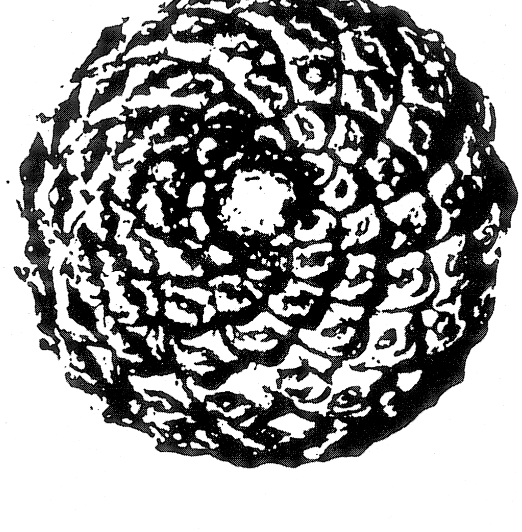
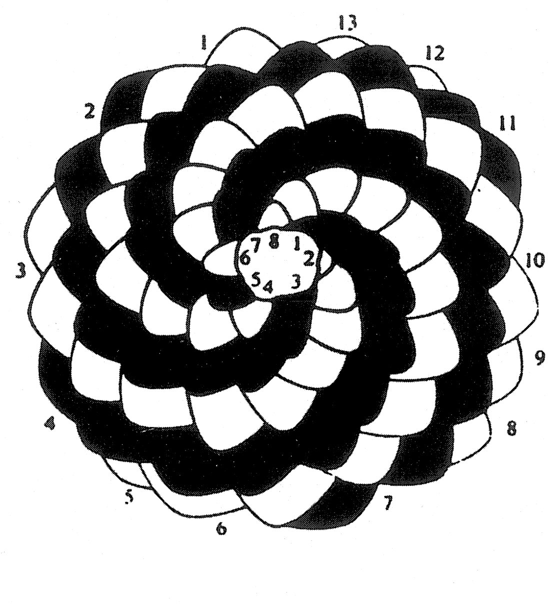
Fibonacci and Lucas Numbers

Verner E. Hoggatt, Jr.





Fibonacci and Lucas Numbers

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ABOUT THIS BOOK

This booklet offers an introduction to some of the interesting properties of Fibonacci and Lucas numbers. In reading this material, the student will have an opportunity to observe how many mathematical generalizations can be derived from some very simple notions.

ABOUT THE AUTHOR

The late Verner E. Hoggatt, Jr. was Professor of Mathematics at San Jose State College in San Jose, California. Together with Brother Alfred Brousseau, he founded *The Fibonacci Quarterly* (the official journal of The Fibonacci Association) in 1963 and was its General Editor until his death in August 1980.

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1 • *Introduction*

Fibonacci

Who was Fibonacci?

Leonardo Fibonacci, mathematical innovator of the thirteenth century, was a solitary flame of mathematical genius during the Middle Ages. He was born in Pisa, Italy, and because of that circumstance, he was also known as Leonardo Pisano, or Leonardo of Pisa. While his father was a collector of customs at Bugia on the northern coast of Africa (now Bougie in Algeria), Fibonacci had a Moorish schoolmaster, who introduced him to the Hindu-Arabic numeration system and computational methods.

After widespread travel and extensive study of computational systems, Fibonacci wrote, in 1202, the *Liber Abaci*, in which he explained the Hindu-Arabic numerals and how they are used in computation. This famous book was instrumental in displacing the clumsy Roman numeration system and introducing methods of computation similar to those used today. It also included some geometry and algebra.

Although he wrote on a variety of mathematical topics, Fibonacci is remembered particularly for the sequence of numbers

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots,$$

to which his name has been applied. This sequence, even today, is the subject of continuing research, especially by the Fibonacci Association, which publishes *The Fibonacci Quarterly*.

We shall study some elementary and interesting aspects of the Fibonacci and related numbers in this booklet.

2 • *Rabbits, Fibonacci Numbers, and Lucas Numbers*

Fibonacci introduced a problem in the *Liber Abaci* by a story that may be summarized as follows. Suppose that

(1) there is one pair of rabbits in an enclosure on the first day of January;
(2) this pair will produce another pair of rabbits on February first and on the first day of every month thereafter; and
(3) each new pair will mature for one month and then produce a new pair on the first day of the third month of its life and on the first day of every month thereafter.

The problem is to find the number of pairs of rabbits in the enclosure on the first day of the following January after the births have taken place on that day.

It will be helpful to make a chart to keep count of the pairs of rabbits. Let A denote an adult pair of rabbits and let B denote a "baby pair" of rabbits. Thus, on January first, we have only an A; on February first we have that A and a B; and on March first, we have the original A, a new B, and the former B, which has become an A:

Date	*Pairs*	*Number of A's*	*Number of B's*
January 1	A	1	0
February 1	A B	1	1
March 1	A B A	2	1

To continue the chart conveniently, we condense our notation as follows. To get the next line of symbols, in any line we replace each A by AB and each B by A. Thus, we have the representation shown in the table at the top of the next page.

Date	*Pairs*	*Number of A's*	*Number of B's*
March 1	ABA	2	1
April 1	ABAAB	3	2
May 1	ABAABABA	5	3
June 1	ABAABABAABAAB	8	5

We now see that the number of A's on July 1 will be the sum of the number of A's on June 1 and the number of B's born on that day (which become A's on July 1). The number of B's on July 1 is the same as the number of A's on June 1. We complete the table for the year:

	Month	*Number of A's*	*Number of B's*	*Total number of pairs*
1	January	1	0	1
	After births on first of			
2	February	1	1	2
3	March	2	1	3
4	April	3	2	5
5	May	5	3	8
6	June	8	5	13
7	July	13	8	21
8	August	21	13	34
9	September	34	21	55
10	October	55	34	89
11	November	89	55	144
12	December	144	89	233
13	January	233	144	377

Thus, we see that under the conditions of the problem, the number of pairs of rabbits in the enclosure one year later would be 377.

We can draw some conclusions by studying the table. It is clear that the number of A's on the following February 1 is 377. Of these, 376 were originally B's, descendants of the original A. Therefore, if we add all the numbers in the column headed "Number of B's," we have

$$S = 0 + 1 + 1 + 2 + 3 + 5 + 8 + 13 + 21 + 34 + 55 + 89 + 144$$
$$= 376.$$

From this, we observe that the sum of the first 12 entries in the column headed "Number of A's" is one less than 377, which would be the 14th entry in that column. This is a specific instance of a general result which we shall establish later in this section.

Further examination of the table on page 3 reveals that each entry in the columns of numbers may be found in accordance with a pattern. For example, the entries in each line after the second may be found as the sum of the two preceding entries in that column. Those in line 3 are:

$$2 = 1 + 1 \qquad 1 = 0 + 1 \qquad 3 = 1 + 2$$

Those in line 4 are:

$$3 = 1 + 2 \qquad 2 = 1 + 1 \qquad 5 = 2 + 3$$

Can we describe this pattern by some kind of formula? Yes, as we shall now show.

In general, ordered sets of numbers such as those in the columns of the table on page 3 are called *sequences*. A sequence may be *finite* or *infinite*. An infinite sequence may be designated by symbols such as

$$u_1, u_2, u_3, \ldots, u_n, \ldots,$$

where the subscripts indicate the order of the *terms*, with n a positive integer. An example of a sequence is the *arithmetic progression*

$$\begin{array}{cccccc} u_1, & u_2 & u_3, & \ldots, & u_n, & \ldots \\ \downarrow & \downarrow & \downarrow & & \downarrow & \\ 2, & 5, & 8, & \ldots, & 2 + (n - 1)3, & \ldots \end{array}$$

where a formula for the nth term is

$$u_n = 2 + (n - 1)3.$$

Another way to specify this sequence would be to state the first term,

$$u_1 = 2,$$

and the formula

$$u_n = u_{n-1} + 3, \qquad n > 1.$$

Such a definition is said to be a *recursive definition*, and the formula is called a *recursion formula* or a *recurrence formula*. (The words "recursive," "recursion," and "recurrence" all come from a Latin verb meaning "to run back.")

We can use an extension of this idea to specify the sequences in the columns of the table on page 3. For example, to specify the sequence in the column headed "Number of A's," we state the first two terms,

$$u_1 = 1, \qquad u_2 = 1,$$

and the recursive, or recurrence, formula

$$\text{(R)} \qquad u_n = u_{n-1} + u_{n-2}, \qquad n > 2.$$

This gives the sequence

$$1, 1, 2, 3, 5, 8, 13, \ldots$$

as we wished. For the column headed "Number of B's," we have $u_1 = 0$, $u_2 = 1$, and the same recurrence formula, yielding the sequence

$$0, 1, 1, 2, 3, 5, 8, 13, \ldots.$$

For the column headed "Total number of pairs," we have $u_1 = 1$, $u_2 = 2$, and the sequence

$$1, 2, 3, 5, 8, 13, \ldots.$$

Because of its source in Fibonacci's rabbit problem, the sequence

$$\mathbf{1, 1, 2, 3, 5, 8, 13, \ldots}$$

is called the **Fibonacci sequence**, and its terms are called **Fibonacci numbers.** We shall denote the nth Fibonacci number by F_n; thus,

$$F_1 = 1, \quad F_2 = 1, \quad F_3 = 2, \quad F_4 = 3, \quad F_5 = 5, \quad F_6 = 8, \ldots.$$

Moreover, we may write these alternative forms:

$$F_1 = F_2 = 1, \qquad F_n = F_{n-1} + F_{n-2}, \qquad n > 2,$$

or

$$F_1 = F_2 = 1, \qquad F_{n+1} = F_n + F_{n-1}, \qquad n > 1,$$

or

$$F_1 = F_2 = 1, \qquad F_{n+2} = F_{n+1} + F_n, \qquad n \geq 1.$$

We can now give a more formal discussion of the Fibonacci rabbit problem. For all positive integral n, we define for the first day of the nth month:

A_n = number of A's (adult pairs of rabbits)
B_n = number of B's (baby pairs of rabbits)
T_n = total number of pairs of rabbits $= A_n + B_n$

Only the A's on the first day of the nth month will produce B's on the first day of the $(n + 1)$st month. Thus,

$$B_{n+1} = A_n, \qquad n \geq 1.$$

In making up the table on page 3, we observed that the number of A's on the first day of the $(n + 2)$nd month is the sum of the number of A's on the first day of the $(n + 1)$st month and the number of B's born on that day. Thus,

$$A_{n+2} = A_{n+1} + B_{n+1},$$

and since $B_{n+1} = A_n$, we have

$$A_{n+2} = A_{n+1} + A_n, \qquad n \geq 1.$$

We also observe from the table that $A_1 = 1$ and $A_2 = 1$. Thus, the sequence

$$A_1, A_2, A_3, \ldots,$$

is the Fibonacci sequence, and

$$A_n = F_n, \qquad n \geq 1.$$

Since $B_{n+1} = A_n$ for $n \geq 1$, we have

$$B_n = A_{n-1} = F_{n-1} \text{ for } n \geq 2.$$

If we now let $n = 1$ in this last formula, we have

$$B_1 = F_0.$$

If we let $n = 1$ in the formula $F_{n+1} = F_n + F_{n-1}$, we have

$$F_2 = F_1 + F_0$$

or

$$F_0 = F_2 - F_1 = 1 - 1 = 0,$$

which checks with $B_1 = 0$ in the table. Thus, *we have now defined F_n for* $n = 0$.

Finally, the total number of pairs on the first day of the nth month is

$$T_n = A_n + B_n = F_n + F_{n-1} = F_{n+1}.$$

We can now establish the following result, already suggested by the specific instance shown at the bottom of page 3:

The sum of the first n Fibonacci numbers is one less than the (n + 2)nd Fibonacci number.

Symbolically:

$$F_1 + F_2 + \cdots + F_n = F_{n+2} - 1, \qquad n \geq 1.$$

We remember that $F_{n+2} = A_{n+2}$ and that A_{n+2} is the number of A's (adult pairs of rabbits) in the enclosure on the first day of the $(n + 2)$nd month.

Originally, we had only one A. Where did the extra A's come from? Each of the extra A's was first a B.

How many more A's do we now have? The number of extra A's is

$$A_{n+2} - 1.$$

Now, one month after being born, each B became an A. If we add the number of B's from the first day of the first month to the first day of the $(n + 1)$st month, the sum is the number of A's other than the original pair that we have on the first day of the $(n + 2)$nd month. Thus,

$$B_1 + B_2 + B_3 + \cdots + B_{n+1} = A_{n+2} - 1.$$

But, remembering that $B_1 = 0$, $B_n = F_{n-1}$, and $A_{n+2} = F_{n+2}$, we have

$$F_1 + F_2 + \cdots + F_n = F_{n+2} - 1, \qquad n \geq 1;$$

as we wished to show.

This formula is an example of a *Fibonacci number identity*. We shall prove this identity again later in three different ways (Section 10).

