shown, if it is true for 3, it is true for 4; if it is true for 4, it is true for 5; and so on. It will be true for all positive integers by reason of the property of the positive integers that to each such integer there is always a next found by adding 1. We can arrive at any integer no matter how large by proceeding step by step. And since the logic of mathematical induction applies for each step, it will apply for any number we may choose to select.

Let us apply this method of proof to some Fibonacci formulas. The formula discovered for the sum of the first n terms of the Fibonacci sequence is:

$$\sum_{k=1}^{n} F_k = F_{n+2} - 1.$$

For $n=1$, the sum is 1, and $F_{1+2} - 1 = 2 - 1$ or 1. This checks.

PROOF BY MATHEMATICAL INDUCTION

For $n=2$, the sum is 2, and $F_{2+2} - 1 = 3 - 1$ or 2.

For $n=3$, the sum is 4, and $F_{3+2} - 1 = 5 - 1$ or 4.

Assume, then, that the formula holds for the first n Fibonacci numbers, i.e.,

$$\sum_{k=1}^{n} F_k = F_{n+2} - 1 \quad \text{for some } n.$$

Add the next number: $F_{n+1} = F_{n+1}$

Then $$\sum_{k=1}^{n+1} F_k = F_{n+3} - 1, \text{ since } F_{n+1} + F_{n+2} = F_{n+3}.$$

This shows that if the formula is true for n, it is true for n + 1. We can then go back to our specific cases and from there proceed to any value of n. Hence the result is proved in general.

THE FORMULA $\qquad F_n^2 - F_{n-1}F_{n+1} = (-1)^{n-1}$

For $n=1$, $\qquad 1^2 - 0 \cdot 1 = 1 = (-1)^0$

For $n=2$, $\qquad 1^2 - 1 \cdot 2 = -1 = (-1)^1$

For $n=3$, $\qquad 2^2 - 1 \cdot 3 = 1 = (-1)^2$

Assume that $F_n^2 - F_{n-1}F_{n+1} = (-1)^{n-1}$ for a given n.

We want to show that as a result of this assumption

$$F_{n+1}^2 - F_n F_{n+2} = (-1)^n .$$

We have on substituting $F_{n+2} = F_{n+1} + F_n$,

$$F_{n+1}^2 - F_n F_{n+2} = F_{n+1}^2 - F_n(F_{n+1} + F_n)$$

$$= F_{n+1}^2 - F_n F_{n+1} - F_n^2 = F_{n+1}(F_{n+1} - F_n) - F_n^2$$

$$= F_{n+1}F_{n-1} - F_n^2 = -(F_n^2 - F_{n+1}F_{n-1}) = -(-1)^{n-1} = (-1)^n .$$

We can now return to our original verifications and proceed from there to any value of n. Thus the formula is true in general.

PROBLEMS

P. 36. Prove by mathematical induction the formula for the sum of the first n terms of the Fibonacci sequence with odd subscripts. (See solution, p. 38)

P. 37. Prove by mathematical induction the formula for the sum of the first n terms of the Fibonacci sequence with even subscripts. (See solution, p. 35)

P.38. Prove by mathematical induction that the formula

$$F_n^2 - F_{n-2}F_{n+2} = (-1)^n$$

holds for all values of n. (See solution, p. 33)

P.39. Prove that

$$\sum_{k=1}^{n} L_k = L_{n+2} - 3 \qquad \text{(No solution provided)}$$

P.40. Prove that

$$\sum_{k=1}^{n} L_{2k-1} = L_{2n} - 2 \qquad \text{(No solution provided)}$$

P.41. Prove that

$$\sum_{k=1}^{n} L_{2k} = L_{2n+1} - 1 \qquad \text{(No solution provided)}$$

P.42. Prove by mathematical induction that $L_n = F_{n-1} + F_{n+1}$.
(See solution, p. 36)

The Summation

$$\sum_{k=1}^{n} L_{4k} = F_{4n+2} - 1$$

For n = 1, we have one term, $L_4 = 7$. Also $F_6 - 1 = 8 - 1 = 7$. Checks.

Assume that $\sum_{k=1}^{n} L_{4k} = F_{4n+2} - 1$ for some given n.

Add $L_{4n+4} = L_{4n+4}$

Then $\sum_{k=1}^{n+1} L_{4k} = F_{4n+2} + L_{4n+4} - 1.$

Clearly, to combine the terms on the right we have to make a transformation. Using $L_{4n+4} = F_{4n+3} + F_{4n+5}$, we have

$$\sum_{k=1}^{n+1} L_{4k} = F_{4n+2} + F_{4n+3} + F_{4n+5} - 1 = F_{4n+4} + F_{4n+5} - 1$$
$$= F_{4n+6} - 1.$$

Completing the mathematical induction, the formula is seen to hold in general.

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P.43. Prove by mathematical induction that

$$\sum_{k=1}^{n} L_{4k-3} = F_{4n-1} - 1.$$

(See solution, p. 41)

P.44. Prove by mathematical induction that

$$\sum_{k=1}^{n} L_{4k-2} = F_{4n}$$

(No solution provided)

P.45. Prove by mathematical induction that

$$\sum_{k=1}^{n} L_{4k-1} = F_{4n+1} - 1.$$

(No solution provided)

This concludes the INTRODUCTION TO MATHEMATICAL INDUCTION. We shall come back to this again after having developed some additional approaches to our subject.

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THE FIBONACCI SHIFT FORMULAS

One of the very convenient properties of Fibonacci numbers is that no matter what terms of the sequence we have combined by addition and subtraction, we can always find an equivalent sum in terms of two successive Fibonacci numbers ANYWHERE IN THE SEQUENCE.

For example, given a combination

$$3F_{n+1} - 5F_{n-3} + 7F_{n+4} + 10F_{n-2}$$

there is an equivalent sum in terms of F_n and F_{n+1} alone. This would be

$$4F_{n+1} + 49F_n .$$

To arrive at the necessary formulas for making this type of transformation we start with our fundamental relation

$$F_n = F_{n-1} + F_{n-2} .$$

Now replace F_{n-1} by $F_{n-2} + F_{n-3}$. We then have

$$F_n = 2F_{n-2} + F_{n-3}$$

Replace F_{n-2} by $F_{n-3} + F_{n-4}$. This gives $F_n = 3F_{n-3} + 2F_{n-4}$.

Continuing this operation of replacement, we find successive formulas:

$$F_n = 5F_{n-4} + 3F_{n-5}$$

$$F_n = 8F_{n-5} + 5F_{n-6}$$

$$F_n = 13F_{n-6} + 8F_{n-7}$$

P.46. What would be the formula for

$$F_n = \_\_\_\_\_\_ F_{n-k} \quad \_\_\_\_\_\_ F_{n-k-1}?$$

(See answer p. 39)

Clearly, this formula, if it holds in general, enables us to shift k steps down (considering the first F_{n-k} on the right).

P.47. Prove the formula found in P.46 by mathematical induction. (Note. There are two letters involved, n and k. Our concern is with what happens when we vary k, so that we go from k to k+1 in our induction.)
(See solution p. 34)

THE FIBONACCI SHIFT FORMULAS

Shifting Upward

We start with the relation $F_{n+2} = F_{n+1} + F_n$

from which $F_n = F_{n+2} - F_{n+1}$.

Then replace F_{n+1} by $F_{n+3} - F_{n+2}$ to obtain

$$F_n = - F_{n+3} + 2 F_{n+2}.$$

Continue this process. $F_n = 2 F_{n+4} - 3 F_{n+3}$

$$F_n = - 3 F_{n+5} + 5 F_{n+4}$$

$$F_n = 5 F_{n+6} - 8 F_{n+5}$$

P.48. What is the formula for

$$F_n = (-1)^{?} \left[\_\_\_\_\_ F_{n+k+1} - \_\_\_\_\_ F_{n+k} \right]$$

(See answer p. 42)

P.49. Prove the formula in P. 48 by mathematical induction. (See solution p. 40)

P.50. Prove that $F_{2n} = F_n L_n$ by letting $k = n$ in the shift-down formula.

P.51. Prove that $F_{2n+1} = F_n^2 + F_{n+1}^2$ by letting $k=n$ in the shift-down formula.

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Shifting in the Lucas Sequence

P.52. Find the shift-down formula in the Lucas sequence. (See solution, p. 33)

P. 53. Find the shift-up formula in the Lucas sequence. (See solution, p. 36)

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EXPLICIT FORMULAS FOR THE FIBONACCI AND LUCAS SEQUENCES

We have been operating thus far with Fibonacci and Lucas numbers by means of the relation:

$$F_{n+1} = F_n + F_{n-1}$$

and the corresponding relation of the Lucas sequence. This is a recursion formula, since it indicates a relation that recurs over and over again at each point of the sequences. However, while we are able to arrive at any Fibonacci or Lucas number we care to name at least in principle using this formula, we are not able to set down one single formula which gives us all Fibonacci or Lucas numbers by substituting for n. This we now propose to investigate.

The key to achieving this result is found in the equation:

$$x^2 - x - 1 = 0.$$

Why this particular equation is selected will become evident in what follows. We first find the roots of this equation. These we shall designate r and s:

EXPLICIT FORMULAS

$$r = \frac{1+\sqrt{5}}{2} \quad ; \qquad s = \frac{1-\sqrt{5}}{2}$$

Note that $r+s=1$, the negative of the coefficient of x and rs equals -1, the constant term in the equation.

We develop the Fibonacci numbers in terms of these roots.

We have $$r-s = \frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2} = \sqrt{5}$$

This quantity $r-s$ or $\sqrt{5}$ will appear in the denominator of the expressions for the Fibonacci numbers.

Clearly, $$F_1 = \frac{r-s}{r-s} = 1;$$

$$F_2 = \frac{r^2-s^2}{r-s} = r+s = 1.$$

We now proceed by mathematical induction. Assume that

$$F_n = \frac{r^n - s^n}{r-s} \quad \text{up to a given } n.$$

Before proceeding, let us look to our original equation and draw some conclusions regarding our roots.

$$x^2 = x+1$$

and since the roots r and s satisfy this equation we must have:

$$r^2 = r+1 \quad \text{and} \quad s^2 = s+1.$$

But we can multiply through these equations by any power of r or s, so that, for example,

$$r^{n+1} = r^n + r^{n-1} \quad \text{and} \quad s^{n+1} = s^n + s^{n-1}.$$

Since by assumption, $$F_{n-1} = \frac{r^{n-1} - s^{n-1}}{r-s}$$

and $$F_n = \frac{r^n - s^n}{r-s}$$

it follows by addition that $$F_{n+1} = \frac{r^{n+1} - s^{n+1}}{r-s}$$

using the property of the r's and s's that we have just indicated. Thus the formula is seen to hold by mathematical induction.

P. 54. Prove by mathematical induction that $L_n = r^n + s^n$.
(See solution, p. 40)

EXPLICIT FORMULAS

AN EASY APPROACH TO FORMULA DEVELOPMENT

This formulation of the Fibonacci and Lucas numbers provides an easy means of proving many formulas and deriving others. For example, since

$$F_{2n} = \frac{r^{2n} - s^{2n}}{r-s} = \frac{(r^n - s^n)}{r-s}(r^n + s^n)$$

it is clear that $F_{2n} = F_n L_n$.

Again, suppose we want to prove the formula

$$F_n^2 - F_{n-k} F_{n+k} = (-1)^{n+k} F_k^2$$

We can proceed to show that both sides are equivalent in terms of r and s.

$$F_n^2 - F_{n-k} F_{n+k} = \frac{(r^n - s^n)^2}{5} - \frac{(r^{n-k} - s^{n-k})(r^{n+k} - s^{n+k})}{5}$$

$$= \frac{r^{2n} - 2r^n s^n + s^{2n}}{5} - \frac{r^{2n} - r^{n-k} s^{n+k} - r^{n+k} s^{n-k} + s^{2n}}{5}$$

As noted before, $rs = -1$. Hence the above can be written

$$\frac{-2(-1)^n + r^{n-k} s^{n-k}(r^{2k} + s^{2k})}{5} = \frac{-2(-1)^n + (-1)^{n-k}(r^{2k} + s^{2k})}{5}$$

$$= (-1)^{n-k}\left[\frac{r^{2k} - 2(-1)^k + s^{2k}}{5}\right]$$

Now the righthand side expands into:

$$(-1)^{n+k}\left[\frac{r^{2k} - 2r^k s^k + s^{2k}}{5}\right]$$

which is an equivalent expression if we remember that n-k and n+k will be odd and even together and that

$$r^k s^k = (-1)^k$$

PROBLEMS

P.55. Prove that $F_{3k} = F_k \left[L_{2k} + (-1)^k \right]$.