Many different sequences may be specified by using formula (R) on page 4 and choosing different numbers for the first two terms. For example, if we take $u_1 = 1$ and $u_2 = 3$, we have

$$\mathbf{1, 3, 4, 7, 11, 18, 29, 47, \ldots,}$$

which we shall call the **Lucas sequence,** in honor of the nineteenth-century French mathematician E. Lucas. Lucas did much work in recurrent sequences and gave the Fibonacci sequence its name. The terms of the Lucas sequence are called **Lucas numbers,** and we shall denote the nth Lucas number by L_n. The Lucas numbers are closely related to the Fibonacci numbers, as we shall show in this booklet.

In general, if we take the first two terms of a sequence defined by (R) as arbitrary integers p and q, that is, $u_1 = p$ and $u_2 = q$, then we have

$$p, \quad q, \quad p + q, \quad p + 2q, \quad 2p + 3q, \quad 3p + 5q, \quad \ldots,$$

which is called a **generalized Fibonacci sequence.** We shall denote the nth term of this sequence by H_n. It may be shown by mathematical induction (see Exercise 17, Section 10) that this generalized Fibonacci sequence is related to the Fibonacci sequence by the formula

$$H_{n+2} = H_2F_{n+1} + H_1F_n, \qquad n \geq 0,\ F_0 = 0,$$

or, expressed in terms of the starting values, p and q,

$$H_{n+2} = qF_{n+1} + pF_n.$$

EXERCISES

1. Compute the first 20 Fibonacci numbers.
2. Compute the first 20 Lucas numbers.
3. Study the results of Exercises 1 and 2, looking for any possible relationships or number patterns.
4. If $H_1 = 1$, $H_2 = 4$, and $H_{n+2} = H_{n+1} + H_n$, $n \geq 1$, compute the first 20 terms of this generalized Fibonacci sequence.

5. Verify that:

a. $L_5 = F_6 + F_4$

b. $F_9 = F_5^2 + F_4^2$

c. $L_7 + L_9 = 5F_8$

d. $H_{20} = (4)F_{19} + (1)F_{18}$ in Exercise 4.

6. Verify that:

a. $F_8 = L_4F_4$

b. $\dfrac{F_{10}}{F_5}$ is an integer.

c. $\dfrac{F_{12}}{F_4}$ is an integer.

d. $F_7F_9 - F_8^2 = 1$

e. $L_3L_5 - L_4^2 = -5$

7. Verify that $F_1 + F_2 + F_3 + F_4 + F_5 + F_6 = F_8 - 1$.

8. Verify that $F_1 + F_2 + F_3 + F_4 + F_5 + F_6 + F_7 + F_8 + F_9 + F_{10} = 11F_7$.

9. Show that:

a. When F_{13} is divided by F_8, the remainder is F_3.

b. When F_{15} is divided by F_8, the remainder is F_1.

3 • *The Golden Section and the Fibonacci Quadratic Equation*

Suppose that we are given a line segment $\overline{AB}$, and that we are to find a point C on it (between A and B) such that the length of the greater part is the mean proportional between the length of the whole segment and the length of the lesser part; that is, in Figure 1,

$$\frac{AB}{AC} = \frac{AC}{CB},$$

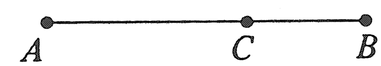


Figure 1

where $AB \neq 0$, $AC \neq 0$, and $CB \neq 0$.

We first find a positive numerical value for the ratio $\frac{AB}{AC}$. For convenience, let

$$x = \frac{AB}{AC} \qquad (x > 0).$$

Then

$$x = \frac{AB}{AC} = \frac{AC + CB}{AC} = 1 + \frac{CB}{AC} = 1 + \frac{1}{\frac{AC}{CB}} = 1 + \frac{1}{\frac{AB}{AC}} = 1 + \frac{1}{x}.$$

From

$$x = 1 + \frac{1}{x}$$

we obtain, by multiplying both members of the equation by x,

$$x^2 = x + 1, \text{ or}$$

(F) $$x^2 - x - 1 = 0.$$

The roots of this quadratic equation are (as you can verify, see Exercise 1, page 13)

$$\alpha = \frac{1 + \sqrt{5}}{2} \qquad \text{and} \qquad \beta = \frac{1 - \sqrt{5}}{2}.$$

(α is the Greek letter *alpha*, and β is the Greek letter *beta*.) You can verify by computation that $\alpha > 0$ and $\beta < 0$; $\alpha \doteq 1.618$ and $\beta \doteq -.618$ (Exercise 2). Thus, we take the positive root, α, as the value of the desired ratio:

$$\frac{AB}{AC} = \frac{1+\sqrt{5}}{2}$$

We can now use this numerical value to devise a method for locating C on $\overline{AB}$. Draw $\overline{BD}$ perpendicular to $\overline{AB}$ at B, but half its length. Draw $\overline{AD}$. Make $\overline{DE}$ the same length as $\overline{BD}$, and $\overline{AC}$ the same length as $\overline{AE}$. Then

$$AB = 2BD, \qquad ED = BD$$

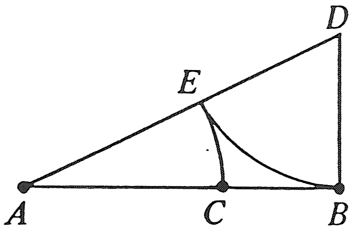


Figure 2

and, by the Pythagorean theorem,

$$AD = \sqrt{5}\, BD;$$

hence:

$$AC = AE = AD - ED = (\sqrt{5} - 1)BD$$

$$\frac{AB}{AC} = \frac{2BD}{(\sqrt{5}-1)BD} = \frac{2(\sqrt{5}+1)}{5-1} = \frac{\sqrt{5}+1}{2}$$

This computation verifies that the construction does indeed locate C on $\overline{AB}$ such that

$$\frac{AB}{AC} = \frac{1+\sqrt{5}}{2}.$$

Since α is a root of equation (F) on page 9, we have

$$\alpha^2 = \alpha + 1.$$

Multiplying both members of this equation by α^n (n can be any integer) yields

(A) $$\alpha^{n+2} = \alpha^{n+1} + \alpha^n.$$

If we let $u_n = \alpha^n, n \geq 1$, then $u_1 = \alpha$ and $u_2 = \alpha^2$, and we have the sequence

$$\alpha, \quad \alpha^2 = \alpha + 1, \quad \alpha^3 = \alpha^2 + \alpha, \quad \ldots,$$

which satisfies the recursive formula (R) on page 4. Similarly, we have

(B) $$\beta^{n+2} = \beta^{n+1} + \beta^n,$$

and the sequence

$$\beta, \quad \beta^2 = \beta + 1, \quad \beta^3 = \beta^2 + \beta, \quad \ldots$$

also satisfies (R).

You can easily verify (Exercise 3) that

$$\alpha + \beta = 1 \quad \text{and} \quad \alpha - \beta = \sqrt{5}.$$

If we now subtract the members of equation (B) from the members of equation (A) and divide each member of the resulting equation by $\alpha - \beta\,(=\sqrt{5} \neq 0)$, we find

$$\frac{\alpha^{n+2} - \beta^{n+2}}{\alpha - \beta} = \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} + \frac{\alpha^n - \beta^n}{\alpha - \beta}.$$

If we now let $u_n = \dfrac{\alpha^n - \beta^n}{\alpha - \beta}$, $n \geq 1$, then we have

$$u_{n+2} = u_{n+1} + u_n$$

and

$$u_1 = \frac{\alpha - \beta}{\alpha - \beta} = 1,$$

$$u_2 = \frac{\alpha^2 - \beta^2}{\alpha - \beta} = \frac{(\alpha - \beta)(\alpha + \beta)}{\alpha - \beta} = \frac{(\sqrt{5})(1)}{\sqrt{5}} = 1.$$

Thus, this sequence u_n is precisely the Fibonacci sequence defined in Section 2, and so

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \qquad n = 1, 2, 3, \ldots.$$

This is called the **Binet form** for the Fibonacci numbers after the French mathematician Jacques-Phillipe-Marie Binet (1786–1856).

Because of the relationship of the roots, α and β, of the equation (F),

$$x^2 - x - 1 = 0,$$

to the Fibonacci numbers, we shall call equation (F) the **Fibonacci quadratic equation.**

We shall call the positive root of (F),

$$\alpha = \frac{1 + \sqrt{5}}{2},$$

the **Golden Section.** [This is often represented by ϕ (Greek letter *phi*) or by some other symbol, but we shall continue to use α in this booklet.]

The point C in Figures 1 and 2, dividing $\overline{AB}$ such that

$$\frac{AB}{AC} = \alpha = \frac{1 + \sqrt{5}}{2},$$

is said to *divide $\overline{AB}$ in the Golden Section.*

Suppose that the rectangle $ABCD$ in Figure 3 is such that if the square $AEFD$ is removed from the rectangle, the lengths of the sides of the remaining rectangle, $BCFE$, have the same ratio as the lengths of the sides of the rectangle $ABCD$. That is,

$$\frac{BC}{EB} = \frac{AB}{DA}.$$

Then if $DA = AE = BC = x$ and $EB = y$, we have

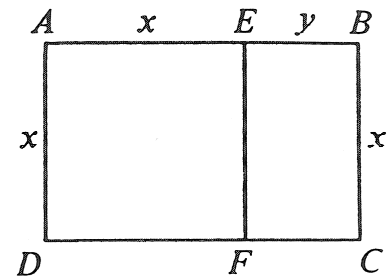


Figure 3

$$\frac{x}{y} = \frac{x+y}{x}, \quad \text{or} \quad \frac{x}{y} = 1 + \frac{y}{x}.$$

Multiplying both members of

$$\frac{x}{y} = 1 + \frac{y}{x}$$

by $\frac{x}{y}$, we find

$$\left(\frac{x}{y}\right)^2 = \frac{x}{y} + 1,$$

or

$$\left(\frac{x}{y}\right)^2 - \frac{x}{y} - 1 = 0,$$

which is in the form of equation (F), the variable now being $\left(\frac{x}{y}\right)$. Since x and y are positive, we seek the positive value of $\frac{x}{y}$. Thus,

$$\frac{x}{y} = \alpha = \frac{1+\sqrt{5}}{2}.$$

That is, the ratio of the length to the width for rectangle $BCFE$ (and also for rectangle $ABCD$) is the number α, the Golden Section. Such a rectangle is called a **Golden Rectangle.**

The proportions of the Golden Rectangle appear often throughout classical Greek art and architecture. As the German psychologists Gustav Theodor Fechner (1801–1887) and Wilhelm Max Wundt (1832–1920) have shown in a series of psychological experiments, most people do unconsciously favor "golden dimensions" when selecting pictures, cards, mirrors, wrapped parcels, and other rectangular objects. For some reason not fully known by either artists or psychologists, the Golden Rectangle holds great aesthetic appeal.

EXERCISES

1. Solve the Fibonacci quadratic equation,

$$x^2 - x - 1 = 0,$$

and verify that the roots are $\alpha = \dfrac{1 + \sqrt{5}}{2}$ and $\beta = \dfrac{1 - \sqrt{5}}{2}$ as stated in the text.

2. Verify that $\alpha > 0$ and $\beta < 0$, using $\sqrt{5} \doteq 2.236$. ($\doteq$ means "is approximately equal to.")

3. Verify that:

   a. $\alpha + \beta = 1$ b. $\alpha - \beta = \sqrt{5}$ c. $\alpha\beta = -1$

4. Verify that $\alpha = 1 + \dfrac{1}{\alpha}\cdot$

5. Using $\alpha^2 = \alpha + 1$, verify that:

   a. $\alpha^3 = 2\alpha + 1$ b. $\alpha^4 = 3\alpha + 2$ c. $\alpha^5 = 5\alpha + 3$

6. Verify that $L_3 = \alpha^3 - \dfrac{1}{\alpha^3}\cdot$

7. Verify that $F_4 = \dfrac{\alpha^4 - \alpha^{-4}}{\sqrt{5}}\cdot$

4 • *Some Geometry Related to the Golden Section*

We shall now consider several geometric problems and their solutions.

PROBLEM 1*

Suppose that we wish to remove from a rectangle, $ABCD$, three right triangles of equal area, $\triangle PAQ$, $\triangle QBC$, and $\triangle CDP$, as shown in Figure 4. How shall we locate points P and Q?

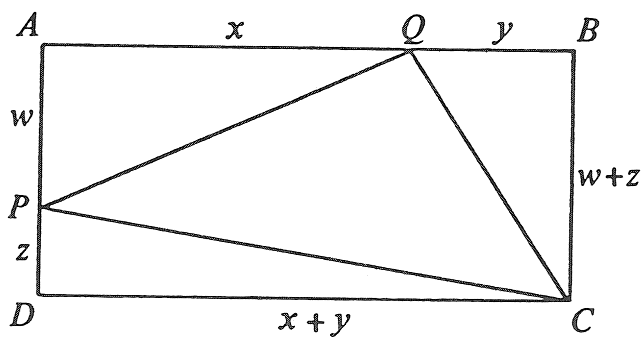


Figure 4

Solution. Let

$$AQ = x, \quad QB = y, \quad AP = w, \quad \text{and} \quad PD = z.$$

Then, since the areas of triangles PAQ, QBC, and CDP are to be equal, we have

$$\tfrac{1}{2}xw = \tfrac{1}{2}y(w + z) = \tfrac{1}{2}z(x + y),$$

or

$$xw = yw + yz = xz + yz.$$

* J. A. H. Hunter, "Triangle Inscribed in a Rectangle," *The Fibonacci Quarterly*, Vol. 1, No. 3 (October, 1963), page 66.