(See solution, p. 41)

P.56. Derive a formula for $F_n L_{n+k} + F_{n+k} L_n$.

(See solution, p. 34)

P.57. Prove that $L_{n+1}^2 + L_n^2 = 5(F_{n+1}^2 + F_n^2)$.

(See solution, p. 38)

P.58. Prove that $F_{5k} = F_k \left[L_{4k} + (-1)^k L_{2k} + 1 \right]$

(No solution provided)

EXPLICIT FORMULAS

P. 59. Find a similar formula for F_{7k}. (See p. 35)

P. 60. Prove that

$$L_{3k} = L_k \left[L_{2k} + (-1)^{k-1} \right]$$

P. 61. Prove that

$$L_{5k} = L_k \left[L_{4k} + (-1)^{k-1} L_{2k} + 1 \right]$$

P. 62. Find a similar formula for L_{7k}. (See p. 39)

P. 63. Derive a formula for $F_n L_{n+k} - F_{n+k} L_n$.
(See solution, p. 37)

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DIVISION PROPERTIES OF FIBONACCI NUMBERS

Some of the most fascinating properties of Fibonacci numbers pertain to their factors and their relation to divisors. It is not difficult to discover regularity in this respect. For example, looking over the table of Fibonacci numbers, it is evident that F_5, F_{10}, F_{15}, F_{20}, etc. are all divisible by 5.

P. 64. What Fibonacci numbers seem to be divisible by:
(a) 2; (b) 3; (c) 7; (d) 13; (e) 4. ?
(See solution, p. 36)

More generally one can examine the remainders that result when the Fibonacci numbers are divided by some given quantity. For example, for the divisor 13, the remainders in order are:

1, 1, 2, 3, 5, 8, 0, 8, 8, 3, 11, 1, 12, 0, 12, 12, 11, 10, 8, 5, 0, 5, 5, 10, 2, 12, 1, 0, 1, 1, 2, 3, 5, 8, 0....

We note that after 28 members of the sequence, we go back to the original set of remainders. Thus we see that for divisor 13, the set of remainders of the Fibonacci sequence is periodic with period 28.

CONGRUENCE APPROACH

If we had to divide the Fibonacci numbers by the divisor in each instance the amount of work involved in studying these remainders and their periodicity would be very great. Fortunately, we can work directly with the remainders themselves. A brief explanation of congruences will make this clear.

In number theory, the divisor about which we have been speaking is called a modulus. We say that two numbers a and b are congruent to each other modulo m if their difference a-b is divisible by m. This is expressed in the following notation (triple equality sign):

$$a \equiv b \pmod{m}$$

Now when a is divided by m we get a quotient q and a remainder r which satisfy the relation

$$a/m = q + r/m$$

On multiplying by m, this becomes

$$a = qm + r$$

Similarly $$b = pm + s$$

If $$a-b = (q-p)m + r - s$$

is divisible by m, it follows that r-s is divisible by m. But since both r and s are less than m, this means that they must be the same. Thus another way of saying that two quantities a and b are congruent modulo m is to state that on being divided by m, they give the same remainder, r, where

$$0 \leq r < m$$

Now suppose we have two numbers

$$c = q_1 m + r_1$$

$$d = q_2 m + r_2$$

If we add the numbers, $$c + d = (q_1 + q_2)m + r_1 + r_2 .$$

Hence, we can find the remainder for the sum of c and d simply by considering the remainders without using c and d at all. This is what we do in working with the Fibonacci sequence.

Let us take an example. Suppose our modulus (divisor) is 17. We start off with the terms of the sequence less than 17: 1, 1, 2, 3, 5, 8, 13. When we come to 21, we subtract out 17, so that our next term is 4. We add the 4 to 13 to obtain 17; subtracting out 17 gives our next remainder 0. We add 0 to 4 to obtain 4; 4 to 0 to obtain 4; 4 to 4 to obtain 8; 8 to 4 to obtain 12; 12 to 8 to obtain 20; subtracting out 17 gives 3; etc. This avoids all the division work into the large Fibonacci numbers. The final sequence would be:

1, 1, 2, 3, 5, 8, 13, 4, 0, 4, 4, 8, 12, 3, 15, 1, 16, 0, 16, 16, 15, 14, 12, 9, 4, 13, 0, 13, 13, 9, 5, 14, 2, 16,

1, 0, 1, 1, 2, 3, 5, 8, ...

PROBLEMS

P.65. Find the set of residues of the Fibonacci sequence modulo (a) 11; (b) 7; (c) 5; (d) 3; (e) 2. (See solution, p. 40)

P.66. What are the periods in the various cases in P.65? (See solution, p. 33)

P.67. Suppose we take the product of two primes, such as 3 and 5. Find the period for modulus 15.

P.68. Find the period for modulus 21.

P.69. Determine the period for modulus 35.

P.70. Determine the period for modulus 77.

P.71. Study this problem: How the period of 15 is related to the periods of 3 and 5; how the period of 21 is related to the periods of 3 and 7; how the period of 35 is related to the periods of 5 and 7; how the period of 77 is related to the periods of 7 and 11. (See solution, p. 39)

P. 72. On the basis of your work in P. 71, given that the period of 23 is 48 and the period of 11 is 10, what would be the period of 253? (See solution, p. 35

P. 73. Given that the period of 2 is 3, the period of 3 is 8, the period of 5 is 20, what would be the period of 30? (See solution, p. 37)

P. 74. Find the periods of $2, 2^2, 2^3, 2^4, \ldots$. How does this seem to work? (See solution, p. 41)

P. 75. Find the periods of $3, 3^2, 3^3, 3^4, \ldots$. Does there seem to be a law? (p. 43)

P. 76. Find the period of 7^2 and compare it to the period of 7. (Solution, p. 44)

PERIODICITY FOR ALL DIVISORS?

We have found periodicity in the Fibonacci sequence for the remainders obtained with both prime and composite divisors of small numbers. Can we be certain that such periodicity exists for all divisors, no matter how large?

To fix our ideas, let us take some particular divisor, such as 97. The remainders we may obtain by dividing by 97 are 0, 1, 2, 3, ..., 96. There are 97 of them in all. Now if two of these remainders are taken and made the start of a Fibonacci sequence of remainders, modulo 97, we evidently get a definite sequence of quantities from them. Otherwise stated, two remainders in a given order determine a sequence of remainders.

There being 97 remainders in all we can have $97 \cdot 97$ or 97^2 pairs in succession. We can also eliminate the pair 0, 0, since this would give an endless set of zeros which is not the case in the Fibonacci sequence we are dealing with. Thus we have 9408 possible pairs at most.

Now when we continue setting down the remainders in our sequence we eventually get to 1,000, then 10,000, then 100,000, etc. terms. Hence, sooner or later we must arrive at a pair in sequence that we had before. Once this takes place periodicity has set in because what happened before will now repeat itself.

This argument which we have applied to the divisor 97 is equally true for any divisor. Hence for any modulus, the set of remainders of the Fibonacci sequence is periodic.

ARE ALL NUMBERS DIVISORS OF THE FIBONACCI SEQUENCE?

Does every integer divide some term of the Fibonacci sequence? Perhaps it may seem trivial to note that every integer divides $F_0 = 0$ in the Fibonacci sequence. But then as a consequence of periodicity, it must divide other terms of the sequence as well.

ENTRY POINT AND PERIOD

If we start with $F_1, F_2, F_3, \ldots$, and the first Fibonacci number divisible by a prime modulus p is F_d, then d is said to be the entry point of the prime p. It must have been noted that the entry point and the period are not the same thing in all cases. There are three situations which can be illustrated by the following.

DIVISION PROPERTIES OF FIBONACCI NUMBERS

(1) Entry point and period are the same.
Example. Modulus 11. 1, 1, 2, 3, 5, 8, 2, 10, 1, 0.

(2) Entry point is half the period
Example. Modulus 7. 1, 1, 2, 3, 5, 1, 6, 0, 6, 6, 5, 4, 2, 6, 1, 0.

(3) Entry point is one-fourth the period.
Example. Modulus 13. 1, 1, 2, 3, 5, 8, 0, 8, 8, 3, 11, 1, 12, 0, 12, 12, 11, 10, 8, 5, 0, 5, 5, 10, 2, 12, 1, 0.

P. 77. Study the entry points and periods as found in Table 3. Can you determine what characterizes each of the three cases mentioned above? (Table 3 is found on p. 55). (See solution, p. 45)

P. 78. There is no rule for finding the entry point for a given prime modulus. Once this is known, however, would it be possible to determine the period? (See solution, p. 46)

P. 79. The periods of 2 and 5 are special. Apart from them, all prime numbers end in 1, 3, 7, or 9. Examine the periods (Table 3) of primes and see whether you can find any law regarding the periods. (Hint. Is there some quantity into which the period divides?) (See solution, p. 47)

DIVISION PROPERTIES OF THE LUCAS NUMBERS

By the same reasoning as was used in the case of Fibonacci numbers, there must be periodicity for any divisor in the Lucas numbers.

P. 80. Are there integers which do not divide any member of the Lucas sequence? Try: 5, 7, 11, 13, ... (See solution, p. 45)

P. 81. The Lucas sequence has terms divisible by 3 and 7. How about 21? Similarly, 3 and 4 divide terms of the sequence, but does 12? Likewise 7 and 4, but does 28? When does a situation of this type arise? (See solution, p. 44)

COMMON FACTORS

If two numbers a and b have a divisor in common, we say it is a common divisor. If we find the largest number that divides both a and b, this is spoken of as the greatest common divisor (abbreviated, g.c.d.). If g is the greatest common divisor of a and b, this is written:

$$(a, b) = g$$

Thus $(6, 15) = 3; \qquad (12, 20) = 4.$

If g is 1, then a and b have no common divisor greater than 1. In this case we say that they are relatively prime.

Now if we take two numbers that have a common divisor such as 6 and 9 and add them together we get another number with this same common divisor. This is provable in general, for if $a = ga'$ and $b = gb'$, then $a + b = g(a' + b')$.

Now, in the Fibonacci sequence: 1, 1, 2, 3, 5, 8, 13, ..., we see that successive terms do not have a common divisor. Will this be true throughout the sequence?

DIVISION PROPERTIES OF FIBONACCI NUMBERS

Yes, it will. For if it were not, then by working backward or forward, it would have to follow that the terms throughout the sequence would have to possess this common divisor greater than 1--which is contrary to the facts at our portion of the sequence. Hence, any two consecutive terms of the Fibonacci sequence are relatively prime; similarly for the Lucas sequence.

P.82. Prove that the terms F_n and F_{n+2} are relatively prime. (See solution, p. 43)

P.83. Prove that F_n and L_n may not have a common factor other than 2. (See solution, p. 35)

FIBONACCI NUMBERS AS DIVISORS

A remarkable property of the Fibonacci sequence is that smaller Fibonacci numbers divide larger members of the sequence. For example, $F_5 = 5$, divides $F_{10} = 55$; $F_8 = 21$, divides $F_{24} = 46368$.

P.84. When does a smaller Fibonacci number divide a larger Fibonacci number? (See solution, p. 39)

P.85. What is the periodicity of the last digit of the Fibonacci numbers? of the last two digits? Of the last three digits? (See solution, p.36)

P.86. Prove that L_{2n} is not divisible by L_n. (See solution, p. 37)

P.87. If the Fibonacci numbers are expressed in base 7, what is the periodicity of the last digit? Of the last two digits? Of the last three digits? (See solution, p. 33)

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GENERAL FIBONACCI SEQUENCES

It was indicated earlier that besides the special Fibonacci and Lucas sequences we have been considering, there are many other sequences of the Fibonacci type in which each term is the sum of the two preceding terms. Actually, it is possible to start with any two integers and produce such a sequence. However, if the two integers have a common factor, all the other terms of the sequence will have this same factor. For example, if we start with 3, 12, the succeeding terms are 15, 27, 42, 69, 111, etc. If the common factor 3 is removed, the sequence that remains is 1, 4, 5, 9, 14, 23, 37, ... To avoid the duplication involved, let us limit our considerations to sequences in which no two successive terms have the same common factor greater than 1.

P.88. Experiment with various sequences developing them forward and backward. For example: The sequences determined by (1, 4); (-23, 12); (-9, -14); (4, -27). Do you arrive at any conclusion regarding the general appearance of such sequences? (See solution, p. 47)

SEQUENCES WITH POSITIVE TERMS (to the right)

Suppose we start with two large positive numbers, such as, 973 and 1458. Evidently if we add them to get the next term to the right, we obtain a still larger positive number; and this will continue. But if we go to the left in the sequence, we obtain a smaller positive quantity 1458-973 or 485; the next term is 973-485 or 488; the next term is 485-488 or -3; the next is 488 - (-3) or 491; the next is -3 -491 or -494. Clearly we have arrived at the alternating part of the sequence.

When we go backward in the positive portion of the sequence, the numbers keep getting smaller; evidently this cannot go on indefinitely. Once we obtain a positive integer that is greater than the one following it, we arrive at a negative term on the next step backward.

This provides us with a convenient point of reference in a sequence with positive terms to the right. THERE IS JUST ONE POSITIVE TERM WHICH IS LESS THAN HALF THE TERM FOLLOWING IT. (Note. The Fibonacci sequence 0, 1, 1, 2, 3, ... is an exception inasmuch as 0 is not a positive term, though it is less than half the term that follows it.) If a is a term of the sequence such that $2a < b$, where b is the term (positive) following a, we shall speak of the sequence as (a, b). Thus the above sequence would be indicated by (485, 973).

P. 89. On the basis of the convention just adopted, what would be the standard method of speaking of the sequences determined by the following pairs of numbers?
(a) 84, 111; (b) -872, 743; (c) 137, 199. (See solution, p. 46)

GENERAL SEQUENCE RELATED TO THE FIBONACCI SEQUENCE

Let us call the terms of our general Fibonacci sequence $T_1, T_2, T_3, \ldots$

Let
$$T_1 = a$$
$$T_2 = b, \quad \text{where } 2a < b$$

Then
$$T_3 = a + b$$
$$T_4 = a + 2b$$
$$T_5 = 2a + 3b$$
$$T_6 = 3a + 5b$$

P. 89A. What is the general term in the above sequence, T_n in terms of a, b, and Fibonacci numbers? (See solution, p. 45)

CHARACTERISTIC NUMBER OF A FIBONACCI SEQUENCE

In studying the Fibonacci and Lucas sequences, we found the following relations:

$$F_n^2 - F_{n-1} F_{n+1} = (-1)^{n-1}$$
$$L_n^2 - L_{n-1} L_{n+1} = (-1)^n \, 5$$

In other words, the number 1 characterizes the Fibonacci sequence according to this type of relation and the number 5 characterizes the Lucas sequence. Does this hold true for other Fibonacci sequences?

GENERAL FIBONACCI SEQUENCES

Suppose we look at a sequence with $T_1 = 1$, $T_2 = 4$. The terms in succession are: 1, 4, 5, 9, 14, 23, 37, 60, 97, 157, 254.....

$$5^2 - 4 \cdot 9 = -11$$

$$9^2 - 5 \cdot 14 = 11$$

$$157^2 - 97 \cdot 254 = 24649 - 24638 = 11$$

It appears that 11 characterizes the Fibonacci sequence (1, 4).

P. 90. By trying several cases as was done above, determine whether the following Fibonacci sequences seem to have characteristic numbers associated with them.
(a) (5, 11); (b) (3, 8); (c) (2, 9).

P. 91. Given that $$T_n^2 - T_{n-1} T_{n+1} = (-1)^n D$$

prove that $$T_{n+1}^2 - T_n T_{n+2} = (-1)^{n+1} D$$

(See solution, p. 47)

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In studying the division properties of the Fibonacci sequence, it was shown that every integer divides an infinity of terms of that sequence; whereas with the Lucas sequence, there were many quantities which did not divide the sequence at all. In particular, 5, the characteristic number of the sequence was not a divisor. Can it be shown that other Fibonacci sequences do not have every integer dividing one or other of their terms?

Using the relation $$T_n^2 - T_{n-1} T_{n+1} = (-1)^n D$$

with $D > 1$, it can be shown that D may not be a factor. For suppose D divides some term of the sequence. By choosing n properly, this could be T_{n-1} in our formula. Then D divides the right-hand side of our relation, it divides the second term on the left-hand side since it divides T_{n-1} but it cannot divide the square of T_n since T_n and T_{n-1} cannot have a factor in common (on the assumption that we started our sequence with two relatively prime numbers as was agreed above). Therefore D may not be a factor of any term of the sequence.

For example, in the sequence (1, 4), let us take the terms modulo 11. We obtain: 1, 4, 5, 9, 3, 1, 4, 5, 9...... Periodicity has set in; there is no zero. Hence 11 is not a factor of any term of the sequence.

P. 92. Verify that the D for each of the sequences in P. 90 does not divide any term of the sequence by establishing that periodicity results before a zero is obtained in the list of remainders.

INVENTORY OF FIBONACCI SEQUENCES FOR A GIVEN MODULUS

In establishing the fact that Fibonacci sequences were periodic we reasoned as follows. For a given modulus, such as 13, there can only be 13 possible remainders: 0, 1, 2, 3, ..., 11, 12. Two consecutive remainders determine what happens in the rest of the sequence. Since thirteen quantities can be paired among themselves in 13 times 13 ways, there can only be 169 possible pairs in sequence. Eliminating 0,0 as a case since it leads to a sequence with all zero remainders, we would then have remaining 168 possible pairs in sequence.

Now, it is not difficult to set down all possible cases that can arise for any given modulus such as 13. Start with any pair of remainders.

0, 1, 1, 2, 3, 5, 8, 0, 8, 8, 3, 11, 1, 12, 0, 12, 12, 11, 10, 8, 5, 0, 5, 5, 10, 2, 12, 1.

This accounts for 28 pairs, examples of which are: 3, 5; 12, 0; 10, 8. Now select any pair not found in the above list such as 0, 2.

0, 2, 2, 4, 6, 10, 3, 0, 3, 3, 6, 9, 2, 11, 0, 11, 11, 9, 7, 3, 10, 0, 10, 10, 7, 4, 11, 2.

Take another pair not previously covered.

0, 4, 4, 8, 12, 7, 6, 0, 6, 6, 12, 5, 4, 9, 0, 9, 9, 5, 1, 6, 7, 0, 7, 7, 1, 8, 9, 4.

Continue this operation.

1, 3, 4, 7, 11, 5, 3, 8, 11, 6, 4, 10, 1, 11, 12, 10, 9, 6, 2, 8, 10, 5, 2, 7, 9, 3, 12, 2.

1, 4, 5, 9, 1, 10, 11, 8, 6, 1, 7, 8, 2, 10, 12, 9, 8, 4, 12, 3, 2, 5, 7, 12, 6, 5, 11, 3.

1, 5, 6, 11, 4, 2, 6, 8, 1, 9, 10, 6, 3, 9, 12, 8, 7, 2, 9, 11, 7, 5, 12, 4, 3, 7, 10, 4.

If these sequences of residues are numbered (1) for 0, 1..., (2) for 0, 2, ..., (3) for 0, 4..., (4) for 1, 3, ...; (5) for 1, 4, ..., (6) for 1, 5, ..., then the following table shows completely which sequence any given pair of residues produces. (First term is from the column at the left; the second from the heading at the top.) Thus, for a sequence generated by 37, 82, since the residues modulo 13 are 11, 4, the table shows that this is part of a sequence of type 6 for modulus 13.

TABLE OF SEQUENCES DETERMINED BY

RESIDUE PAIRS MODULO 13

	0	1	2	3	4	5	6	7	8	9	10	11	12
0		1	2	2	3	1	3	3	1	3	2	2	1
1	1	1	1	4	5	6	3	5	3	6	5	4	1
2	2	4	2	1	2	5	6	4	4	6	5	2	1
3	2	5	5	2	4	1	2	6	4	6	2	1	4
4	3	6	6	6	3	5	2	4	3	3	4	2	5
5	1	3	4	4	3	1	6	5	1	5	1	5	6
6	3	5	4	6	4	5	3	3	6	2	2	6	3
7	3	3	6	2	2	6	3	3	5	4	6	4	5
8	1	6	5	1	5	1	5	6	1	3	4	4	3
9	3	5	2	4	3	3	4	2	5	3	6	6	6
10	2	4	1	2	6	4	6	2	1	4	2	5	5
11	2	1	2	5	6	4	4	6	5	2	1	2	4
12	1	1	4	5	6	3	5	3	6	5	4	1	1

P. 93. Find all the sequences for modulus 11. Make a table showing which sequence is generated by any given pair of residues.

GENERAL FIBONACCI SEQUENCES

PERIODICITY

From the relation $$T_n = F_{n-2}\, a + F_{n-1}\, b$$

we can see that the period of any Fibonacci sequence for a modulus must be the same as that of the Fibonacci sequence or some divisor of the period for the Fibonacci sequence. That this latter case arises is seen from the Lucas sequence where the period of the modulus 5 is four, whereas for the Fibonacci sequence it is 20.

P. 94. Examine the Fibonacci sequence (1, 4) for the following prime moduli: 2, 3, 5, 7, 11, 13, 17, 19. Determine (a) the period for the given prime; (b) whether the prime divides any term of the sequence. (See solution, p. 43)

FORMULAS FOR THE GENERAL FIBONACCI SEQUENCES

In the case of the Fibonacci and Lucas sequences we found a number of formulas that pertained to them specifically. Can we find formulas that apply to Fibonacci sequences in general?

We have already noted the following:

$$T_{n+1} = T_n + T_{n-1}$$

$$T_n = F_{n-2}\, a + F_{n-1}\, b$$

if $T_1 = a$ and $T_2 = b$.

$$T_n^2 - T_{n-1}\, T_{n+1} = (-1)^n D$$

where D is characteristic of the Fibonacci sequence in question. The $(-1)^n$ would hold on the assumption that $2a < b$, i.e., we have set up the sequence according to the standard form previously mentioned.

P. 95. What is the sum of the first n terms of a general Fibonacci sequence? Try out particular examples and see whether you can find what this sum might be. Prove your formula by induction. (See solution, p. 49)

P. 96. Find the formula for

$$\sum_{k=1}^{n} T_{2k}$$ (See solution, p. 51)

P. 97. Determine the formula for

$$\sum_{k=1}^{n} T_{2k-1}$$ (See solution, p. 50)

P. 98. Determine the formula for

$$\sum_{k=1}^{n} T_{4k-1}$$ (See solution, p. 46)

P. 99. Determine the formula for

$$\sum_{k=1}^{n} T_{4k-2}$$ (See solution, p. 44)

GENERAL FIBONACCI SEQUENCES

P. 100. Determine the formula for

$$\sum_{k=1}^{n} T_{4k-3}$$

(See solution, p. 50)

P. 101. Determine the formula for

$$\sum_{k=1}^{n} T_{4k}$$

(See solution, p. 46)

P. 102. What is the formula for

$$T_{m-1} T_n - T_m T_{n-1}$$

(See solution, p. 49)

P. 103. Find the formula for

$$T_n^2 - T_{n-k} T_{n+k}$$

(See solution, p. 51)

P. 104. Show that $T_n T_{n+1} - T_{n-2} T_{n-1} = T_{n-1}^2 + T_n^2$

(See solution, p. 45)

P. 105. Show that

$$T_{m-k} T_n - T_m T_{n-k} = (-1)^{m+k} F_k F_{n-m} D$$

(See solution, p. 44)

THE ASSOCIATED "LUCAS" SEQUENCE

In studying the Fibonacci and Lucas sequences we found the relation

$$L_n = F_{n-1} + F_{n+1}$$

between the terms of the Lucas sequence and the Fibonacci sequence. Starting with a sequence, such as (1, 4) we can set up an associated "Lucas" sequence by means of the relation

$$R_n = T_{n-1} + T_{n+1}$$

The following table shows this for the case (1, 4).