From

$$yw + yz = xz + yz$$

we have

$$yw = xz,$$

or

$$\frac{w}{z} = \frac{x}{y}.$$

Also, from

$$xw = y(w + z)$$

we have

$$\frac{x}{y} = \frac{w + z}{w} = 1 + \frac{z}{w} = 1 + \frac{1}{\frac{w}{z}}.$$

Since $\frac{w}{z} = \frac{x}{y}$, we have

$$\frac{x}{y} = 1 + \frac{1}{\frac{x}{y}},$$

or

$$\left(\frac{x}{y}\right)^2 - \frac{x}{y} - 1 = 0.$$

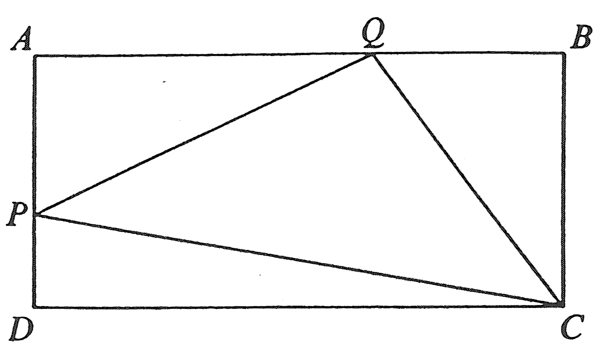
Again choosing the positive root, we have

$$\frac{x}{y} = \alpha = \frac{1 + \sqrt{5}}{2}.$$

But also

(A) $$\frac{w}{z} = \frac{x}{y} = \alpha,$$

and so the points P and Q must divide sides $\overline{AD}$ and $\overline{AB}$, respectively, in the Golden Section. Thus:



$$\frac{AP}{PD} = \frac{AQ}{QB} = \alpha$$

PROBLEM 2*

If the rectangle $ABCD$ in Problem 1 had been a Golden Rectangle, then the additional condition

(B) $$\frac{x+y}{w+z} = \alpha$$

would be imposed. Show that in this case $\triangle PQC$ would be an isosceles right triangle with the right angle at vertex Q.

Solution. From (A) on page 15 we have

$$x = \alpha y \qquad \text{and} \qquad w = \alpha z,$$

and so, from (B) above, we have

$$\alpha = \frac{x+y}{w+z} = \frac{\alpha y + y}{\alpha z + z} = \frac{(\alpha+1)y}{(\alpha+1)z}, \qquad \text{or} \qquad y = \alpha z.$$

But we had $w = \alpha z$, and so

$$w = y; \qquad \text{that is,} \qquad \overline{AP} \cong \overline{QB}.$$

Since we were given originally in Problem 1 that

$$\tfrac{1}{2}xw = \tfrac{1}{2}y(w+z),$$

the fact that $w = y$ implies that

$$x = w + z, \qquad \text{that is,} \qquad \overline{AQ} \cong \overline{BC}.$$

Therefore, right triangles PAQ and QBC are congruent. Thus,

$$\overline{PQ} \cong \overline{QC} \qquad \text{and} \qquad \angle AQP \cong \angle BCQ.$$

Moreover,

$$m^\circ\angle BCQ + m^\circ\angle CQB = 90.$$

Since

$$m^\circ\angle AQP + m^\circ\angle PQC + m^\circ\angle CQB = 180^\circ$$

and

$$m^\circ\angle AQP = m^\circ\angle BCQ,$$

we have

$$m^\circ\angle PQC = 90.$$

Therefore, since in $\triangle PQC$ we now have $\overline{PQ} \cong \overline{QC}$ and $m^\circ\angle PQC = 90$, $\triangle PQC$ is an isosceles right triangle, as pictured in Figure 5.

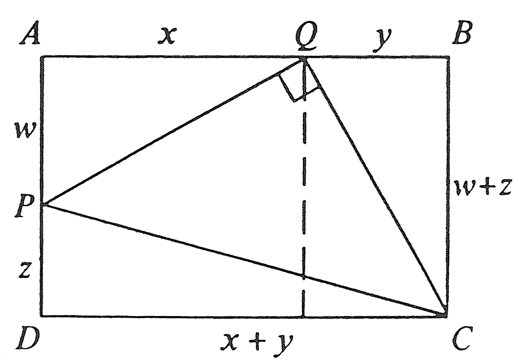


Figure 5

* H. E. Huntley, "Fibonacci Geometry," *The Fibonacci Quarterly*, Vol. 2, No. 2 (April, 1964), page 104.

PROBLEM 3

Do two triangles exist which have measures of five of their six parts (three angles and three sides) equal and yet are *not* congruent? Your first impulsive answer may be a resounding No! However, we propose to show that this is indeed possible.

Solution. Clearly, if among the five parts are three sides, then the triangles must be congruent. The only possibility then is to have the three angles of one triangle congruent to the three angles of the other triangle (thus, the triangles are similar) and two sides of one triangle congruent to two sides of the other. (Notice that it is *not* specified that these sides be corresponding sides.) One example of such a pair of triangles (in this case with integral sides) is shown in Figure 6, where

$$\frac{27}{18} = \frac{18}{12} = \frac{12}{8}.$$

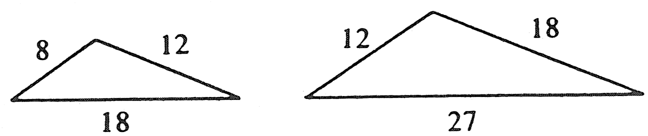


Figure 6

Let us see how to find pairs of triangles having just five parts congruent. First, we know that the triangles must be similar; that is, the measures of the sides must be related as shown in Figure 7, where r is the ratio of similarity:

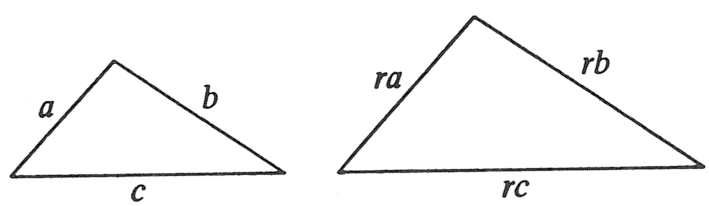


Figure 7

The additional conditions are

$$b = ra \quad \text{and} \quad c = rb = r^2a.$$

Thus, the measures of the sides of the two triangles will be

$$a, \ ra, \ r^2a \quad \text{and} \quad ra, \ r^2a, \ r^3a,$$

as shown in Figure 8:

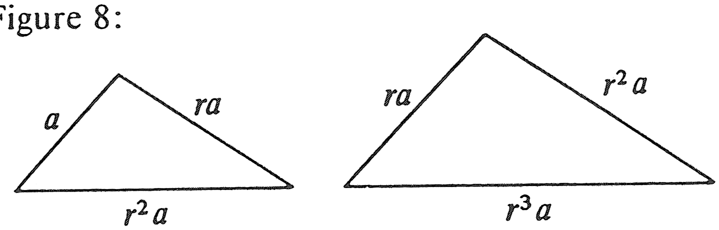


Figure 8

If $a = 8$ and $r = \frac{3}{2}$, you will find the measures of the sides of the triangles shown in Figure 6. Try other values to find other triangles having this property.

PROBLEM 4

If $a = 4$ and $r = 2$ in Problem 3, then the measures of the sides of the triangles would be

$$4, \quad 8, \quad 16 \qquad \text{and} \qquad 8, \quad 16, \quad 32.$$

Can there be triangles with sides of these measures? No, because

$$4 + 8 < 16 \qquad \text{and} \qquad 8 + 16 < 32.$$

What restrictions must be placed on the values of r, assuming for the present that $r > 1$?

Solution. If a, ra, and r^2a are to be the measures of the sides of a triangle, then the sum of each two must be greater than the third. Thus, we have three inequalities:

$$\begin{array}{lll} \text{(i)} \quad a + ra > r^2a & \text{or, since } a > 0, & 1 + r > r^2 \\ \text{(ii)} \quad ra + r^2a > a & \text{or, since } a > 0, & r + r^2 > 1 \\ \text{(iii)} \quad r^2a + a > ra & \text{or, since } a > 0, & r^2 + 1 > r \end{array}$$

We are looking for the solution set of these three inequalities.

On the assumption that $r > 1$, we have

$$r^2 > r > 1,$$

and inequalities (ii) and (iii) hold. Thus, we need to consider inequality (i), which we shall write as

$$r^2 < r + 1, \qquad \text{or} \qquad r^2 - r - 1 < 0.$$

Recall that

$$x^2 - x - 1 = 0$$

is the Fibonacci quadratic equation with roots

$$\alpha = \frac{1 + \sqrt{5}}{2} \qquad \text{and} \qquad \beta = \frac{1 - \sqrt{5}}{2}.$$

Therefore, we can write

$$r^2 - r - 1 = \left(r - \frac{1 + \sqrt{5}}{2}\right)\left(r - \frac{1 - \sqrt{5}}{2}\right).$$

For the second factor, we have

$$r - \frac{1 - \sqrt{5}}{2} = r - \frac{1}{2} + \frac{\sqrt{5}}{2},$$

and this is positive for $r > 1$.

Therefore, to have

$$r^2 - r - 1 < 0$$

we must have

$$r - \frac{1+\sqrt{5}}{2} < 0, \qquad \text{or} \qquad r < \frac{1+\sqrt{5}}{2},$$

that is, $r < \alpha$, and so if $r > 1$, we must have

$$1 < r < \alpha$$

in order to have a pair of triangles with just five parts congruent.

PROBLEM 5

Can a pair of right triangles have just five parts congruent?

Solution. Suppose that $r > 1$. Then r^2a is the measure of the longest side. If the triangles are to be right triangles, we have by the Pythagorean theorem that

$$(r^2a)^2 = a^2 + (ra)^2.$$

Thus,

$$r^4a^2 - r^2a^2 - a^2 = 0$$

or, since $a \neq 0$,

$$r^4 - r^2 - 1 = 0.$$

Therefore,

$$r^2 = \alpha,$$

and so the positive value of r in this case is $\sqrt{\alpha}$.

How can we construct such a pair of right triangles? Recall that in a right triangle, the altitude from the vertex of the right angle to the hypotenuse separates the given triangle into two triangles that are similar to each other and also to the given triangle. That is, in Figure 9, where angle C is a right angle,

$$\triangle ACD \sim \triangle CBD \sim \triangle ABC.$$

Notice that $\triangle ACD$ and $\triangle CBD$ are similar and have one side, $\overline{CD}$, in common. If we had $\overline{AD}$ congruent to $\overline{BC}$, we would have two right triangles that have just five parts congruent. These are shown as

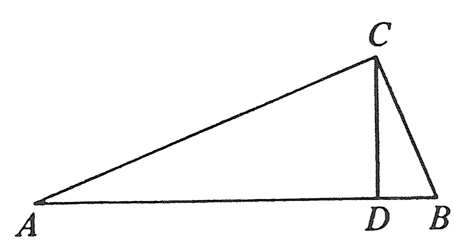


Figure 9

$$\triangle A'C'D' \qquad \text{and} \qquad \triangle C'B'D'$$

in Figure 10 on the next page.

If we use the letters shown in Figure 10 to represent the measures of the sides, we have

$$\frac{x+y}{x} = \frac{z}{w} = \frac{x}{y}.$$

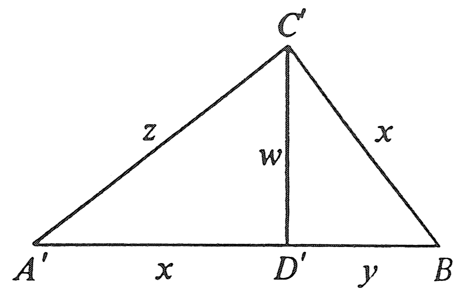


Figure 10

From

$$\frac{x}{y} = \frac{x+y}{x}$$

we have

$$\left(\frac{x}{y}\right)^2 - \frac{x}{y} - 1 = 0, \qquad \text{and so} \qquad \frac{x}{y} = \alpha.$$

Thus, the point D' must divide $\overline{A'B'}$ in the Golden Section.

Let us find the ratio of similarity for $\triangle A'C'D'$ and $\triangle C'B'D'$, that is, the value of

$$r = \frac{w}{y} = \frac{x}{w} = \frac{z}{x}.$$

Since $x = \alpha y$, we have

$$\frac{w}{y} = \frac{\alpha y}{w}, \qquad \text{or} \qquad \frac{w^2}{y^2} = \alpha.$$

Therefore, since $r > 0$,

$$r = \frac{w}{y} = \sqrt{\alpha},$$

as predicted by our computation on page 19.