$T_1 = 1,$	$R_1 = 7$
$T_2 = 4,$	$R_2 = 6$
$T_3 = 5,$	$R_3 = 13$
$T_4 = 9$	$R_4 = 19$
$T_5 = 14$	$R_5 = 32$
$T_6 = 23$	$R_6 = 51$
$T_7 = 37$	$R_7 = 83$
$T_8 = 60$	$R_8 = 134$

Clearly, the R sequence will also be a Fibonacci sequence with the relation

$$R_{n+1} = R_n + R_{n-1}$$

GENERAL FIBONACCI SEQUENCES

P. 106. If the Fibonacci sequence T has a characteristic number D associated with it, what is the characteristic number of the associated Lucas sequence as above defined? (See solution, p. 43)

P. 107. Show that $R_n^2 - R_{n-2k}^2 = 5(T_n^2 - T_{n-2k}^2)$

(See solution, p. 47)

P. 108. Prove that $R_{n+1}R_{p+1} + R_nR_p = 5(T_{n+1}T_{p+1} + T_n T_p)$

(See solution, p. 51)

P. 109. Prove that $T_{2n+1} = F_n R_{n+1} + (-1)^n T_1$

(See solution, p. 49)

P. 110. Show that $R_n^2 + R_{n+1}^2 = 5(T_n^2 + T_{n+1}^2)$

(See solution, p. 45)

P. 111. Prove that $F_n T_m - F_m T_n = (-1)^{n-1} F_{m-n} T_0$

(See solution, p. 44)

THE FIBONACCI SEQUENCE AND PASCAL'S TRIANGLE

Beginning with this section, a number of special topics will be taken up in which various phases of mathematics will be shown in relation to the Fibonacci sequence. We start with the well-known Pascal Triangle. The properties of this table of numbers are many, but we single out the following.

(1) The elements of each row represent the coefficients in the expansion of $(1+x)^n$.

(2) The coefficient of x^r in the expansion is ${}_nC_r$, written also as $\binom{n}{r}$ being the number of combinations of n things taken r at a time.

$$ {}_nC_r = \frac{n!}{r!\,(n-r)!} $$

(3) The law of formation of the table is this: Add two consecutive elements of any row to obtain the element of the following row immediately below the element to the right. This corresponds to the relation:

$$ {}_{n+1}C_r = {}_nC_r + {}_nC_{r-1}. $$

What is the connection with Fibonacci numbers? If diagonals going upward to the right in the triangle have their terms added the sums are the successive Fibonacci numbers.

This is shown in the table on the following page.

THE FIBONACCI SEQUENCE AND PASCAL'S TRIANGLE

Diagonal sums									
1 —	1								
1 —	1	1							
2 —	1	2	1						
3 —	1	3	3	1					
5 —	1	4	6	4	1				
8 —	1	5	10	10	5	1			
13 —	1	6	15	20	15	6	1		
21 —	1	7	21	35	35	21	7	1	
34 —	1	8	28	56	70	56	28	8	1

P. 112. What is the formula for the nth Fibonacci number in terms of the elements of Pascal's triangle? (See solution, p. 43)

P. 113. Prove this relation by mathematical induction. (See solution, p. 50)

P. 114. Prove that

$$F_{2n} = \sum_{k=1}^{n} F_k \; {}_nC_k$$

(See solution, p. 48)

P. 115. Prove that

$$L_{2n} = \sum_{k=1}^{n} L_k \; {}_nC_k$$

(No solution provided)

P. 116. Prove that

$$F_n = 2^{1-n} \left[{}_nC_1 + 5 \; {}_nC_3 + 5^2 \; {}_nC_5 + 5^3 \; {}_nC_7 \ldots \right]$$

(See solution, p. 46)

P. 117. Find a formula for L_n similar to that for F_n in P. 116.

(See solution, p. 50)

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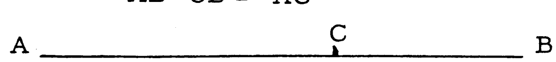
THE GOLDEN SECTION

Given a line AB. It is required to find a point C on the line so that

$$AB \cdot CB = AC^2$$



Looked at another way, $AB/AC = AC/BC$. It is this ratio that is known as the Golden Section ratio.

Let $AB/AC = r$. Then $AC = AB/r$; $BC = AC/r = AB/r^2$.

Since $AB = AC + BC = AB/r + AB/r^2$

we have the relation $1 = 1/r + 1/r^2$ or $r^2 = r + 1$.

Writing this as a quadratic equation $r^2 - r - 1 = 0$ and solving

$$r = \frac{1+\sqrt{5}}{2} \quad \text{or} \quad r = \frac{1-\sqrt{5}}{2}$$

The first is positive and greater than 1; the second is negative. Evidently, the first is the ratio in question. The value of

$$r = \frac{1+\sqrt{5}}{2} \quad \text{is} \quad 1.618033989\ldots\ldots\ldots\ldots$$

The reciprocal of r is $\frac{2}{1+\sqrt{5}} = \frac{\sqrt{5}-1}{2} = .618033989\ldots\ldots$

P. 118. Construct an isosceles triangle with an angle of 36 degrees at the vertex. What is the ratio of side to base? (See solution, p. 46)

P. 119. Construct a regular pentagon inscribed in a circle. Draw its diagonals. Find relations involving the Golden Section among sides, diagonals, and portions of diagonals. (See solution, p. 50)

The user of this discovery book should have noted that the equation for finding the value of the Golden Section ratio is exactly the same in form as that used to determine an explicit formula for the Fibonacci and Lucas numbers (p. 12). There we started with

$$x^2 - x - 1 = 0,$$

found roots $r = \frac{1+\sqrt{5}}{2}$ and $s = \frac{1-\sqrt{5}}{2}$

and proved that $F_n = \frac{r^n - s^n}{\sqrt{5}}$

$$L_n = r^n + s^n.$$

The r in these formulas is precisely the Golden Section ratio (often written as the Greek letter, phi, φ).

We are able to draw some interesting consequences from this connection. $s = -.618033989\ldots\ldots$ being less than one and negative will have even powers plus and odd powers minus; also as n increases the nth power of s will get smaller and smaller, so that for large values of n,

F_n equals approximately $r^n/\sqrt{5}$

L_n equals approximately r^n

For example, r^{25} is 167761.00000596086.....and L_{25} is 167761.

THE GOLDEN SECTION

Let us draw some consequences from the above. What is the limiting ratio approached by F_n / F_{n-1} as n gets larger and larger ?

$$\lim_{n \to \infty} F_n/F_{n-1} = \lim_{n \to \infty} (r^n - s^n) / (r^{n-1} - s^{n-1})$$

Divide all terms in numerator and denominator by r^{n-1}. The right-hand side becomes:

$$\lim_{n \to \infty} (r - s^n / r^{n-1}) / (1 - s^{n-1} / r^{n-1})$$

Now we have already noted that as n gets larger, s^n gets smaller and smaller tending to zero in the limit, while the nth power of r gets larger since r is greater than 1. Thus in the limit

$$\lim_{n \to \infty} F_n / F_{n-1} = r, \text{ the GOLDEN RATIO.}$$

P. 120. Derive the value of the $\lim_{n \to \infty} L_n / L_{n-1}$. (No solution provided)

It is a curious fact that no matter with what two numbers we start (apart from two zero's) to begin a Fibonacci sequence, the ratio of successive terms will in the limit approximate to r, the Golden Ratio. Let us take what looks like an extreme example: 2, 83. We shall list the terms and in a second column the ratio of each term to the preceding term.

n	T_n	T_n/T_{n-1}
1	2	--
2	83	41.5
3	85	1.0241
4	168	1.9765
5	253	1.5060
6	421	1.6640
7	674	1.6010
8	1095	1.6246
9	1769	1.61553
10	2864	1.61899

Can it be proved in general that the ratio of successive terms of any Fibonacci sequence must converge to the Golden Ratio? In our study of general Fibonacci sequences, we found that starting with a and b as our first and second terms,

$$T_n = F_{n-2}\, a + F_{n-1}\, b$$

$$T_{n-1} = F_{n-3}\, a + F_{n-2}\, b$$

P. 121. Prove that $\lim_{n \to \infty} T_n/T_{n-1} = r$. (See solution, p. 45)

GENERATING FUNCTIONS

A generating function is an expression which on being expanded formally into a series unfolds a sequence by means of the coefficients in the series. For example,

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \ldots\ldots.$$

by the operation of division. The coefficients 1, 1, 1, 1, ...thus have $1/(1-x)$ as their generating function. A less trivial example which will illustrate one method of finding the coefficients which constitute the sequence is the following.

Of what set of quantities is $1/(1-x)^2$ the generating function? Since we do not know what they are we can set up the following relation:

$$\frac{1}{1-2x+x^2} = a_0 + a_1x + a_2x^2 + a_3x^3 \quad \ldots\ldots\ldots$$

We multiply both sides by $1-2x+x^2$ and obtain a relation in which 1 equals the product of

$$(1-2x+x^2) \text{ by } a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots\ldots.$$

Since this is an identity, the coefficients of like powers of x on either side must be equal. Hence

$$\begin{aligned} a_0 &= 1 \\ a_1 - 2a_0 &= 0 \\ a_2 - 2a_1 + a_0 &= 0 \\ a_3 - 2a_2 + a_1 &= 0 \\ a_4 - 2a_3 + a_2 &= 0 \\ \ldots\ldots&\ldots\ldots\ldots \end{aligned}$$

Apart from the initial steps, we evidently have a recursion relation

$$a_{n+1} = 2a_n - a_{n-1}$$

for finding later coefficients from earlier ones. Solving:

$$a_0 = 1, \quad a_1 = 2, \quad a_2 = 3, \quad a_3 = 4, \quad a_4 = 5, \quad a_5 = 6, \text{ etc.}$$

Clearly, we have generated the integers: 1, 2, 3, 4, 5, 6, ...so that $1/(1-x)^2$ is the generation function of the positive integers.

P. 122. Determine what is generated by

$$\frac{1}{1-x-x^2}$$

(See solution, p. 43)

P. 123. Find the quantities generated by

$$\frac{1+2x}{1-x-x^2}$$

(See solution, p. 48)

GENERATING FUNCTIONS

P. 124. Find a generating function for the Fibonacci sequence (1, 4).

P. 125. Determine a generating function for the Fibonacci sequence (2, 7).

P. 126. Taking the derivative of both sides of the relation found in P. 122, derive another generating function relation. (See solution, p. 51)

P. 127. Find the sequence generated by:

$$\frac{1-x}{1-2x-2x^2+x^3}$$

(See solution, p. 43)

P. 128. Modify the generating function in P. 127 so as to obtain Lucas numbers instead of Fibonacci numbers. (See solution, p. 48)

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MATRICES AND FIBONACCI NUMBERS

Given a second-order matrix $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ which we shall designate Q.

We raise this to powers by the usual method of matrix multiplication.

$$Q^2 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \qquad Q^3 = \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}$$

If we identify the quantities in the matrix as Fibonacci numbers, it appears that

$$Q^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}$$

Multiplying by Q again gives $\begin{pmatrix} F_{n+2} & F_{n+1} \\ F_{n+1} & F_n \end{pmatrix}$

so that the result holds by mathematical induction.

The determinant of the original matrix is $\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = -1.$

Therefore, the determinant of Q^n is $(-1)^n$, since the determinant of a product of matrices is the product of the determinants of the matrices in the product. Applying this to the nth power of Q, we obtain

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n$$

a well-known relation.

Again:

$$Q^m Q^n = Q^{m+n}$$

so that
$$\begin{pmatrix} F_{m+1} & F_m \\ F_m & F_{m-1} \end{pmatrix} \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} = \begin{pmatrix} F_{m+n+1} & F_{m+n} \\ F_{m+n} & F_{m+n-1} \end{pmatrix}$$

Since for equality, two matrices must have the same elements, we obtain the following identities:

$$F_{m+n+1} = F_{m+1} F_{n+1} + F_m F_n$$
$$F_{m+n} = F_{m+1} F_n + F_m F_{n-1}$$
$$F_{m+n} = F_m F_{n+1} + F_{m-1} F_n$$
$$F_{m+n-1} = F_m F_n + F_{m-1} F_{n-1}$$

Letting $m = n$ in the first, we have the special relation

$$F_{2n+1} = F_{n+1}^2 + F_n^2$$

In the second, $m = n$ gives us
$$F_{2n} = F_{n+1} F_n + F_n F_{n-1} = F_n L_n$$

P. 129. Using matrices derive relations on the basis of

$$Q^{3n} = (Q^n)^3$$

P. 130. Consider a matrix which we may designate P

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix}$$

Find the various powers of P and determine the elements of P^n in terms of Fibonacci numbers. Prove your result by mathematical induction.

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CONTINUED FRACTIONS

There are many interesting patterns in the continued fractions of ratios involving Fibonacci numbers. For example,

$$55/34 = 1 + 21/34$$
$$34/21 = 1 + 13/21$$
$$21/13 = 1 + 8/13$$
$$13/8 = 1 + 5/8$$
$$8/5 = 1 + 3/5$$
$$5/3 = 1 + 2/3$$
$$3/2 = 1 + 1/2$$

Thus $55/34 = (1, 1, 1, 1, 1, 1, 1, 2)$ which can also be written $(1, 1, 1, 1, 1, 1, 1, 1, 1)$.

P. 131. Find the continued fraction representation of

$$L_{10}/L_9$$

(See solution, p. 51)

P. 132. What, in general, would be the continued fraction representation of

$$L_n/L_{n-1}?$$

(See solution, p. 46)

P. 133. What is the continued fraction representation of F_n/F_{n-2}?

(See solution, p. 48)

P. 134. Find the continued fraction representation of L_n/F_n.

(See solution, p. 51)

P. 135. What is the continued fraction representation of F_n/F_{n-3}?

(See solution, p. 46)

It is also possible to represent irrational numbers by continued fractions. Such a number which is of special interest is the Golden Section ratio, φ.

$$\frac{1+\sqrt{5}}{2} = 1 + \frac{\sqrt{5}-1}{2}$$

Then $$\frac{2}{\sqrt{5}-1} = \frac{\sqrt{5}+1}{2}$$

so that we are right back at our starting point. Thus the continued fraction representation of φ is an infinite continued fraction with all its elements 1.

EPILOGUE

The Fibonacci numbers are named after Leonardo Pisano who in his Liber Abaci (1202) proposed the famous rabbit problem that gave rise to this series of integers. Over the years these numbers have inspired the production of hundreds of mathematical papers either dealing with them directly or in their many adaptations and relations. In December, 1962, the Fibonacci Association was organized for the purpose of collecting a Fibonacci bibliography, engaging in research along these lines and producing a publication, the Fibonacci Quarterly.

It is hoped that the present introduction will have served to provide a speaking acquaintance with this particular field of mathematics. Those who wish to pursue the topic at greater length will find many leads as well as the opportunity for publication in the Fibonacci Quarterly.

The Editor as of this writing is: Dr. Verner E. Hoggatt, Jr., Department of Mathematics, San Jose State College, San Jose, Calif. The Managing Editor is: Brother U. Alfred, Department of Mathematics, St. Mary's College, Calif.

#4 p. 1	.....-6765, 4181, -2584, 1597, -987, 610, -377, 233, -144, 89, -55, 34, -21, 13, -8, 5, -3, 2, -1, 1,0,1,1,......
#11, p. 3	F_{2n}
#16 p. 5	$F_n^2 - F_{n-k} F_{n+k} = (-1)^{n+k} F_k^2$
P. 20 p. 6	$L_n^2 - L_{n-3} L_{n+3} = (-1)^n 20$
P. 30 p. 7	$F_n L_{n+1} - F_{n+1} L_n = (-1)^{n-1} 2$
P. 38 p. 10	For n=1, $F_1^2 - F_{-1} F_3 = 1 - 1 \cdot 2 = -1 = (-1)^1$ We wish to show that $F_{n+1}^2 - F_{n-1} F_{n+3} = (-1)^{n+1}$ if the formula is assumed to hold for n. Substituting $F_{n+3} = F_{n+2} + F_{n+1}$, we have $F_{n+1}^2 - F_{n-1} F_{n+3} = F_{n+1}^2 - F_{n-1} F_{n+2} - F_{n-1} F_{n+1}$ Substituting $F_{n+2} = F_{n+1} + F_n$ and $F_{n-1} = F_{n+1} - F_n$, this becomes $F_{n+1}^2 - F_{n+1}^2 + F_n^2 - F_{n-1} F_{n+1} = (-1)^{n-1} = (-1)^{n+1}$ by a previous formula.
P. 52 p. 12	$L_n = F_{k+1} L_{n-k} + F_k L_{n-k-1}$
P. 66 p. 16	(a) 10; (b) 16; (c) 20; (d) 8; (e) 3.
P. 87 p. 19	If the Fibonacci numbers are expressed in base 7, the period of the last digit will be the same as the period modulo 7 or 16. For the last two digits, the period will be that of 49 or 112. For the last three digits, the period will be that of seven cubed or 784.

#3 p. 1	$F_{n-1} = F_{n+1} - F_n$
P. 2 p. 4	$F_{2n+1} - 1$
P. 16 p. 6	F_{4n}
#17 p. 6	$L_n = F_{n-1} + F_{n+1}$
P. 27 p. 7	$L_n^2 + L_{n+1}^2 = 5\,F_{2n+1}$
P. 47 p. 11	Assume $F_n = F_{k+1}\,F_{n-k} + F_k\,F_{n-k-1}$ Replace F_{n-k} by $F_{n-k-1} + F_{n-k-2}$. This gives $F_n = F_{k+1}\,F_{n-k-1} + F_{k+1}\,F_{n-k-2} + F_k\,F_{n-k-1}$ $= (F_{k+1} + F_k)\,F_{n-k-1} + F_{k+1}\,F_{n-k-2}$ $= F_{k+2}\,F_{n-k-1} + F_{k+1}\,F_{n-k-2}$ which is the original formula with k replaced by k + 1.
P. 56 p. 14	$F_n\,L_{n+k} + F_{n+k}\,L_n =$ $\frac{(r^n - s^n)}{\sqrt{5}}\,(r^{n+k} + s^{n+k}) + \frac{(r^{n+k} - s^{n+k})}{\sqrt{5}}\,(r^n + s^n) =$ $\frac{r^{2n+k} + r^n s^{n+k} - r^{n+k} s^n - s^{2n+k} + r^{2n+k} + r^{n+k} s^n}{\sqrt{5}}$ $+ \frac{-r^n s^{n+k} - s^{2n+k}}{\sqrt{5}} = \frac{2(r^{2n+k} - s^{2n+k})}{\sqrt{5}} =$ $2\,F_{2n+k}$

# 2 p. 1	To obtain the Fibonacci number that precedes two successive Fibonacci numbers, subtract the first from the second.
P. 5 p. 4	$F_{2n} F_{2n+1}$
P. 13 p. 5	$L_{2n+1} - 1$
P. 22 p. 6	$L_n^2 - L_{n-5} L_{n+5} = (-1)^n 125$
P. 32 p. 7	$F_n L_{n+3} - F_{n+3} L_n = (-1)^{n-1} 4$
P. 37 p. 9	Only the portion of the induction in which we proceed from n to n+1 is given here. $\sum_{k=1}^{n} F_{2k} = F_{2n+1} - 1$ $F_{2n+2} = F_{2n+2}$ $\sum_{k=1}^{n+1} F_{2k} = F_{2n+3} - 1$
P. 59 p. 15	$F_{7k} = F_k \left[L_{6k} + (-1)^k L_{4k} + L_{2k} + (-1)^k \right]$
P. 72 p. 17	240
P. 83 p. 19	$L_n = F_{n+1} + F_{n-1} = F_n + 2F_{n-1}$. If F_n and L_n have a common factor greater than 1, it must either divide 2 or F_{n-1}. But it cannot divide F_{n-1}, since F_n and F_{n-1} are relatively prime. Therefore 2 must be the highest possible common factor. F_{3k} and L_{3k} are the terms having this highest common factor 2.

# 1 p. 1	$F_{n+1} = F_n + F_{n-1}$
#13 p. 4	$F_n^2 - F_{n-1} F_{n+1} = (-1)^{n-1}$
P. 9 p. 5	15127 , -9349, 5778, -3571, 2207, -1364, 843, -521 322, -199, 123, -76, 47, -29, 18, -11, 7, -4, 3, -1, 2.
P. 16 p. 6	F_{4n} .
P. 33 p. 7	$F_n L_{n+4} - F_{n+4} L_n = (-1)^{n-1} 6$
P. 42 p. 10	For n=1, $L_1 = 1$, $F_0 + F_2 = 0 + 1 = 1$. Checks Assume $L_n = F_{n-1} + F_{n+1}$ $L_{n-1} = F_{n-2} + F_n$ (Formula holds to n) Add $L_{n+1} = F_n + F_{n+2}$ Complete the induction.
P. 53 p. 12	$L_n = (-1)^{k-1} \left[F_k L_{n+k+1} - F_{k+1} L_{n+k} \right]$
P. 64 p. 15	(a) F_{3k}; (b) F_{4k}; (c) F_{8k}; (d) F_{7k}; (e) F_{6k}.
P. 85 p. 19	The periodicity of the last digit of the Fibonacci numbers is given by the period modulo 10. Since the period of 2 is 3 and the period of 5 is 20, the period modulo 10 is 60. For the last two digits, the modulus is 100. The period of 4 is 6; the period of 25 is 100. Hence the period of the last two digits is the least common multiple of 6 and 100 or 300. For the last three digits, the modulus is 1000. The period of 8 is 12; the period of 125 is 500. Hence the period of the last three digits is the least common multiple of 12 and 500 or 1500.

#8, p. 2	The sum of the first n Fibonacci numbers is one less than the (n+2)nd Fibonacci number.
P. 3 p. 4	$F_{2n-1} F_{2n}$
P. 12 p. 5	$L_{2n} -2$
P. 19 p. 6	$L_n^2 - L_{n-2} L_{n+2} = (-1)^{n-1} 5$
P. 29 p. 7	$L_n L_{n+1} - L_{n-1} L_{n+2} = (-1)^n 5$
P. 35 p. 7	$F_n L_{n+k} - F_{n+k} L_n = (-1)^{n-1} 2F_k$
P. 63 p. 15	$F_n L_{n+k} - F_{n+k} L_n =$ $\frac{(r^n - s^n)}{\sqrt{5}} (r^{n+k} + s^{n+k}) - \frac{(r^{n+k} - s^{n+k})}{5} (r^n + s^n) =$ $\frac{r^{2n+k} - r^{n+k} s^n + r^n s^{n+k} - s^{2n+k} - r^{2n+k} - r^{n+k} s^n}{\sqrt{5}}$ $+ \frac{r^n s^{n+k} + s^{2n+k}}{\sqrt{5}} = 2(-1)^{n+1} \frac{(r^k - s^k)}{\sqrt{5}} =$ $2 (-1)^{n+1} F_k$
P. 73 p. 17	The least common multiple of 3, 8, and 20 is 120.
P. 86 p. 19	$L_{2n} = r^{2n} + s^{2n}$ and $L_n = r^n + s^n$ Clearly, one does not divide the other in general. There is also the formula: $L_{2n} = L_n^2 + 2(-1)^{n+1}$. Except for $L_n = 1$, there will be a remainder on dividing both sides by L_n.

#9, p. 3	$F_{n+2} - 1$
#14 p. 4	$F_n^2 - F_{n-2} F_{n+2} = (-1)^n$
P. 7 p. 5	(a) 3, 12, 15, 27, 42, 69, 111, 180, 291, 471. (b) -7, 4, -3, 1, -2, -1, -3, -4, -7, -11. (c) 6, -13, -7, -20, -27, -47, -74, -121, -195, -316
P. 21 p. 6	$L_n^2 - L_{n-4} L_{n+4} = (-1)^{n-1} 45$
P. 31 p. 7	$F_n L_{n+2} - F_{n+2} L_n = (-1)^{n-1} 2$
P. 36 p. 9	Only the portion of the induction in which we proceed from n to n+1 is given here. $\sum_{k=1}^{n} F_{2k-1} = F_{2n}$ $F_{2n+1} = F_{2n+1}$ ――――――――――― $\sum_{k=1}^{n} F_{2k-1} = F_{2n+2}$
P. 57 p. 14	$L_{n+1}^2 + L_n^2 = r^{2n+2} + 2r^{n+1}s^{n+1} + s^{2n+2} + r^{2n} + 2r^n s^n + s^{2n}$ Since $r^{n+1}s^{n+1} = (-1)^{n+1}$ and $r^n s^n = (-1)^n$, these terms add to zero. Thus the sum becomes $r^{2n+2} + s^{2n+2} + r^{2n} + s^{2n}$ The right-hand side $5(F_n^2 + F_{n+1}^2)$ equals $5\left(\frac{r^{2n} - 2r^n s^n + s^{2n} + r^{2n+2} - 2r^{n+1}s^{n+1} + s^{2n+2}}{5}\right)$ $= r^{2n} + s^{2n} + r^{2n+2} + s^{2n+2}$ Thus the quantities are equal to each other.