Some of the possible shapes for triangles having just five parts congruent are shown in Figure 11. Those sketched are right and oblique triangles. In order for such triangles to have only acute angles, we must have

$$1 < r < \sqrt{\alpha}, \qquad \sqrt{\alpha} \doteq 1.27.$$

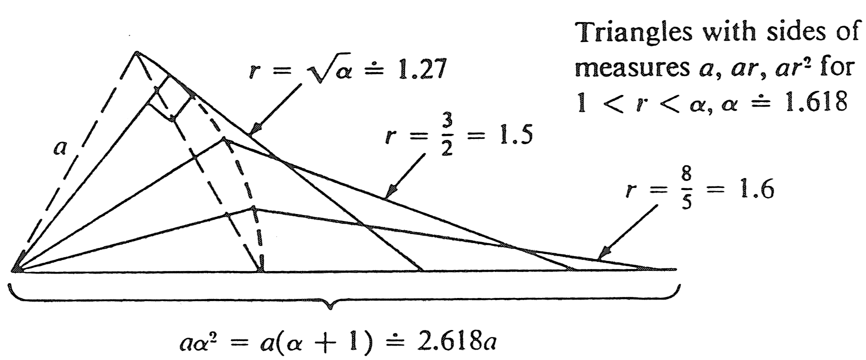


Figure 11

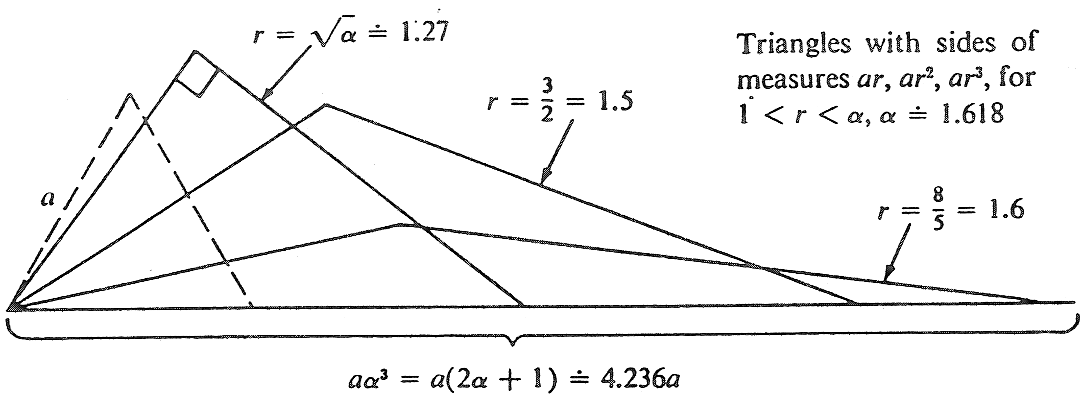


Figure 11 (continued)

Up to now we have restricted ourselves to $r > 1$. If $r > 1$ is the ratio of similarity of the larger triangle to the smaller triangle, then

$$r' = \frac{1}{r}$$

is the ratio of similarity of the smaller triangle to the larger triangle. Thus, in general, we may have

$$\frac{1}{\alpha} < r < 1 \qquad \text{as well as} \qquad 1 < r < \alpha,$$

that is, approximately

$$.618 < r < 1 \qquad \text{or} \qquad 1 < r < 1.618.$$

The triangles pictured in Figure 12 have ratios that are the reciprocals of those for the triangles in Figure 11. Notice that each pair of triangles in Figure 11 is similar to the pair in Figure 12 that has the reciprocal ratio (compare Exercise 6 on page 25). (If r were 1, then the triangles would be congruent equilateral triangles.)

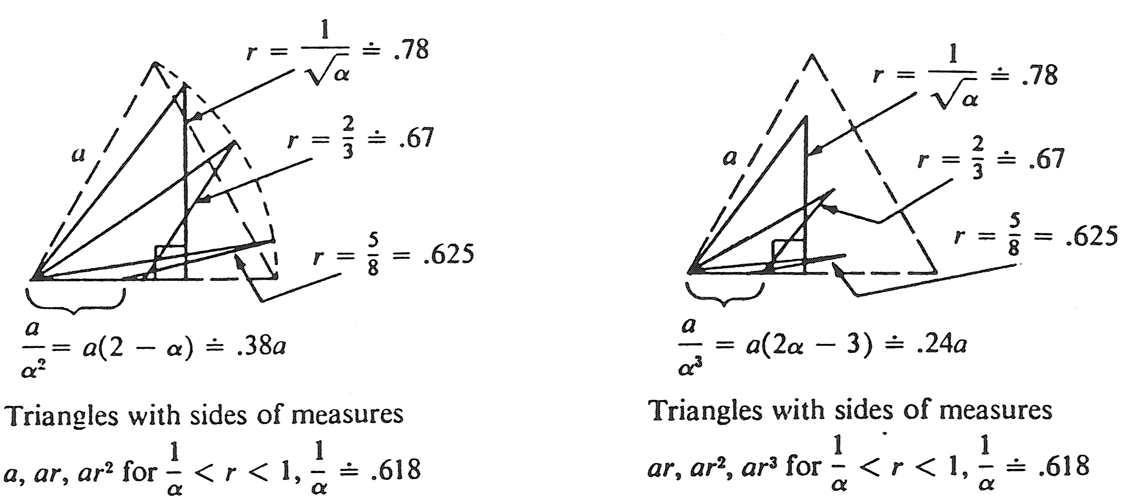


Figure 12

In Section 3 we defined a Golden Rectangle. We shall now define a Golden Triangle. In Figure 10, the ratio of the area of $\triangle A'B'C'$ to the area of $\triangle A'C'D'$ can be found as follows:

$$\text{Area } \triangle A'B'C' = \tfrac{1}{2}w(x+y)$$

$$\text{Area } \triangle A'C'D' = \tfrac{1}{2}wx$$

$$\frac{\text{Area } \triangle A'B'C'}{\text{Area } \triangle A'C'D'} = \frac{x+y}{x} = \frac{\alpha y + y}{\alpha y} = \frac{\alpha+1}{\alpha} = \frac{\alpha^2}{\alpha} = \alpha$$

A triangle that has this property is called a **Golden Triangle**; that is, a **Golden Triangle** is one such that when a triangle similar to it is removed from it, the ratio of the area of the Golden Triangle to the area of the *remaining triangle* is α. That is, in Figure 10 when $\triangle C'B'D'$ is removed from $\triangle A'B'C'$,

$$\frac{\text{Area } \triangle A'B'C'}{\text{Area } \triangle A'C'D'} = \alpha.$$

We also note that

$$\frac{\text{Area } \triangle A'C'D'}{\text{Area } \triangle C'B'D'} = \frac{\frac{1}{2}wx}{\frac{1}{2}wy} = \frac{x}{y} = \alpha,$$

and

$$\frac{\text{Area } \triangle A'B'C'}{\text{Area } \triangle C'B'D'} = \frac{\text{Area } \triangle A'B'C'}{\text{Area } \triangle A'C'D'} \cdot \frac{\text{Area } \triangle A'C'D'}{\text{Area } \triangle C'B'D'} = \alpha \cdot \alpha = \alpha^2.$$

PROBLEM 6

Show that an isosceles triangle with vertex angle measuring 36° is a Golden Triangle.

Solution. The base angles measure 72° each. If one of these base angles is bisected (see Figure 13), two isosceles triangles are formed. One triangle, $\triangle CDB$, is similar to the given triangle, $\triangle ABC$, while the other, $\triangle ACD$, is not.

In $\triangle ABC$ and $\triangle CDB$,

$$\frac{x+y}{x} = \frac{x}{y},$$

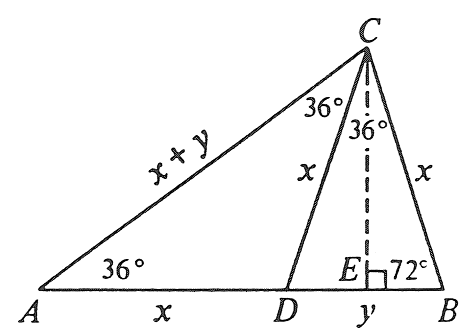


Figure 13

and again we have

$$\left(\frac{x}{y}\right)^2 - \frac{x}{y} - 1 = 0$$

and the positive result is

$$\frac{x}{y} = \alpha.$$

To find the areas of $\triangle ABC$ and $\triangle ADC$, draw altitude $\overline{CE}$. Then:

$$CE = \sqrt{x^2 - \frac{y^2}{4}} \qquad (CE \neq 0)$$

$$\text{Area } \triangle ABC = \tfrac{1}{2}(x + y)(CE)$$

$$\text{Area } \triangle ADC = \tfrac{1}{2}x(CE)$$

$$\frac{\text{Area } \triangle ABC}{\text{Area } \triangle ADC} = \frac{x+y}{x} = \alpha.$$

Thus, $\triangle ABC$ is a Golden Triangle.

We also note that

$$\frac{\text{Area } \triangle ADC}{\text{Area } \triangle CDB} = \frac{x}{y} = \alpha$$

and

$$\frac{\text{Area } \triangle ABC}{\text{Area } \triangle CDB} = \frac{x+y}{y} = \frac{x+y}{x} \cdot \frac{x}{y} = \alpha^2.$$

Since the central angle of a regular decagon is 36° (see Figure 14), we know from Problem 6 that the ratio of the radius r to the measure s of the side of an inscribed decagon is α.

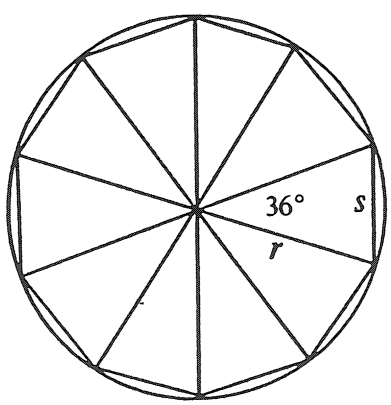


Figure 14

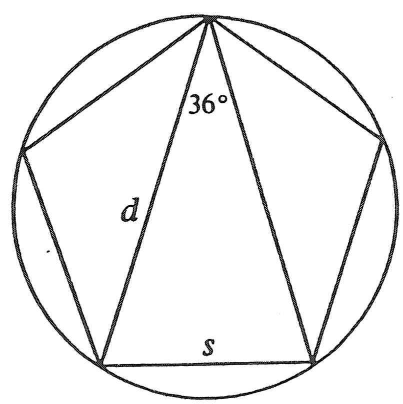


Figure 15

Also, in a regular inscribed pentagon, the angle between two adjacent diagonals at one vertex is 36° (see Figure 15), and so the ratio of the measure d of a diagonal to the measure s of a side is also α.

Notice that in Figure 13, $\dfrac{AB}{BC} = \alpha$ and that in Figure 10, $\dfrac{A'B'}{B'C'} = \alpha$. Thus, in each case the ratio of the measure of the longest side to the measure of the shortest side is α.

The Golden Triangle appears on pages 61–62 of Tobias Dantzig's *The Bequest of the Greeks* (New York: Charles Scribner's Sons, 1955) and also on page 42 of N. N. Vorobyov's *The Fibonacci Numbers* (Boston: D. C. Heath and Company, 1963). Also, see the article, "Golden Triangles, Rectangles, and Cuboids," by Marjorie Bicknell and Verner E. Hoggatt, Jr., in *The Fibonacci Quarterly*, Vol. 7, No. 1 (February, 1969), pages 73–91.

PROBLEM 7

Inscribe a square in a semicircle.

Solution. Figure 16 shows the completed construction where $AMNB$ is a square. The construction makes use of the fact that right triangles

$$OAM, \quad ODC, \quad OEF, \quad \text{and} \quad OBN$$

are similar.

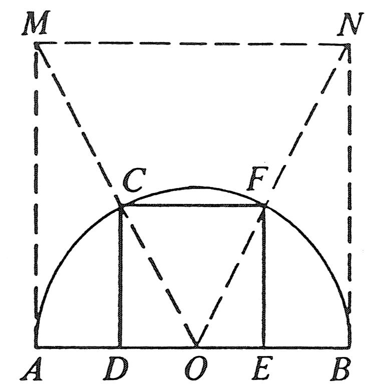


Figure 16

Now consider Figure 17, in which $\overline{AF}$ and $\overline{FB}$ have been drawn, forming similar right triangles ABF, AFE, and FBE. Thus,

$$\frac{t+s}{s} = \frac{s}{t},$$

from which we obtain

$$\left(\frac{s}{t}\right)^2 - \frac{s}{t} - 1 = 0,$$

and so the positive result is

$$\frac{s}{t} = \alpha.$$

Thus, point E divides $\overline{DB}$ in the Golden Section.