#7, p. 2	$F_{-n} = (-1)^{n-1} F_n$
#12 p. 4	$F_n F_{n+1}$
P. 10 p. 5	$L_{-k} = (-1)^k L_k$
P. 17 p. 6	$F_{4n+1} - 1$
P. 34 p. 7	$F_n L_{n+5} - F_{n+5} L_n = (-1)^{n-1} 10$
P. 46 p. 11	$F_n = F_{k+1} F_{n-k} + F_k F_{n-k-1}$
P. 62 p. 15	$L_{7k} = L_k \left[L_{6k} + (-1)^{k-1} L_{4k} + L_{2k} + (-1)^{k-1} \right]$
P. 71 p. 16	The period of the product of two primes is the least common multiple of the periods of the primes taken individually. Thus for 77, the periods of 7 and 11 are 16 and 10 respectively. The period of 77 is the least common multiple of 16 and 10 or 80. The residues for 77 are shown herewith. 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 12, 67, 2, 69, 71, 63, 57, 43, 23, 66, 12, 1, 13, 14, 27, 41, 68, 32, 24, 55, 1, 56, 57, 36, 16, 52, 68, 43, 34, 0, 34, 34, 68, 25 16, 41, 57, 21, 1, 22, 23, 45, 68, 36, 27, 63, 13, 76, 12, 11, 23, 34, 57, 14, 71, 8, 2, 10, 12, 22, 34, 56, 13, 69, 5, 74, 2, 76, 1, 0.
P. 84 p. 19	A smaller Fibonacci number divides a larger when the subscript of the smaller divides the subscript of the larger. Let there be a Fibonacci number F_d and a larger Fibonacci number F_{kd}. Then $F_{kd} = \frac{r^{kd} - s^{kd}}{\sqrt{5}} = \frac{(r^d)^k - (s^d)^k}{\sqrt{5}}$ The numerator has a factor $r^d - s^d$, so that F_d must be a factor of F_{kd}.

#6, p.2	Fibonacci numbers with corresponding positive and negative odd subscripts are the same; those with corresponding positive and negative even subscripts are opposite in sign.
P. 4 p. 4	F_{2n}^2
P. 14 p. 5	$F_{4n+2} - 1$
P. 23 p. 6	$L_n^2 - L_{n-k} L_{n+k} = (-1)^{n+k-1} 5 F_k^2$
P. 28 p. 7	$F_n F_{n+1} - F_{n-1} F_{n+2} = (-1)^{n-1}$
P. 49 p. 12	Assume $F_n = (-1)^{k-1} \left[F_k F_{n+k+1} - F_{k+1} F_{n+k} \right]$ Replace F_{n+k} by $F_{n+k+2} - F_{n+k+1}$ $F_n = (-1)^{k-1} \left[-F_{k+1} F_{n+k+2} + F_{k+1} F_{n+k+1} + F_k F_{n+k+1} \right]$ $= (-1)^k \left[F_{k+1} F_{n+k+2} - F_{k+2} F_{n+k+1} \right]$
P. 54 p. 13	For $n = 1$, $L_1 = r + s = 1$. Assume $L_n = r^n + s^n$ up to n $L_{n-1} = r^{n-1} + s^{n-1}$ Add $L_{n+1} = r^{n+1} + s^{n+1}$
P. 65 p. 16	(a) 1, 1, 2, 3, 5, 8, 2, 10, 1, 0. (b) 1, 1, 2, 3, 5, 1, 6, 0, 6, 6, 5, 4, 2, 6, 1, 0. (c) 1, 1, 2, 3, 0, 3, 3, 1, 4, 0, 4, 4, 3, 2, 0, 2, 2, 4, 1, 0. (d) 1, 1, 2, 0, 2, 2, 1, 0. (e) 1, 1, 0.

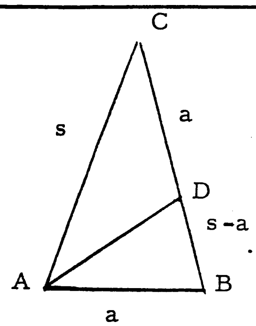
#10, p. 3	Headings: F_{2k-1} $\qquad \sum_{k=1}^{n} F_{2k-1}$
P. 6 p. 4	$F_{2n+1}^{2} - 1$
P. 15 p. 6	$F_{4n-1} - 1$
P. 24 p. 7	$5F_n = L_{n-1} + L_{n+1}$
P. 26 p. 7	$F_{2n+1} = F_n^2 + F_{n+1}^2$
P. 43 p. 10	For n=1, $L_{4-3} = L_1 = 1$ $F_{4-1} - 1 = F_3 - 1 = 2 - 1 = 1$. Checks. Assume $\sum_{k=1}^{n} L_{4k-3} = F_{4n-1} - 1$ up to n. $L_{4n+1} = F_{4n} + F_{4n+2}$ (P. 42) $\sum_{k=1}^{n+1} L_{4k-3} = F_{4n+1} + F_{4n+2} - 1$ $= F_{4n+3} - 1$ Complete the induction.
P. 55 p. 14	$F_{3k} = \frac{r^{3k} - s^{3k}}{\sqrt{5}} \qquad \frac{(r^k - s^k)}{\sqrt{5}} (r^{2k} + r^k s^k + s^{2k})$ $= F_k \left[L_{2k} + (-1)^k \right]$
P. 74 p. 17	The period of 2 is 3; of 2^2, is 6; of 2^3, is 12; of 2^4, is 24. The general formula for the period of 2^n would appear to be $3 \cdot 2^{n-1}$

#5, p. 2	$F_0 = 0$ $F_{-1} = 1$ $F_{-2} = -1$ $F_{-3} = 2$ $F_{-4} = -3$ $F_{-5} = 5$ $F_{-6} = -8$ $F_{-7} = 13$ $F_{-8} = -21$ $F_{-9} = 34$ $F_{-10} = -55$ $F_{-11} = 89$ $F_{-12} = -144$ $F_{-13} = 233$ $F_{-14} = -377$ $F_{-15} = 610$ $F_{-16} = -987$ $F_{-17} = 1597$ $F_{-18} = -2584$ $F_{-19} = 4181$ $F_{-20} = -6765$
#15 p. 5	$F_n^2 - F_{n-3} F_{n+3} = (-1)^{n-1} 4$ $F_n^2 - F_{n-4} F_{n+4} = (-1)^n 9$ $F_n^2 - F_{n-5} F_{n+5} = (-1)^{n-1} 25$ $F_n^2 - F_{n-6} F_{n+6} = (-1)^n 64$ $F_n^2 - F_{n-7} F_{n+7} = (-1)^{n-1} 169$
P. 11 p. 5	$L_{n+2} -3$
P. 18 p. 6	$L_n^2 - L_{n-1} L_{n+1} = (-1)^n 5$
P. 25 p. 7	$F_{2n} = F_n L_n$
P. 48 p. 12	$F_n = (-1)^{k-1} \left[F_k F_{n+k+1} - F_{k+1} F_{n+k} \right]$

P. 75 p. 17	Period of 3^n is $8 \cdot 3^{n-1}$.
P. 82 p. 19	$F_{n+2} = F_{n+1} + F_n$ If F_{n+2} and F_n have a divisor g greater than 1, this quantity must divide F_{n+1} as well. This would contradict the fact that F_n and F_{n+1} are relatively prime.
P. 94 p. 23	In the sequence (1, 4), 2 has a period of 3 and is a divisor; 3 has a period of 8 and is a divisor; 5 has a period of 20 and is a divisor; 7 has a period of 16 and is a divisor; 11 has a period of 5 and does not divide any term of the sequence; 13 has a period of 28 and does not divide any term of the sequence; 17 has a period of 36 and divides no term of the sequence; 19 has a period of 18 and divides certain terms of the sequence.
P. 106 p. 25	Let $T_1 = a$, $T_2 = b$. Then $T_0 = b-a$; $T_3 = a+b$. $D = T_2^2 - T_1 T_3 = b^2 - a^2 - ab.$ For the associated Lucas sequence, $R_1 = 2b-a, \quad R_2 = 2a+b, \quad R_3 = a+3b.$ $R_2^2 - R_1 R_3 = (2a+b)^2 - (2b-a)(a+3b)$ $= 5(a^2 + ab - b^2)$ Thus the characteristic number for the associated Lucas sequence is 5 times that of the Fibonacci sequence.
P. 112 p. 26	$F_n = \sum_{k=1}^{[(n+1)/2]} {}_{n-k}C_{k-1}$ where the square bracket on (n+1)/2 means the "greatest integer in."
P. 122 p. 29	$\frac{1}{1 - x - x^2} = F_1 + F_2 x + F_3 x^2 + F_4 x^3 + F_5 x^4 + \ldots\ldots$
P. 127 p. 30	The function generates: $F_1^2 + F_2^2 x + F_3^2 x^2 + F_4^2 x^3 + F_5^2 x^4 \ldots\ldots$

P. 76 p. 17	Period of 7 is 16; period of 7^2 is $7 \cdot 16$ or 112.
P. 81 p. 18	Despite the fact that the primes 3 and 7 divide terms of the Lucas sequence, 21 does not. The Lucas sequence is divided by 4, but neither 12 nor 28 divide any term of the sequence. This situation arises as follows. In the case of 21, the prime 3 divides terms with subscripts of the form $2+4k$; the prime 7 divides terms with subscripts of the form $4+8k$. If a term is divisible by 21, this means that both 3 and 7 must divide it, so that $2+4k$ must equal $4+8k'$ for some values of k and k'. But this is clearly seen to be impossible.
P. 99 p. 23	$$\sum_{k=1}^{n} T_{4k-2} = F_{2n} T_{2n}$$
P. 105 p. 24	We make an induction on k. When $k=0$, $T_m T_n - T_m T_n = 0$ checks with the formula. When $k=1$, $T_{m-1} T_n - T_m T_{n-1} = (-1)^{m-1} F_{n-m} D$ by P. 102. This also checks with the proposed formula. Assume the relation holds up to k. Then $$T_{m-k} T_n - T_m T_{n-k} = (-1)^{m+k} F_k F_{n-m} D$$ $$T_{m-k+1} T_n - T_m T_{n-k+1} = (-1)^{m+k-1} F_{k-1} F_{n-m} D$$ Subtract the first from the second. This gives $$T_{m-k-1} T_n - T_m T_{n-k-1} = (-1)^{m+k+1} F_{k+1} F_{n-m} D$$ which is the expected relation for k+1. The induction can now be completed on k.
P. 111 p. 25	The solution would proceed as follows: (1) Prove that $F_n T_{n+1} - F_{n+1} T_n = (-1)^{n-1} T_0$ (2) Prove that $F_n T_{n+2} - F_{n+2} T_n = (-1)^{n-1} T_0$ Adding: $F_n T_{n+3} - F_{n+3} T_n = (-1)^{n-1} F_3 T_0$ Continued addition will arrive at the desired formula.

P. 77 p. 18	Let k(p) be the period for prime p and Z(p), the entry point for prime p. The three cases are as follows: (1) If $k(p) = Z(p)$, then $k(p) = 2(2m+1)$ in form. (2) If $k(p) = 2\,Z(p)$, then $k(p) = 2^r(2m+1)$, $r \geq 3$. (3) If $k(p) = 4\,Z(p)$, then $k(p) = 4\,(2m+1)$.
P. 80 p. 18	A prime does not enter the Lucas sequence if the entry point of the prime in the Fibonacci sequence is odd. Examples are: 5, 13, 17, 37, 53, 61, 73, ...
P. 89A p. 20	$T_n = F_{n-2}\,a + F_{n-1}\,b$.
P. 104 p. 24	$T_n(T_{n+1} - T_{n-1}) = T_n^2$. or $T_n T_{n+1} = T_n T_{n-1} + T_n^2$ Then $T_n T_{n+1} - T_{n-2} T_{n-1} = T_n T_{n-1} - T_{n-1} T_{n-2} + T_n^2$ $= T_{n-1}(T_n - T_{n-2}) + T_n^2 = T_{n-1}^2 + T_n^2$
P. 110 p. 25	$R_n^2 + R_{n-1}^2 = (T_{n+1} + T_{n-1})^2 + (T_n + T_{n-2})^2$ $= (2\,T_{n+1} + T_n)^2 + (2\,T_n - T_{n+1})^2$ $= 5\,(T_n^2 + T_{n+1}^2)$
P. 121 p. 28	$\lim_{n \to \infty} T_n / T_{n-1} = \lim_{n \to \infty} \dfrac{F_{n-2}\,a + F_{n-1}\,b}{F_{n-3}\,a + F_{n-2}\,b}$ $= \lim_{n \to \infty} \dfrac{F_{n-2}\left[a + (F_{n-1}/F_{n-2})\,b\right]}{F_{n-3}\left[a + (F_{n-2}/F_{n-3})\,b\right]}$ $= r \left[\dfrac{(a + b/r)}{a + b/r}\right] = r.$

P. 78 p. 18	There are three cases: (1) If $Z(p) = 2m+1$, then $k(p) = 4(2m+1)$. (2) If $Z(p) = 2(2m+1)$, then $k(p) = 2(2m+1)$. (3) If $Z(p) = 2^r(2m+1)$, $r \geq 2$, then $k(p) = 2^{r+1}(2m+1)$
P. 89 p. 20	(a) (27, 84); (b) (485, 1099); (c) (62, 137)
P. 98 p. 23	$\sum_{k=1}^{n} T_{4k-1} = F_{2n} T_{2n+1}$
P. 101 p. 24	$\sum_{k=1}^{n} T_{4k} = F_{2n} T_{2n+2}$
P. 116 p. 26	$F_n = \frac{r^n - s^n}{\sqrt{5}}$, where $r = \frac{1+\sqrt{5}}{2}$, $s = \frac{1-\sqrt{5}}{2}$ The formula is obtained by expanding r^n and s^n by the binomial theorem and combining like terms.
P. 118 p. 27	 On bisecting the base angle of 72° two isosceles triangles are formed. Triangle ABD is similar to CAB. Hence $s/a = a/(s-a)$ or $s^2 - as - a^2 = 0$ from which $s/a = \frac{1 \pm \sqrt{5}}{2}$ The plus value indicates that $s/a = \varphi$, so that it is simply necessary to find two lines in this ratio to produce the required triangle.
P. 132 p. 32	$L_n / L_{n-1} = (1_{n-2}, 3)$ meaning n-2 one's and a 3.
P. 135 p. 32	A series of 4's ending in 3, 5, or 4.

P.79 p.18	If a prime, p, is of the form $10x \pm 1$, the period $k(p)$ divides $p-1$; if it is of the form $10x \pm 3$, the period $k(p)$ divides $2p+2$.
P.88 p.19	In all such sequences, there is a portion to the right in which the signs of all the terms are either all plus or all minus. There is a portion to the left in which the signs are alternating. Since the sequences with minus signs to the right may be obtained by multiplying all the terms of a corresponding sequence with plus signs to the right by minus one, we shall limit ourselves to sequences in which the terms to the right are positive.
P. 91 p. 21	Given that $T_n^2 - T_{n-1} T_{n+1} = (-1)^n D$ To prove that: $T_{n+1}^2 - T_n T_{n+2} = (-1)^{n+1} D$ Proof. Replace T_{n+2} by $T_{n+1} + T_n$. This gives $T_{n+1}^2 - T_n T_{n+1} - T_n^2 = T_{n+1}(T_{n+1} - T_n) - T_n^2$ $= T_{n+1} T_{n-1} - T_n^2 = -(T_n^2 - T_{n+1} T_{n-1}) = (-1)^{n+1} D$ Therefore each Fibonacci sequence will have its own characteristic D associated with it.
P. 107 p. 25	The proof of P. 107 depends on the proof of P. 110. It is necessary to work with two formulas simultaneously, namely: $R_n^2 - R_{n-2k}^2 = 5(T_n^2 - T_{n-2k}^2)$ and $R_n^2 + R_{n-2k-1}^2 = 5(T_n^2 + T_{n-2k-1}^2)$ Now, from P. 110, $R_n^2 + R_{n-1}^2 = 5(T_n^2 + T_{n-1}^2)$ $R_{n-1}^2 + R_{n-2}^2 = 5(T_{n-1}^2 + T_{n-2}^2)$ Subtracting the second equation from the first gives: $R_n^2 - R_{n-2}^2 = 5(T_n^2 - T_{n-2}^2)$ Adding $R_{n-2}^2 + R_{n-3}^2 = 5(T_{n-2}^2 + T_{n-3}^2)$ gives $R_n^2 + R_{n-3}^2 = 5(T_n^2 + T_{n-3}^2)$ More.

P. 107 p. 25 (cont.)	Assume the two formulas hold up to k as given. Then $R_n^2 - R_{n-2k}^2 = 5(T_n^2 - T_{n-2k}^2)$ $R_{n-2k}^2 + R_{n-2k-1}^2 = 5(T_{n-2k}^2 + T_{n-2k-1}^2)$ Adding $R_n^2 + R_{n-2k-1}^2 = 5(T_n^2 + T_{n-2k-1}^2)$ Then $R_{n-2k-1}^2 + R_{n-2k-2}^2 = 5(T_{n-2k-1}^2 + T_{n-2k-2}^2)$ Subtract. $R_n^2 - R_{n-2k-2}^2 = 5(T_n^2 - T_{n-2k-2}^2)$ so that we have advanced from k to k+1. The induction on k can now be completed.
P. 114 p. 26	The binomial expansion gives: $(1+a)^n = \sum_{k=0}^{n} a^k \, {}_nC_k$ We have $r^2 = 1+r$ and $s^2 = 1+s$. Therefore $r^{2n} = (1+r)^n = \sum_{k=0}^{n} r^k \, {}_nC_k$ $s^{2n} = (1+s)^n = \sum_{k=0}^{n} s^k \, {}_nC_k$ Hence $F_{2n} = \frac{r^{2n} - s^{2n}}{\sqrt{5}} = \sum_{k=0}^{n} \frac{(r^k - s^k)}{\sqrt{5}} \, {}_nC_k$ $= \sum_{k=0}^{n} F_k \, {}_nC_k$
P. 123 p. 29	$\frac{1+2x}{1-x-x^2} = L_1 + L_2 x + L_3 x^2 + L_4 x^3 + \ldots\ldots$
P. 128 p. 30	$\frac{1+7x-4x^2}{1-2x-2x^2+x^3}$ generates the squares of the Lucas numbers.
P. 133 p. 32	$F_n / F_{n-2} = (2, 1_{n-5}, 2)$

<table>
<tr><td>P. 95
p. 23</td><td>

$$T_1 = T_3 - T_2$$
$$T_1 + T_2 = T_3 = T_4 - T_2$$
$$T_1 + T_2 + T_3 = T_4 - T_2 + T_3 = T_5 - T_2$$

It appears that the formula is

$$\sum_{k=1}^{n} T_k = T_{n+2} - T_2 .$$ Assume true to n.

Add $$T_{n+1} = T_{n+1}$$

$$\sum_{k=1}^{n+1} T_k = T_{n+3} - T_2$$

Complete the induction.

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<tr><td>P. 102
p. 24</td><td>

$$T_{m-1} T_n - T_m T_{n-1} = (-1)^{m-1} F_{n-m} D$$

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<tr><td>P. 109
p. 25</td><td>

We need to use two relations to make the induction:.

$$T_{2n} = F_n R_n + (-1)^n T_0$$
$$T_{2n+1} = F_n R_{n+1} + (-1)^n T_1$$

For n=0, we have $T_0 = T_0$ and $T_1 = T_1$.

Assume the relations to hold up to n.

$$T_{2n-1} = F_{n-1} R_n + (-1)^{n-1} T_1$$
$$T_{2n} = F_n R_n + (-1)^n T_0$$

Add: $$T_{2n+1} = F_{n+1} R_n + (-1)^n (T_0 - T_1)$$

Here we must invoke an unproved relation:

$$F_{n+1} R_n = F_n R_{n+1} + (-1)^n R_0$$

Making this substitution, we have:

$$T_{2n+1} = F_n R_{n+1} + (-1)^n (R_0 + T_0 - T_1)$$
$$= F_n R_{n+1} + (-1)^n T_1$$

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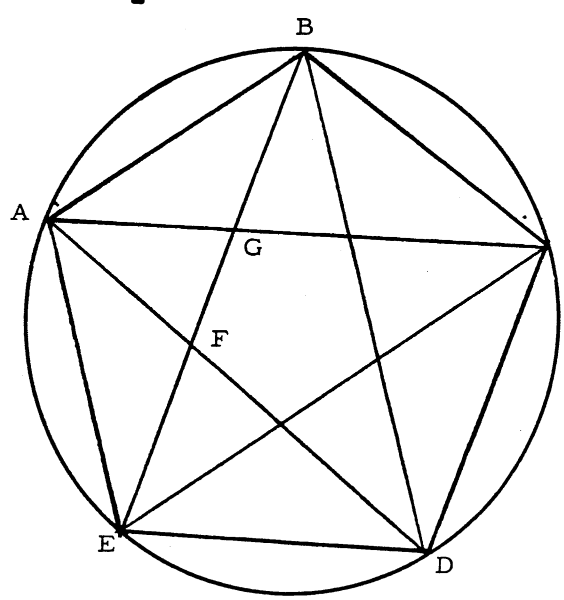
P. 97 p. 23	$\sum_{k=1}^{n} T_{2k-1} = T_{2n} - T_0$
P. 100 p. 24	$\sum_{k=1}^{n} T_{4k-3} = F_{2n} T_{2n-1}$
P. 113 p. 26	This relation follows from the very nature of the Pascal Triangle. For example, if we add the quantities (p. 26) in the table that give 21 and 34, we get 1, 8, 21, 20, and 5 the quantities in the next diagonal.
P. 117 p. 26	$L_n = 2^{1-n} \left[{}_nC_0 + 5\,{}_nC_2 + 5^2\,{}_nC_4 + 5^3\,{}_nC_6 \cdots \right]$
P. 119 p. 27	 Since the central angle of a regular decagon is 36°, the ratio of the radius to the side of the regular decagon is φ. Thus the points of division on the circle can be determined. Connecting alternate points gives the regular inscribed pentagon. In the pentagon, BED is an isosceles triangle with vertical angle of 36°, so that the ratio of the diagonal to the side is φ. For the same reason it follows that $EF = ED/\varphi$ and $FG = AF/\varphi = ED/\varphi^2$.

P. 96 p. 23	$\sum_{k=1}^{n} T_{2k} = T_{2n+1} - T_1$
P. 103 p.24	$T_n^2 - T_{n-k} T_{n+k} = (-1)^{n+k-1} F_k^2 D$
P. 108 p. 25	We proceed by a double induction. $R_2 R_2 + R_1 R_1 = 5 (T_2 T_2 + T_1 T_1)$ by P. 110 $R_2 R_3 + R_1 R_2 = R_2^2 + R_1^2 + R_1 R_0 + R_1 R_2$ Using $R_0 + R_2 = 5 T_1$, this becomes $5 (T_2^2 + T_1^2 + T_1 R_1) = 5 (T_2^2 + T_1^2 + T_1 T_0 + T_1 T_2)$ $= 5 (T_1 T_2 + T_2 T_3)$ Keeping the R_2 in the first product and the R_1 in the second constant, it is now possible to build up the second subscripts to any point desired. Thus $R_2 R_{p+1} + R_1 R_p = 5 (T_2 T_{p+1} + T_1 T_p)$ (1) Again, $R_3 R_2 + R_2 R_1 = 5(T_3 T_2 + T_2 T_1)$ by rearranging one of the above relations. Also $R_3^2 + R_2^2 = 5 (T_3^2 + T_2^2)$ Keeping the first element in each product constant, we can now arrive at $R_3 R_{p+1} + R_2 R_p = 5 (T_3 T_{p+1} + T_2 T_p)$ Using (1) and (2), we can now build up the subscripts of the first elements and thus arrive at $R_{n+1} R_{p+1} + R_n R_p = 5 (T_{n+1} T_{p+1} + T_n T_p)$
P. 126 p. 30	$\frac{1+2x}{(1-x-x^2)^2} = F_2 + 2F_3 x + 3 F_4 x^2 + 4 F_5 x^3 \ldots\ldots$
P. 131 p. 32	$L_{10}/L_9 = 123/76 = (1, 1, 1, 1, 1, 1, 1, 1, 3)$
P. 134 p. 32	$L_n/F_n = 2, 4, 4, \ldots$ ending in 5, 3, or 4 .