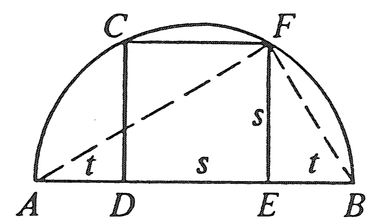


Figure 17

Two articles by Marvin Holt, which give further excellent material on the Golden Section and geometry, are "Mystery Puzzler and Phi" in *The Fibonacci Quarterly*, Vol. 3, No. 2 (April, 1965), pages 135–138, and "The Golden Section" in the *Pentagon*, Spring, 1964, pages 80–104.

The Golden Cuboid is discussed in an article of that title by H. Huntley in *The Fibonacci Quarterly*, Vol. 2, No. 3 (October, 1964), page 184.

Another interesting reference is *Patterns in Space* by Colonel R. S. Beard (available from Brother Alfred Brousseau, St. Mary's College, California 94575). This book describes many appearances of the Golden Section in variations of the regular solids.

EXERCISES

1. Show that there can be no triangle having three distinct Fibonacci numbers as measures of its sides.

2. Show that a pair of triangles which have the measures of five parts equal, but which are not congruent, cannot be isosceles.

3. Show that

a. if $a < b$, then $y = (x - a)(x - b)$ is negative for $a < x < b$ and non-negative otherwise;

b. if $a = b$, then $y = (x - a)^2 \geq 0$ for all x.

4. Using the results of Exercise 3a, show that, in general,

a. inequality (i) on page 18 holds when

$$r^2 - r - 1 < 0, \text{ that is, for } \beta < r < \alpha.$$

b. inequality (ii) on page 18 holds when

$$r^2 + r - 1 > 0, \text{ that is, for } r < -\alpha \text{ or } r > -\beta.$$

c. inequalities (i) and (ii) both hold when

$$-\beta < r < \alpha, \qquad \text{or} \qquad \frac{1}{\alpha} < r < \alpha,$$

since $\alpha\beta = -1$ (Exercise 3c, page 13).

5. Using the result of Exercise 3b, show that inequality (iii) on page 18 holds for $r > 0$, that is, that

$$r^2 + 1 \geq 2r > r \quad \text{for} \quad r > 0.$$

6. The measures of the sides of one triangle are a, ar, and ar^2 and those of a second triangle are a, $\frac{a}{r}$, and $\frac{a}{r^2}$. By suitably pairing the sides, show that the triangles are similar and find the ratio of similarity.

7. In Figure 16 extend $\overline{CF}$ to meet $\overline{BN}$ in point G. Show that rectangle $DCGB$ is a Golden Rectangle.

8. In Figure 17 show that $\frac{AF}{FB} = \alpha$.

9. In Figure 17 show that:

a. $\frac{s^2 + 2st}{s^2 + t^2} = \alpha$ **b.** $\frac{t^2 - 2st}{s^2 + t^2} = \beta$ **c.** $\frac{s^2 + 4st - t^2}{s^2 + t^2} = \sqrt{5}$

Hint: Recall that $\alpha^2 = \alpha + 1$, $\alpha + \beta = 1$, $\alpha - \beta = \sqrt{5}$.

5 • *Some Fibonacci Algebra*

Recall from Section 3 that $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$ are the roots of

(F) $$x^2 - x - 1 = 0,$$

and so $\alpha^2 = \alpha + 1$ and $\beta^2 = \beta + 1$. Also, $\alpha + \beta = 1$ and $\alpha - \beta = \sqrt{5}$. Moreover,

(A) $$\alpha^{n+2} = \alpha^{n+1} + \alpha^n$$

and

(B) $$\beta^{n+2} = \beta^{n+1} + \beta^n$$

and by using these equations, we found that the Fibonacci numbers can be expressed in the so-called Binet form:

(C) $$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \frac{\alpha^n - \beta^n}{\sqrt{5}}, \qquad n = 1, 2, 3, \ldots$$

Now suppose that we add the members of equation (B) to the members of equation (A), giving

$$(\alpha^{n+2} + \beta^{n+2}) = (\alpha^{n+1} + \beta^{n+1}) + (\alpha^n + \beta^n).$$

If we let $u_n = \alpha^n + \beta^n$, then we have

$$u_{n+2} = u_{n+1} + u_n$$

and

$$u_1 = \alpha + \beta = 1,$$
$$u_2 = \alpha^2 + \beta^2 = \alpha + 1 + \beta + 1 = (\alpha + \beta) + 2 = 1 + 2 = 3.$$

Thus, this sequence u_n is the sequence of Lucas numbers defined in Section 2, and so we have a Binet form for the Lucas numbers:

(D) $$L_n = \alpha^n + \beta^n, \qquad n = 1, 2, 3, \ldots$$

Now look at the following comparison of the Fibonacci numbers and the Lucas numbers:

F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9	F_{10}	...
1	1	2	3	5	8	13	21	34	55	...
1	3	4	7	11	18	29	47	76	123	...
L_1	L_2	L_3	L_4	L_5	L_6	L_7	L_8	L_9	L_{10}	...

Notice that

$$F_1 + F_3 = L_2, \quad F_2 + F_4 = L_3, \quad \text{and so on.}$$

It can be proved (Exercise 2, page 29) that in general

$$L_n = F_{n-1} + F_{n+1},$$

from which, since $F_{n+1} = F_n + F_{n-1}$, it follows that

(E) $$L_n = F_n + 2F_{n-1}.$$

You can verify this latter statement for specific examples; that is, you can show that $L_6 = F_6 + 2F_5$, and so on.

We now have F_n and L_n expressed in terms of α^n and β^n. We can also find α^n and β^n in terms of F_n and L_n. If we note that $\alpha - \beta = \sqrt{5}$, then, from the Binet forms, we have

$$\sqrt{5}\, F_n = \alpha^n - \beta^n,$$
$$L_n = \alpha^n + \beta^n.$$

Adding, we find

$$2\alpha^n = L_n + \sqrt{5}\, F_n$$

or

$$\alpha^n = \frac{L_n + \sqrt{5}\, F_n}{2};$$

subtracting, we find

$$\beta^n = \frac{L_n - \sqrt{5}\, F_n}{2}.$$

Recall that in Section 2 we had occasion to define F_0 as $F_2 - F_1$ (page 6). Similarly, we can define L_0 as $L_2 - L_1 = 3 - 1 = 2$. (Notice that this agrees with the definition by the Binet form, since $\alpha^0 + \beta^0 = 1 + 1 = 2$.) Since $L_1 = \alpha + \beta = 1$, we can now write expression (E) for L_n (above) as

(E′) $$L_n = L_1F_n + L_0F_{n-1}.$$

The method of defining F_0 and L_0 suggests that we can also define F_{-1}, L_{-1}, and so on, by applying the formulas

$$F_{n-1} = F_{n+1} - F_n$$

and

$$L_{n-1} = L_{n+1} - L_n$$

repeatedly. Thus, we have:

...	F_{-4}	F_{-3}	F_{-2}	F_{-1}	F_0	F_1	F_2	F_3	F_4	...
...	−3	2	−1	1	0	1	1	2	3	...
...	7	−4	3	−1	2	1	3	4	7	...
...	L_{-4}	L_{-3}	L_{-2}	L_{-1}	L_0	L_1	L_2	L_3	L_4	...

We can derive a formula for F_{-n}, $n > 0$, by assuming that the Binet form also holds for negative values of the exponents (compare derivation on pages 10–11):

$$F_{-n} = \frac{\alpha^{-n} - \beta^{-n}}{\alpha - \beta} = \frac{\left(\frac{1}{\alpha}\right)^n - \left(\frac{1}{\beta}\right)^n}{\alpha - \beta}$$

Since $\alpha\beta = -1$ (Exercise 3c, page 13), we have

$$\frac{1}{\alpha} = -\beta \quad \text{and} \quad \frac{1}{\beta} = -\alpha.$$

Therefore

$$F_{-n} = \frac{(-\beta)^n - (-\alpha)^n}{\alpha - \beta} = \frac{(-1)^n(\beta^n - \alpha^n)}{\alpha - \beta} = (-1)^{n+1}\left(\frac{\alpha^n - \beta^n}{\alpha - \beta}\right),$$

and so

$$F_{-n} = (-1)^{n+1}F_n.$$

Similarly, you can show (Exercise 3, page 29) that for $n > 0$,

$$L_{-n} = (-1)^n L_n.$$

Now suppose that we compute the first 14 successive ratios $\frac{F_{n+1}}{F_n}$ and $\frac{L_{n+1}}{L_n}$. The values of the successive ratios as shown at the top of the next page suggest that in both cases the value of the ratio becomes closer and closer to α as we take larger and larger values of n. However, we shall not undertake to prove this here. We can also observe that the first Fibonacci ratio is less than α, the second is greater than α, and so on, while the first Lucas ratio is greater than α, the second is less than α, and so on. Moreover,

$$\frac{F_2}{F_1} < \alpha < \frac{L_2}{L_1}, \qquad \frac{F_3}{F_2} > \alpha > \frac{L_3}{L_2}, \qquad \text{and so on.}$$

$\frac{F_{n+1}}{F_n}$	$\frac{L_{n+1}}{L_n}$
$\frac{1}{1} = 1.0000$	$\frac{3}{1} = 3.0000$
$\frac{2}{1} = 2.0000$	$\frac{4}{3} \doteq 1.3333$
$\frac{3}{2} = 1.5000$	$\frac{7}{4} = 1.7500$
$\frac{5}{3} \doteq 1.6667$	$\frac{11}{7} \doteq 1.5714$
$\frac{8}{5} = 1.6000$	$\frac{18}{11} \doteq 1.6363$
$\frac{13}{8} = 1.6250$	$\frac{29}{18} \doteq 1.6111$
$\frac{21}{13} \doteq 1.6154$	$\frac{47}{29} \doteq 1.6207$
$\frac{34}{21} \doteq 1.6190$	$\frac{76}{47} \doteq 1.6170$
$\frac{55}{34} \doteq 1.6176$	$\frac{123}{76} \doteq 1.6184$
$\frac{89}{55} \doteq 1.6182$	$\frac{199}{123} \doteq 1.6179$
$\frac{144}{89} \doteq 1.6180$	$\frac{322}{199} \doteq 1.6181$
$\frac{233}{144} \doteq 1.6181$	$\frac{521}{322} \doteq 1.6180$
$\frac{377}{233} \doteq 1.6180$	$\frac{843}{521} \doteq 1.6180$
$\frac{610}{377} \doteq 1.6180$	$\frac{1364}{843} \doteq 1.6180$

$$\alpha \doteq 1.61803398875\ldots$$

EXERCISES

Using the Binet form:

1. Show that $F_{2n} = F_n L_n$, $n \geq 1$.
2. Show that $L_n = F_{n-1} + F_{n+1}$, $n \geq 1$. *Hint:* Use $\alpha\beta = -1$.
3. Show that $L_{-n} = (-1)^n L_n$ for $n > 0$.
4. Show that $5F_n^2 = L_{2n} - 2(-1)^n$.
5. Show that $L_n^2 = L_{2n} + 2(-1)^n$.
6. Show that $F_{n+1}L_n - L_{n+1}F_n = 2(-1)^n$.

Using previously found results:

7. Show that $\dfrac{F_{n+1}}{F_n} - \dfrac{L_{n+1}}{L_n} = \dfrac{2(-1)^n}{F_{2n}}$.
8. Show that $F_{-n} = (-1)^{n+1}F_n$ holds also when $n < 0$.
9. Show that $L_{-n} = (-1)^n L_n$ holds also when $n < 0$.
10. Show that $5F_n^2 = L_n^2 - 4(-1)^n$.
11. Show from $L_n = F_{n+1} + F_{n-1}$ that $L_n = F_{n+2} - F_{n-2}$.

6 • Shortcuts to Large F_n and L_n

Since we shall be dealing extensively with inequalities in this section, we shall recall the following propositions.

For real numbers a, b, c, and d, we have:

(a) If $a < b$ and $c > 0$, then $ac < bc$ and $\frac{a}{c} < \frac{b}{c}$.

(b) If $a < b$ and $c < 0$, then $ac > bc$ and $\frac{a}{c} > \frac{b}{c}$.

(c) If $a < b$, then $a + c < b + c$ and $a - c < b - c$.

(d) If $0 < \frac{a}{b} < \frac{c}{d}$, then $\frac{b}{a} > \frac{d}{c}$, and conversely.

(e) If $\frac{a}{b} < \frac{c}{d}$ and $\frac{c}{d} < \frac{e}{f}$, then $\frac{a}{b} < \frac{e}{f}$.

(f) If $|b| < 1$, then $|b|^n < 1$ for $n = 1, 2, 3, \ldots$.

(g) If $a = b + c$ and $c > 0$, then $b < a$.

We shall also use an idea suggested by the following problem. Suppose that we have s pounds of sugar (s not necessarily an integer), and we ask how many one-pound sacks of sugar can be made from this quantity. Then we are interested in finding the greatest integer not greater than s. We denote this integer by $[s]$. Thus,

$$[7.2] = 7 \qquad \text{and} \qquad [7.9] = 7.$$

Similarly, we have

$$[-5.4] = -6 \qquad \text{and} \qquad [\tfrac{1}{2}] = 0.$$

We are now able to devise methods for finding values of F_n and L_n (or at least estimates of them) without doing all the additions from the beginning. The first theorem that we shall prove is the following.