TABLE 1

THE FIRST HUNDRED FIBONACCI NUMBERS AND THEIR PRIME FACTORIZATIONS

Note. In this table, an underlined factor means that it is entering the sequence for the first time.

n	F_n	Factors of F_n
1	1	
2	1	
3	2	2
4	3	3
5	5	5
6	8	2^3
7	13	13
8	21	3 • 7
9	34	2 • 17
10	55	5 • 11
11	89	89
12	144	$2^4 \cdot 3^2$
13	233	233
14	377	13 • 29
15	610	2 • 5 • 61
16	987	3 • 7 • 47
17	1597	1597
18	2584	2^3 • 17 • 19
19	4181	37 • 113
20	6765	3 • 5 • 11 • 41
21	10946	2 • 13 • 421
22	17711	89 • 199
23	28657	28657
24	46368	$2^5 \cdot 3^2$ • 7 • 23
25	75025	5^2 • 3001
26	121393	233 • 521
27	196418	2 • 17 • 53 • 109
28	317811	2 • 13 • 29 • 281
29	514229	514229
30	832040	2^3 • 5 • 11 • 31 • 61
31	1346269	557 • 2417
32	2178309	3 • 7 • 47 • 2207
33	3524578	2 • 89 • 19801
34	5702887	1597 • 3571
35	9227465	5 • 13 • 141961
36	14930352	$2^4 \cdot 3^3$ • 17 • 19 • 107
37	24157817	73 • 149 • 2221
38	39088169	37 • 113 • 9349
39	63245986	2 • 233 • 135721
40	102334155	3 • 5 • 7 • 11 • 41 • 2161

TABLE 1. FIBONACCI NUMBERS

n	F_n	Factors of F_n
41	165580141	$2789 \cdot 59369$
42	267914296	$2^3 \cdot 13 \cdot 29 \cdot 211 \cdot 421$
43	433494437	433494437
44	701408733	$3 \cdot 43 \cdot 89 \cdot 199 \cdot 307$
45	1134903170	$2 \cdot 5 \cdot 17 \cdot 61 \cdot 109441$
46	1836311903	$139 \cdot 461 \cdot 28657$
47	2971215073	2971215073
48	4807526976	$2^6 \cdot 3^2 \cdot 7 \cdot 23 \cdot 47 \cdot 1103$
49	7778742049	$13 \cdot 97 \cdot 6168709$
50	12586269025	$5^2 \cdot 11 \cdot 101 \cdot 151 \cdot 3001$
51	20365011074	$2 \cdot 1597 \cdot 6376021$
52	32951280099	$3 \cdot 233 \cdot 521 \cdot 90481$
53	53316291173	$953 \cdot 55945741$
54	86267571272	$2^3 \cdot 17 \cdot 19 \cdot 53 \cdot 109 \cdot 5779$
55	139583862445	$5 \cdot 89 \cdot 661 \cdot 474541$
56	225851433717	$3 \cdot 7 \cdot 7 \cdot 13 \cdot 29 \cdot 281 \cdot 14503$
57	365435296162	$2 \cdot 37 \cdot 113 \cdot 797 \cdot 54833$
58	591286729879	$59 \cdot 19489 \cdot 514229$
59	956722026041	$353 \cdot 2710260697$
60	1548008755920	$2^4 \cdot 3^2 \cdot 5 \cdot 11 \cdot 31 \cdot 41 \cdot 61 \cdot 2521$
61	2504730781961	$4513 \cdot 555003497$
62	4052739537881	$557 \cdot 2417 \cdot 3010349$
63	6557470319842	$2 \cdot 13 \cdot 17 \cdot 421 \cdot 35239681$
64	10610209857723	$3 \cdot 7 \cdot 47 \cdot 1087 \cdot 2207 \cdot 4481$
65	17167680177565	$5 \cdot 233 \cdot 14736206161$
66	27777890035288	$2^3 \cdot 89 \cdot 199 \cdot 9901 \cdot 19801$
67	44945570212853	$269 \cdot 116849 \cdot 1429913$
68	72723460248141	$3 \cdot 67 \cdot 1597 \cdot 3571 \cdot 63443$
69	117669030460994	$2 \cdot 137 \cdot 829 \cdot 18077 \cdot 28657$
70	190392490709135	$5 \cdot 11 \cdot 13 \cdot 29 \cdot 71 \cdot 911 \cdot 141961$
71	308061521170129	$6673 \cdot 46165371073$
72	498454011879264	$2^5 \cdot 3^3 \cdot 7 \cdot 17 \cdot 19 \cdot 23 \cdot 107 \cdot 103681$
73	806515533049393	$9375829 \cdot 86020717$
74	1304969544928657	$73 \cdot 149 \cdot 2221 \cdot 54018521$
75	2111485077978050	$2 \cdot 5^2 \cdot 61 \cdot 3001 \cdot 230686501$
76	3416454622906707	$3 \cdot 37 \cdot 113 \cdot 9349 \cdot 29134601$
77	5527939700884757	$13 \cdot 89 \cdot 988681 \cdot 4832521$
78	8944394323791464	$2^3 \cdot 79 \cdot 233 \cdot 521 \cdot 859 \cdot 135721$
79	14472334024676221	$157 \cdot 92180471494753$
80	23416728348467685	$3 \cdot 5 \cdot 7 \cdot 11 \cdot 41 \cdot 47 \cdot 1601 \cdot 2161 \cdot 3041$
81	37889062373143906	$2 \cdot 17 \cdot 53 \cdot 109 \cdot 2269 \cdot 4373 \cdot 19441$
82	61305790721611591	$2789 \cdot 59369 \cdot 370248451$
83	99194853094755497	99194853094755497
84	160500643816367088	$2^4 \cdot 3^2 \cdot 13 \cdot 29 \cdot 83 \cdot 211 \cdot 281 \cdot 421 \cdot 1427$
85	259695496911122585	$5 \cdot 1597 \cdot 9521 \cdot 3415914041$
86	420196140727489673	$6709 \cdot 144481 \cdot 433494437$

TABLE 1. FIBONACCI NUMBERS

F_n	F_n	Factors of F_n
87	679891637638612258	2 · 173 · 514229 · 3821263937
88	1100087778366101931	3 · 7 · 43 · 89 · 199 · 263 · 307 · 881 · 967
89	1779979416004714189	1069 · 1665088321800481
90	2880067194370816120	2^3 · 5 · 11 · 17 · 19 · 31 · 61 · 181 · 541 · 109441
91	4660046610375530309	13 · 13 · 233 · 741469 · 159607993
92	7540113804746346429	3 · 139 · 461 · 4969 · 28657 · 275449
93	12200160415121876738	2 · 557 · 2417 · 4531100550901
94	19740274219868223167	2971215073 · 6643838879
95	31940434634990099905	5 · 37 · 113 · 761 · 29641 · 67735001
96	51680708854858323072	$2^7 \cdot 3^2$ · 7 · 23 · 47 · 769 · 1103 · 2207 · 3167
97	83621143489848422977	193 · 389 · 3084989 · 361040209
98	135301852344706746049	13 · 29 · 97 · 6168709 · 599786069
99	218922995834555169026	2 · 17 · 89 · 197 · 19801 · 18546805133
100	354224848179261915075	$3 \cdot 5^2$ · 11 · 41 · 101 · 151 · 401 · 3001 · 570601

TABLE 2
THE FIRST FIFTY LUCAS NUMBERS
AND THEIR PRIME FACTORIZATIONS

n	L_n	Factors of L_n
1	1	
2	3	3
3	4	2^2
4	7	7
5	11	11
6	18	$2 \cdot 3^2$
7	29	29
8	47	47
9	76	$2^2 \cdot 19$
10	123	3 · 41
11	199	199
12	322	2 · 7 · 23
13	521	521
14	843	3 · 281
15	1364	$2^2 \cdot 11 \cdot 31$
16	2207	2207
17	3571	3571
18	5778	$2 \cdot 3^3 \cdot 107$
19	9349	9349
20	15127	7 · 2161
21	24476	$2^2 \cdot 29 \cdot 211$
22	39603	3 · 43 · 307
23	64079	139 · 461
24	103682	2 · 47 · 1103
25	167761	11 · 101 · 151
26	271443	3 · 90481

TABLE 2. LUCAS NUMBERS

n	L_n	Factors of L_n
27	439204	$2^2 \cdot 19 \cdot 5779$
28	710647	$7^2 \cdot 14503$
29	1149851	$59 \cdot 19489$
30	1860498	$2 \cdot 3^2 \cdot 41 \cdot 2521$
31	3010349	3010349
32	4870847	$1087 \cdot 4481$
33	7881196	$2^2 \cdot 199 \cdot 9901$
34	12752043	$3 \cdot 67 \cdot 63443$
35	20633239	$11 \cdot 29 \cdot 71 \cdot 911$
36	33385282	$2 \cdot 7 \cdot 23 \cdot 103681$
37	54018521	54018521
38	87403803	$3 \cdot 29134601$
39	141422324	$2^2 \cdot 79 \cdot 521 \cdot 859$
40	228826127	$47 \cdot 1601 \cdot 3041$
41	370248451	370248451
42	599074578	$2 \cdot 3^2 \cdot 83 \cdot 281 \cdot 1427$
43	969323029	$6709 \cdot 144481$
44	1568397607	$7 \cdot 263 \cdot 881 \cdot 967$
45	2537720636	$2^2 \cdot 11 \cdot 19 \cdot 31 \cdot 181 \cdot 541$
46	4106118243	$3 \cdot 4969 \cdot 275449$
47	6643838879	6643838879
48	10749957122	$2 \cdot 769 \cdot 2207 \cdot 3167$
49	17393796001	$29 \cdot 599786069$
50	28143753123	$3 \cdot 41 \cdot 401 \cdot 570601$

TABLE 3. ENTRY POINTS AND PERIODS OF FIBONACCI AND LUCAS SEQUENCES FOR PRIMES LESS THAN 270

Note. In this table k(p) indicates the period which is the same for both Fibonacci and Lucas sequences except for 5, where it is 20 for the Fibonacci sequence and 4 for the Lucas sequence; Z(F, p) is the entry point of the Fibonacci sequence; Z(L, p) is the entry point of the Lucas sequence.

p	k(p)	Z(F, p)	Z(L, p)
2	3	3	3
3	8	4	2
5	20	5	--
7	16	8	4
11	10	10	5
13	28	7	--
17	36	9	--
19	18	18	9
23	48	24	12
29	14	14	7

TABLE 3. PERIODS AND ENTRY POINTS

p	k(p)	Z(F, p)	Z(L, p)
31	30	30	15
37	76	19	--
41	40	20	10
43	88	44	22
47	32	16	8
53	108	27	--
59	58	58	29
61	60	15	--
67	136	68	34
71	70	70	35
73	148	37	--
79	78	78	39
83	168	84	42
89	44	11	--
97	196	49	--
101	50	50	25
103	208	104	52
107	72	36	18
109	108	27	--
113	76	19	--
127	256	128	64
131	130	130	65
137	276	69	--
139	46	46	23
149	148	37	--
151	50	50	25
157	316	79	--
163	328	164	82
167	336	168	84
173	348	87	--
179	178	178	89
181	90	90	45
191	190	190	95
193	388	97	--
197	396	99	--
199	22	22	11
211	42	42	21
223	448	224	112
227	456	228	114
229	114	114	57
233	52	13	--
239	238	238	119
241	240	120	60
251	250	250	125
257	516	129	--
263	176	88	44
269	268	67	--