THEOREM I

$$F_n = \left[\frac{\alpha^n}{\sqrt{5}} + \frac{1}{2}\right] \quad \text{for} \quad n = 1, 2, 3, \ldots .$$

Proof. We have the Binet form

$$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}}, \qquad n = 1, 2, 3, \ldots,$$

where $\alpha = \dfrac{1 + \sqrt{5}}{2} \doteq \dfrac{1 + 2.236}{2} \doteq 1.618$ and $\beta = \dfrac{1 - \sqrt{5}}{2} \doteq -.618.$
We can write

$$F_n = \frac{\alpha^n}{\sqrt{5}} - \frac{\beta^n}{\sqrt{5}} = \left(\frac{\alpha^n}{\sqrt{5}} + \frac{1}{2}\right) - \left(\frac{1}{2} + \frac{\beta^n}{\sqrt{5}}\right).$$

Since $0 < |\beta| < 1$, we find by (f) on page 30 that

$$0 < |\beta|^n < 1.$$

Since $1 < \dfrac{\sqrt{5}}{2}$, we find by (e) that

$$0 < |\beta|^n < \frac{\sqrt{5}}{2}.$$

Since $\sqrt{5} > 0$, we find by (a) that

(A) $$0 < \frac{|\beta|^n}{\sqrt{5}} < \frac{1}{2}.$$

Then by (c) we find (by adding $\frac{1}{2}$) that

$$\frac{1}{2} < \frac{1}{2} + \frac{|\beta|^n}{\sqrt{5}} < 1.$$

Although $\beta < 0$, we have $|\beta|^n = \beta^n$ when n is even, and so

(B) $$\text{if } n \text{ is even,} \quad \frac{1}{2} < \frac{1}{2} + \frac{\beta^n}{\sqrt{5}} < 1.$$

On the other hand, we have $|\beta|^n = -\beta^n$ when n is odd, and so from (A),

$$-\frac{1}{2} < \frac{\beta^n}{\sqrt{5}} < 0.$$

Then by (c) we find (by adding $\frac{1}{2}$) that

(C) $$\text{if } n \text{ is odd,} \quad 0 < \frac{1}{2} + \frac{\beta^n}{\sqrt{5}} < \frac{1}{2}.$$

Thus, from (B) and (C) we have, in general,

$$0 < \frac{1}{2} + \frac{\beta^n}{\sqrt{5}} < 1.$$

But we saw on page 31 that F_n can be expressed as

$$F_n = \left(\frac{\alpha^n}{\sqrt{5}} + \frac{1}{2}\right) - \left(\frac{1}{2} + \frac{\beta^n}{\sqrt{5}}\right);$$

and so, since we have shown (page 31) that $\left(\frac{1}{2} + \frac{\beta^n}{\sqrt{5}}\right)$ is positive and less than 1, we have

$$F_n < \frac{\alpha^n}{\sqrt{5}} + \frac{1}{2} < F_n + 1,$$

or

$$F_n = \left[\frac{\alpha^n}{\sqrt{5}} + \frac{1}{2}\right] \quad \text{for} \quad n = 1, 2, 3, \ldots,$$

and the theorem is proved.

Similarly, the following theorem can be proved, but we shall not give the proof here.

THEOREM II

$$L_n = [\alpha^n + \tfrac{1}{2}] \qquad \text{for } n = 2, 3, 4, \ldots.$$

It is shown in *The Fibonacci Numbers* by N. N. Vorobyov* that F_n is the integer nearest to $\frac{\alpha^n}{\sqrt{5}}$, that is,

$$\left|F_n - \frac{\alpha^n}{\sqrt{5}}\right| < \frac{1}{2}.$$

Using this result, F_n can be computed if logarithms to a sufficient number of places are used. A similar development can be given to show that

$$|L_n - \alpha^n| < \tfrac{1}{2}.$$

PROBLEM 1

Find $F_{16} \doteq \frac{\alpha^{16}}{\sqrt{5}}$.

Solution.
$$\log \frac{\alpha^{16}}{\sqrt{5}} \doteq 16 \log \alpha - \log 2.236$$

Since we are going to multiply $\log \alpha$ by 16, we should find $\log \alpha$ to more

* For one translation see the reference on page 23.

decimal places than we plan to use for the remainder of the computation.

$$\alpha = \frac{1+\sqrt{5}}{2}$$

$$\doteq \frac{1+2.23607}{2} \doteq 1.6180$$

From a five-place table of common, or base 10, logarithms we find

$$\log \alpha \doteq 0.20898.$$

Thus, we have:

$$\log \frac{\alpha^{16}}{\sqrt{5}} \doteq 16(0.20898) - 0.3494$$

$$\doteq 3.3437 - 0.3494 \doteq 2.9943$$

$$\frac{\alpha^{16}}{\sqrt{5}} \doteq 987.0$$

Therefore, $F_{16} = 987$. You can check this answer in the list on page 83.

For larger indices we can find only the first three digits accurately if we are using four-place logarithms.

PROBLEM 2

Estimate F_{30}.

Solution.

$$\log \frac{\alpha^{30}}{\sqrt{5}} \doteq 30(0.20898) - 0.3494 \doteq 5.9200$$

$$\frac{\alpha^{30}}{\sqrt{5}} \doteq 831800$$

Therefore, $F_{30} \doteq 832{,}000$. You can find the exact value in the list on page 83.

PROBLEM 3

Find $L_{14} \doteq \alpha^{14}$.

Solution.

$$\log \alpha^{14} \doteq 14(0.20898) \doteq 2.9257$$

$$\alpha^{14} \doteq 842.8$$

Therefore, $L_{14} = 843$. You can check this value in the list on page 83.

We shall now develop methods for finding F_{n+1} from F_n and L_{n+1} from L_n *even if we do not know the value of n.* To achieve this, we shall prove the following new theorem.

THEOREM III

$$F_{n+1} = [\alpha F_n + \tfrac{1}{2}], \qquad n = 2, 3, 4, \ldots.$$

Proof. Since

$$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}},$$

we have

$$\alpha F_n = \frac{\alpha^{n+1} - \alpha\beta^n}{\sqrt{5}} = \frac{\alpha^{n+1} - \alpha\beta^n - \beta^{n+1} + \beta^{n+1}}{\sqrt{5}}$$

$$= \frac{\alpha^{n+1} - (\alpha\beta)\beta^{n-1} - \beta^{n+1} + \beta^{n+1}}{\sqrt{5}}.$$

Since $\alpha\beta = -1$, we have

$$\alpha F_n = \frac{\alpha^{n+1} - \beta^{n+1} + \beta^{n+1} + \beta^{n-1}}{\sqrt{5}}$$

$$= F_{n+1} + \beta^{n-1}\left(\frac{\beta^2 + 1}{\sqrt{5}}\right)$$

But

$$\beta^2 + 1 = \beta + 2 = \frac{1 - \sqrt{5} + 4}{2} = \frac{5 - \sqrt{5}}{2} = \sqrt{5}\left(\frac{\sqrt{5} - 1}{2}\right) = -\sqrt{5}\,\beta.$$

Therefore,

$$\alpha F_n = F_{n+1} + \beta^{n-1}(-\beta) = F_{n+1} - \beta^n,$$

and

(A) $$\alpha F_n + \tfrac{1}{2} = F_{n+1} + (\tfrac{1}{2} - \beta^n).$$

Now, since $|\beta| < .62$, we have $|\beta|^2 < \frac{1}{2}$, and so

$$|\beta|^n < \tfrac{1}{2} \quad \text{for} \quad n \geq 2.$$

Also, $|\beta^n| = |\beta|^n$, and so

$$|\beta^n| < \tfrac{1}{2} \qquad \text{or} \qquad -\tfrac{1}{2} < \beta^n < \tfrac{1}{2}.$$

Therefore, by (b) on page 30, we have

$$\tfrac{1}{2} > -\beta^n > -\tfrac{1}{2}, \qquad \text{or} \qquad -\tfrac{1}{2} < -\beta^n < \tfrac{1}{2},$$

and by (c) we have

$$0 < \tfrac{1}{2} - \beta^n < 1.$$

Now since $\frac{1}{2} - \beta^n > 0$, we find from equation (A) and (g) on page 30 that

$$F_{n+1} < \alpha F_n + \tfrac{1}{2}.$$

Moreover, since $\frac{1}{2} - \beta^n < 1$, we have

$$F_{n+1} + (\tfrac{1}{2} - \beta^n) < F_{n+1} + 1.$$

Applying these to equation (A), we have

$$F_{n+1} < \alpha F_n + \tfrac{1}{2} < F_{n+1} + 1,$$

or

$$F_{n+1} = [\alpha F_n + \tfrac{1}{2}], \qquad n = 2, 3, 4, \ldots,$$

and the theorem is proved.

COROLLARY*

$$F_{n+1} = \left[\frac{F_n + 1 + \sqrt{5F_n^2}}{2}\right], \qquad n \geq 2.$$

Proof. From Theorem III, we have

$$F_{n+1} = \left[\alpha F_n + \tfrac{1}{2}\right] = \left[F_n\left(\frac{1+\sqrt{5}}{2}\right) + \frac{1}{2}\right]$$
$$= \left[\frac{F_n + \sqrt{5}\,F_n + 1}{2}\right] = \left[\frac{F_n + 1 + \sqrt{5F_n^2}}{2}\right].$$

This corollary shows that we can compute F_{n+1} from F_n without using either n or α.

We could prove in a similar way the corresponding theorem and corollary for Lucas numbers.

THEOREM IV

$$L_{n+1} = [\alpha L_n + \tfrac{1}{2}], \qquad n \geq 4.$$

COROLLARY*

$$L_{n+1} = \left[\frac{L_n + 1 + \sqrt{5L_n^2}}{2}\right], \qquad n \geq 4.$$

* A slightly different form of these corollaries appears as Theorem 4 in V. E. Hoggatt, Jr., and D. A. Lind, "The Heights of the Fibonacci Polynomials and an Associated Function," *The Fibonacci Quarterly*, Vol. 5, No. 2 (April, 1967), page 144.

PROBLEM 4

Given that 610 is a Fibonacci number, use the Corollary to Theorem III to find the next one.

Solution.

$$F_{n+1} = \left[\frac{610 + 1 + \sqrt{5(610)^2}}{2}\right]$$
$$= \left[\frac{611 + \sqrt{1860500}}{2}\right]$$
$$= \left[\frac{611 + 1364.0}{2}\right]$$
$$= \left[\frac{1975.0}{2}\right] = 987$$

You can check this value in the list on page 83.

Alternatively, you may compute as follows:

$$F_{n+1} = \left[\frac{610 + 1 + \sqrt{5}\,(610)}{2}\right]$$
$$= \left[\frac{611 + (2.236)(610)}{2}\right]$$
$$= \left[\frac{611 + 1363.96}{2}\right]$$
$$= \left[\frac{1974.96}{2}\right] = 987$$

For larger Fibonacci numbers you may need to find $\sqrt{5}$ to more decimal places.

EXERCISES

In Exercises 1–4, use four-place tables of logarithms except for $\log \alpha = 0.20898$.

1. Find F_{12}, using the method of Problem 1.
2. Find L_{12}, using the method of Problem 3.
3. Find the first three digits of F_{34}, using the method of Problem 2.
4. Find the first three digits of L_{33}.
5. Given that 1597 is a Fibonacci number, find the next one.
6. Given that 2207 is a Lucas number, find the next one.

7 • *Divisibility Properties of the Fibonacci and Lucas Numbers*

Let us look at the first few Fibonacci numbers:

F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9	F_{10}	F_{11}	F_{12}	F_{13}	F_{14}	F_{15} ...
1	1	2	3	5	8	13	21	34	55	89	144	233	377	610 ...

We observe the following:

1. Every third F_n is even; that is, $F_3 = 2$ divides $F_6 = 8$, $F_9 = 34$, $F_{12} = 144$, $F_{15} = 610$, ...
2. $F_4 = 3$ divides $F_8 = 21$, $F_{12} = 144$, ...
3. $F_5 = 5$ divides $F_{10} = 55$, $F_{15} = 610$, ...
4. $F_6 = 8$ divides $F_{12} = 144$, ...
5. $F_7 = 13$ divides $F_{14} = 377$, ...

These examples suggest the following theorem.

THEOREM I

Every Fibonacci number F_k divides every Fibonacci number F_{nk} for $n = 1, 2, 3, \ldots$; or if r is divisible by s, then F_r is divisible by F_s.

First, consider, for example,

$$F_{10} = \frac{\alpha^{10} - \beta^{10}}{\alpha - \beta}.$$

The difference of equal powers of two numbers can be factored, often in various ways. Here we may divide $\alpha^{10} - \beta^{10}$ by $\alpha^2 - \beta^2$ or by $\alpha^5 - \beta^5$ as shown on the following page.

$$
\begin{aligned}
F_{10} &= \frac{\alpha^{10} - \beta^{10}}{\alpha - \beta} \\
&= \frac{\alpha^2 - \beta^2}{\alpha - \beta}(\alpha^8 + \alpha^6\beta^2 + \alpha^4\beta^4 + \alpha^2\beta^6 + \beta^8) \\
&= F_2[(\alpha^8 + \beta^8) + \alpha^2\beta^2(\alpha^4 + \beta^4) + \alpha^4\beta^4]
\end{aligned}
$$

Since $\alpha\beta = -1$, we have

(A) $$F_{10} = F_2(L_8 + L_4 + 1).$$

Also, we have

$$
\begin{aligned}
F_{10} &= \frac{\alpha^{10} - \beta^{10}}{\alpha - \beta} \\
&= \frac{\alpha^5 - \beta^5}{\alpha - \beta}(\alpha^5 + \beta^5), \text{ or}
\end{aligned}
$$

(B) $$F_{10} = F_5L_5.$$

Thus, since $(L_8 + L_4 + 1)$ in equation (A) is an integer, F_2 is a factor of F_{10}; also F_5 is a factor of F_{10} as shown by equation (B).

In general, to see why F_{nk} is divisible by F_k, consider the factorization

(C)
$$
\begin{aligned}
F_{nk} &= \frac{\alpha^{nk} - \beta^{nk}}{\alpha - \beta} \\
&= \frac{\alpha^k - \beta^k}{\alpha - \beta}(\alpha^{(n-1)k} + \alpha^{(n-2)k}\beta^k \\
&\qquad + \alpha^{(n-3)k}\beta^{2k} + \cdots + \alpha^k\beta^{(n-2)k} + \beta^{(n-1)k})
\end{aligned}
$$

In the right-hand parenthesis, the first and last terms may be paired to form a Lucas number:

$$\alpha^{(n-1)k} + \beta^{(n-1)k} = L_{(n-1)k}$$

The second and next-to-last terms may be paired to form a product of $(-1)^k$ and a Lucas number:

$$
\begin{aligned}
\alpha^{(n-2)k}\beta^k + \alpha^k\beta^{(n-2)k} &= \alpha^k\beta^k(\alpha^{(n-3)k} + \beta^{(n-3)k}) \\
&= (\alpha\beta)^k L_{(n-3)k} = (-1)^k L_{(n-3)k}
\end{aligned}
$$

And so on. Notice that the number of terms in the right-hand parenthesis of (C) is n. If n is even, then the terms match up in symmetric pairs to make Lucas numbers, clearly adding up to an integer. See, for example, equation (B), where

$$F_{10} = F_{2\cdot 5} = F_5L_5,$$

with $k = 5$ and $n = 2$, which is even. If n is odd, then the terms still match up in symmetric pairs except for the middle term, which is of the form $(\alpha\beta)^{\frac{(n-1)k}{2}}$, clearly an integer. Again the right-hand parenthetical expression in (C) is an integer. See, for example, equation (A) where

$$F_{10} = F_{5 \cdot 2} = F_2(L_8 + L_4 + 1),$$

with $k = 2$ and $n = 5$, which is odd.

Another interesting theorem is given in terms of the greatest common divisor of two positive integers. The greatest common divisor of two positive integers a and b is denoted by

$$(a, b) = d,$$

meaning that d is the greatest positive integer dividing both a and b. For example,

$$(14, 2) = 2, \quad (24, 15) = 3, \quad \text{and} \quad (6765, 610) = 5.$$

This theorem can now be stated as:

THEOREM II

$$(F_m, F_n) = F_{(m, n)}.$$

This means that the greatest common divisor of two Fibonacci numbers is a *Fibonacci number* whose subscript (index) is the greatest common divisor of the subscripts (indices) of the other two Fibonacci numbers. Thus,

$$(F_{15}, F_{14}) = (610, 377) = F_{(15, 14)} = F_1 = 1;$$

$$(F_9, F_6) = (34, 8) = F_{(9, 6)} = F_3 = 2;$$

$$(F_{12}, F_6) = (144, 8) = F_{(12, 6)} = F_6 = 8. \quad \text{Here } F_6 \text{ divides } F_{12}.$$

Theorem II can be proved by using the Euclidean Algorithm* or as the solution to a Diophantine equation.†

Theorem I and Theorem II may be combined as:

THEOREM III

F_n is divisible by F_m *if and only if* n is divisible by m.

* N. N. Vorobyov, *Fibonacci Numbers* (Boston: D. C. Heath and Co., 1963), pp. 22–24.

† Glenn Michael, "A New Proof for an Old Property," *The Fibonacci Quarterly*, Vol. 2, No. 1 (February, 1964), pp. 57–58.

Let us now look at the first few Lucas numbers:

L_1	L_2	L_3	L_4	L_5	L_6	L_7	L_8	L_9	L_{10}	L_{11}	L_{12}	L_{13}	L_{14}	L_{15} ...
1	3	4	7	11	18	29	47	76	123	199	322	521	843	1364...

We immediately notice that every third L_n is even, as was the case for the Fibonacci numbers.

We next state without proof two interesting theorems.

THEOREM IV (L. Carlitz)*

$$L_n \text{ divides } F_m \textit{ if and only if } m = 2kn, \qquad n > 1.$$

For example, $L_3 = 4$ divides $F_6 = 8$, $F_{12} = 144$, and so on.

THEOREM V (L. Carlitz)*

$$L_n \text{ divides } L_m \textit{ if and only if } m = (2k - 1)n, \qquad n > 1.$$

For example, $L_3 = 4$ divides $L_9 = 76$, $L_{15} = 1364$, and so on.

It is easy to see that two consecutive Fibonacci numbers have no common factor greater than one. If F_{n+2} and F_{n+1} had a common factor d, then $F_n = F_{n+2} - F_{n+1}$ would also be divisible by d. Thus, we could progress down to $F_2 = 1$, which d would have to divide. Since d is a positive integer, d must be 1. A similar argument applies to Lucas numbers. Thus, we have:

THEOREM VI

$$(F_{n+2}, F_{n+1}) = 1.$$

THEOREM VII

$$(L_{n+2}, L_{n+1}) = 1.$$

EXERCISES

Refer to the list of Fibonacci and Lucas numbers on page 83.

1. Verify that F_7 divides F_{14}, F_{21}, and F_{28}.

2. Verify that F_{10} divides F_{20}, F_{30}, and F_{40}.

* L. Carlitz, "A Note on Fibonacci Numbers," *The Fibonacci Quarterly*, Vol. 2, No. 1 (February, 1964), pages 15–28.

3. Verify that F_{24} is divisible by F_3, F_4, F_6, F_8, and F_{12}.

4. Verify that F_{30} is divisible by F_3, F_5, F_6, F_{10}, and F_{15}.

5. Verify that L_4 divides F_8 and F_{16}.

6. Verify that L_7 divides F_{14} and F_{28}.

7. Verify that L_4 divides L_{12} and L_{20}.

8. Verify that L_5 divides L_{15} and L_{25}.

9. Verify that

$$\begin{aligned} F_{12} &= F_3(L_9 - L_3) = F_3L_3L_6 \\ &= F_4(L_8 + 1) \\ &= F_6L_6. \end{aligned}$$

10. Verify that

$$\begin{aligned} F_{18} &= F_3(L_{15} - L_9 + L_3) \\ &= F_6(L_{12} + 1) \\ &= F_9L_9. \end{aligned}$$

11. Find the greatest common divisor of F_{16} and F_{24}.

12. Find the greatest common divisor of F_{24} and F_{36}.

8 • *Periodicity of the Fibonacci and Lucas Numbers*

What else can we discover from the list of Fibonacci numbers?

F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9	F_{10}	F_{11}	F_{12}	F_{13}	F_{14}	$F_{15}\ldots$
1	1	2	3	5	8	13	21	34	55	89	144	233	377	610 . . .

Let us consider the sequence of primes. We observe that

2 divides F_3, F_6, and so on,
3 divides F_4, F_8, and so on,
5 divides F_5, F_{10}, and so on,
7 divides F_8, and so divides F_{16}, and so on
11 divides F_{10}, and so divides F_{20}, and so on
13 divides F_7, F_{14}, and so on.

We can speculate as to whether *every* prime number divides some Fibonacci number (and hence divides infinitely many of them). Later in this section we shall prove that *every integer* divides some Fibonacci number.

To do this, we shall use a kind of periodicity property of the Fibonacci numbers. But first we consider the following.

You are, no doubt, familiar with periodic decimals, such as the decimal representation of

$$\tfrac{1}{7}.$$

```
        .1 4 2 8 5 7 1
  7 ) 1.0 0 0 0 0 0 0
        7
       ---
        3 0
        2 8
        ---
          2 0
          1 4
          ---
            6 0
            5 6
            ---
              4 0
              3 5
              ---
                5 0
                4 9
                ---
                  1 0
                    7
                  ---
                    3
```

When 1 is divided by 7, the only possible remainders are

$$0, 1, 2, 3, 4, 5, 6.$$

If we had a zero remainder, our division would be exact. However, when we get any other of those seven digits, the division continues as shown, leading to the periodic decimal

$$\tfrac{1}{7} = .142857142857142857\ldots.$$

Every rational number has a periodic decimal expansion and every periodic decimal expansion represents a rational number. (Even $\frac{1}{2} = .5000\ldots = .4999\ldots$.)

In our present discussion we shall use a similar argument, but this time we shall be discussing *repeating ordered pairs of remainders.*

When F_0 through F_{31} are each divided by 7, we obtain the following sequence displaying the quotients and remainders.

$F_0 =$	$0 =$	$0 \cdot 7 + 0$	$F_{16} =$	$987 =$	$141 \cdot 7 + 0$
$F_1 =$	$1 =$	$0 \cdot 7 + 1$	$F_{17} =$	$1597 =$	$228 \cdot 7 + 1$
$F_2 =$	$1 =$	$0 \cdot 7 + 1$	$F_{18} =$	$2584 =$	$369 \cdot 7 + 1$
$F_3 =$	$2 =$	$0 \cdot 7 + 2$	$F_{19} =$	$4181 =$	$597 \cdot 7 + 2$
$F_4 =$	$3 =$	$0 \cdot 7 + 3$	$F_{20} =$	$6765 =$	$966 \cdot 7 + 3$
$F_5 =$	$5 =$	$0 \cdot 7 + 5$	$F_{21} =$	$10946 =$	$1563 \cdot 7 + 5$
$F_6 =$	$8 =$	$1 \cdot 7 + 1$	$F_{22} =$	$17711 =$	$2530 \cdot 7 + 1$
$F_7 =$	$13 =$	$1 \cdot 7 + 6$	$F_{23} =$	$28657 =$	$4093 \cdot 7 + 6$
$F_8 =$	$21 =$	$3 \cdot 7 + 0$	$F_{24} =$	$46368 =$	$6624 \cdot 7 + 0$
$F_9 =$	$34 =$	$4 \cdot 7 + 6$	$F_{25} =$	$75025 =$	$10717 \cdot 7 + 6$
$F_{10} =$	$55 =$	$7 \cdot 7 + 6$	$F_{26} =$	$121393 =$	$17341 \cdot 7 + 6$
$F_{11} =$	$89 =$	$12 \cdot 7 + 5$	$F_{27} =$	$196418 =$	$28059 \cdot 7 + 5$
$F_{12} =$	$144 =$	$20 \cdot 7 + 4$	$F_{28} =$	$317811 =$	$45401 \cdot 7 + 4$
$F_{13} =$	$233 =$	$33 \cdot 7 + 2$	$F_{29} =$	$514229 =$	$73461 \cdot 7 + 2$
$F_{14} =$	$377 =$	$53 \cdot 7 + 6$	$F_{30} =$	$832040 =$	$118862 \cdot 7 + 6$
$F_{15} =$	$610 =$	$87 \cdot 7 + 1$	$F_{31} =$	$1346269 =$	$192324 \cdot 7 + 1$

You can see by the zero remainders that 7 divides F_0, F_8, F_{16}, and F_{24} in this list.

Notice the pattern of the remainders (since each F_n is the sum of the two preceding Fibonacci numbers):

$$0 + 1 = 1,\ 1 + 1 = 2,\ \ldots,\ 3 + 5 = 8 = 7 + 1,\ 1 + 6 = 7 + 0,\ \ldots$$

That is, each remainder after the second is the sum of the two preceding remainders, decreased as necessary by 7.

Consider the sequences of remainders

$$1,\ 6,\ 0,\ 6 \qquad \text{(beginning with } F_6\text{)}$$

and

$$6,\ 1,\ 0,\ 1 \qquad \text{(beginning with } F_{14}\text{)}.$$

Notice that the pair (1, 6) gives a different sequence from that following the pair (6, 1). Since there are seven possible remainders (0, 1, 2, 3, 4, 5, 6) when numbers are divided by 7, there can be at most 7 × 7, or 49, different *ordered pairs* of remainders. Therefore, in a set of 49 + 1, or 50, ordered pairs, at least two of the pairs must be the same. In our example, not all the possible ordered pairs of remainders appear, and there are many repetitions between the first ordered pair of remainders, (0, 1) for F_0 and F_1, and the fiftieth, (1, 1) for F_{49} and F_{50}.

We shall now show that the *first* repeated pair of remainders is (0, 1) when we consider Fibonacci numbers with positive or zero subscripts, as we did in the table above for $m = 7$. In general, let the pairs of remainders (r_k, r_{k+1}) be obtained by dividing the Fibonacci numbers F_k and F_{k+1} by some integer m. Consider the sequence of pairs

$$(r_0, r_1), (r_1, r_2), (r_2, r_3), \ldots, (r_n, r_{n+1}).$$

We shall say that pairs (a_1, b_1) and (a_2, b_2) are equal if and only if $a_1 = a_2$ and $b_1 = b_2$. Clearly, in the first $m^2 + 1$ of these pairs obtained from the division of consecutive Fibonacci numbers by m, there must be at least two which are equal. (Conceivably, there could be more than two equal pairs as in the example above.)

Let us assume that the first pair to be repeated is (r_k, r_{k+1}) for $k \geq 0$. Then in our sequence of pairs there is a later pair (r_n, r_{n+1}) equal to (r_k, r_{k+1}) with $m^2 + 1 \geq n + 1 > k + 1$. But since the pairs are equal, $r_k = r_n$ and $r_{k+1} = r_{n+1}$. Since

$$F_{k-1} = F_{k+1} - F_k \quad \text{and} \quad F_{n-1} = F_{n+1} - F_n,$$

upon dividing by m, we get

$$r_{k-1} = r_{k+1} - r_k \quad \text{and} \quad r_{n-1} = r_{n+1} - r_n.$$

Since $r_{k+1} = r_{n+1}$ and $r_k = r_n$,

$$r_{k-1} = r_{n-1},$$

although it may be necessary to add m to r_{k-1} and r_{n-1} to make them positive. Thus, the pairs (r_{k-1}, r_k) and (r_{n-1}, r_n) are also equal. Therefore, every repeated pair has a predecessor that is also repeated except the first (0, 1), which has no predecessor. This first (0, 1) is the first pair in our list; see the table made for divisor $m = 7$ on page 43.

Let $(r_n, r_{n+1}) = (0, 1)$, $n > 0$, be the second appearance of (0, 1) in the sequence of ordered pairs of remainders. Then $r_n = 0$, so that F_n is divisible by m where $1 < n + 1 \leq m^2 + 1$, or $0 < n \leq m^2$. Therefore, we have proved the following theorem.

THEOREM I

Every integer m divides some Fibonacci number ($>F_0$) whose subscript does not exceed m^2.

As we progress up the sequence of remainders from the second (0, 1), we simply repeat the pairs in the same order as before so that F_{2n} is also divisible by m. We define n as the *period* of m in the Fibonacci sequence and denote it by K_m. For example,

$$K_7 = 16,$$

as you can see by looking back at the remainder pairs obtained by dividing the Fibonacci numbers by 7 (page 43). The pair (0, 1) occurred first with F_0 and F_1. The next occurrence of these remainders was for F_{16} and F_{17}, and F_{16} was divisible by 7. The pairs of remainders now repeat in cycles of 16; hence, $K_7 = 16$.

However, we noticed that F_8 also has a zero remainder and is divisible by 7. The subscript of the first positive Fibonacci number divisible by m is called the *rank of apparition* or *entry point* of the number m in the Fibonacci numbers. Thus, the entry point of 7 in the Fibonacci numbers is 8.

We can assert that 7 divides a Fibonacci number if and only if *its subscript* is divisible by 8. We note that the *only if* follows from the *periodicity* and not from Theorem III in Section 7, since 7 is not a Fibonacci number. (We also note that 7 does *not* divide F_7.) If a prime divides some Fibonacci number F_d, then it divides every Fibonacci number F_{nd}, since F_d divides F_{nd}. (Note that, since the number 7, which we chose to illustrate periodicity, happens to be a Lucas number, for this particular divisor we could have applied Theorem IV in Section 7.)

Now let us turn to the Lucas numbers. It can easily be shown that 5 does not divide any Lucas number:

$L_0 = 2 = 0 \cdot 5 + 2$	$L_4 = 7 = 1 \cdot 5 + 2$
$L_1 = 1 = 0 \cdot 5 + 1$	$L_5 = 11 = 2 \cdot 5 + 1$
$L_2 = 3 = 0 \cdot 5 + 3$	$L_6 = 18 = 3 \cdot 5 + 3$
$L_3 = 4 = 0 \cdot 5 + 4$	$L_7 = 29 = 5 \cdot 5 + 4$

Thus, in considering the remainder pairs when the Lucas numbers are divided by 5, we come to a repeated remainder pair (2, 1) again at L_4 and L_5. We note that in the period of length 4 there were no zeros. (Zero is not a Lucas number, and so the remainder 0 may not occur at all.) Thus, we see that *no Lucas number* is divisible by 5.

Of course, some Lucas numbers are divisible by other integers, for example 6 (see Exercise 4, page 47).

In the booklet *An Introduction to Fibonacci Discovery* by Brother U. Alfred (available from the author, St. Mary's College, California 94575) there is a nice table (Table 3, page 55) of entry points and periods of primes 2 to 269 in both the Fibonacci numbers and the Lucas numbers.

The Fibonacci and Lucas numbers have interesting remainder properties. We state the following theorems without proof.

THEOREM II

If F_n is divided by F_m ($n > m$), then either the remainder R is a Fibonacci number or $F_m - R$ is a Fibonacci number.

For example,

$$89 = 2 \cdot 34 + 21, \quad \text{or} \quad F_{11} = 2 \cdot F_9 + F_8;$$
$$144 = 6 \cdot 21 + 18, \quad \text{or} \quad F_{12} = 6 \cdot F_8 + 18.$$

In the second case, although 18 is not a Fibonacci number, the difference

$$F_8 - 18 = 21 - 18 = 3 = F_4$$

is a Fibonacci number. We can write

$$F_{12} = 6 \cdot F_8 + (F_8 - F_4).$$

Of course, $F_0 = 0$ is a Fibonacci number, and thus when F_{10} is divided by F_5, the zero remainder is a Fibonacci number.

With one exception, the property given in Theorem II is shared by the Lucas numbers. We have this theorem.

THEOREM III

If L_n is divided by L_m, $n > m$, then the remainder R is zero, or R is a Lucas number, or $L_m - R$ is a Lucas number.

For example,

$$76 = 19 \cdot 4 + 0, \quad \text{or} \quad L_9 = 19 \cdot L_3 + 0,$$

where $R = 0$, which is not a Lucas number but is the first possibility given in the theorem. Also,

$$18 = 1 \cdot 11 + 7, \quad \text{or} \quad L_6 = 1 \cdot L_5 + L_4;$$
$$47 = 6 \cdot 7 + 5, \quad \text{or} \quad L_8 = 6 \cdot L_4 + (L_4 - L_0).$$

A proof of the "Remainder Property" occurs in a short article by John H. Halton in *The Fibonacci Quarterly*, Vol. 2, No. 3 (October, 1964), pages 217–218, titled "Fibonacci Residues." The Lucas property is discussed in a paper by Laurence Taylor in *The Fibonacci Quarterly*, Vol. 5, No. 3 (October, 1967), pages 298–304, titled "Residues of Fibonacci-Like Sequences"; in the same paper, we find that the Fibonacci and Lucas sequences are the only sequences which satisfy the recurrence relation $u_{n+2} = u_{n+1} + u_n$ and have the properties given in Theorems II and III.

EXERCISES

1. Show that $K_{13} = 28$ for the Fibonacci sequence, and find the entry point of 13 in that sequence.
2. Show that 13 does not divide any Lucas number, but $K_{13} = 28$ also for the Lucas sequence.
3. Show that $K_{10} = 60$ for the Fibonacci sequence, and find the entry point of 10 in that sequence.
4. Show that $K_6 = 24$ for the Lucas sequence, and state the entry point of 6 in that sequence.
5. Show that $K_{10} = 12$ for the Lucas sequence, and state why 10 has no entry point in that sequence.

9 • *Pascal's Triangle and the Fibonacci Numbers*

If we consider the expansions of the binomial $(x + y)^n$ for $n = 0, 1, 2, 3, 4, 5, \ldots$, we can write them in the form:

$$\begin{aligned}
(x + y)^0 &= x^0y^0 \\
(x + y)^1 &= x^1y^0 + x^0y^1 \\
(x + y)^2 &= x^2y^0 + 2x^1y^1 + x^0y^2 \\
(x + y)^3 &= x^3y^0 + 3x^2y^1 + 3x^1y^2 + x^0y^3 \\
(x + y)^4 &= x^4y^0 + 4x^3y^1 + 6x^2y^2 + 4x^1y^3 + x^0y^4 \\
(x + y)^5 &= x^5y^0 + 5x^4y^1 + 10x^3y^2 + 10x^2y^3 + 5x^1y^4 + x^0y^5 \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

Recalling that $x^0 = y^0 = 1$ (x and y nonzero), we can write the array of coefficients, which is called *Pascal's triangle*, as follows:

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}$$

Before we proceed further, let us introduce some special symbolism. If we write

$$1 + 2 + 3 + 4 + \cdots + n,$$

we mean that starting with one, we add the consecutive integers until we come to n, which is the last number added. By using a notation based on

$\sum$, the Greek letter sigma for s, the first letter in *summation*, we can write

$$\sum_{i=1}^{n} i = 1 + 2 + 3 + \cdots + n,$$

where the *index* is i and the *limits of summation* are 1 and n. The index progresses from $i = 1$ to $i = n$ in unit increases. A *summand* is a quantity obtained by putting a certain value of i into a formula that yields the quantities to be added. For example, in

$$\sum_{i=1}^{6} i = 1 + 2 + 3 + 4 + 5 + 6 = 21$$

the summands are 1, 2, 3, 4, 5, and 6.

You may recall from your study of arithmetic and geometric progressions the general formulas

$$\sum_{i=1}^{n} i = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2},$$

$$\sum_{i=0}^{n-1} r^i = 1 + r + r^2 + \cdots + r^{n-1} = \frac{1 - r^n}{1 - r}, \qquad r \neq 1.$$

Returning now to Pascal's triangle, let $\binom{n}{m}$ represent the mth term in the nth row ($n \geq m \geq 0$). Thus, for $n = 4$, we have

$$\begin{aligned}(x + y)^4 &= \binom{4}{0} x^4 + \binom{4}{1} x^3y + \binom{4}{2} x^2y^2 + \binom{4}{3} xy^3 + \binom{4}{4} y^4 \\ &= (1)x^4 + (4)x^3y + (6)x^2y^2 + (4)xy^3 + (1)y^4.\end{aligned}$$

The numbers represented by the symbols $\binom{n}{m}$ are called the *binomial coefficients*. In the expansion of $(x + y)^4$ the numbers are the binomial coefficients for the fourth-power expansion.

In general, from a study of Pascal's triangle, we can define

$$\binom{0}{0} = \binom{n}{0} = \binom{n}{n} = 1, \quad \binom{n}{m} = \binom{n-1}{m} + \binom{n-1}{m-1} \quad \text{for } n \geq m \geq 1.$$

We see that these interesting numbers are thus defined by a recurrence formula for each positive integer n. For fixed n, the binomial coefficients are the entries across the nth level of Pascal's triangle. We can now write

$$(x + y)^4 = \sum_{i=0}^{4} \binom{4}{i} x^{4-i}y^i \quad (\text{since } y^0 = x^0 = 1)$$

and, in general (as can be proved by mathematical induction),

$$\text{(A)} \qquad (x + y)^n = \sum_{i=0}^{n} \binom{n}{i} x^{n-i}y^i.$$

Although we can generate any number of the binomial coefficients by writing down successive levels of Pascal's triangle, it is possible to calculate them directly. Let $1 \cdot 2 \cdot 3 \cdot \cdots \cdot n = n!$ (called "n factorial") and $0! = 1$. Then it can be shown that

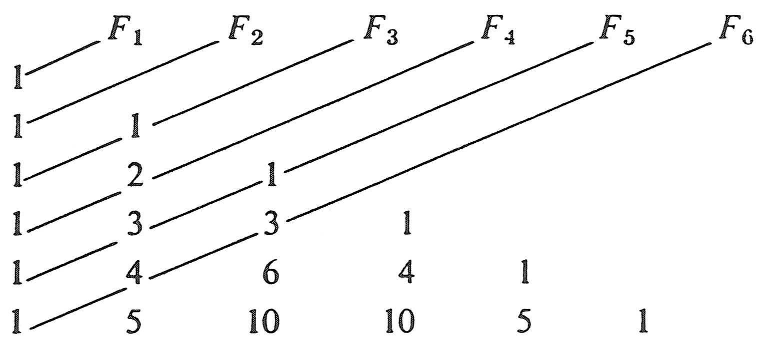
$$\binom{n}{m} = \frac{n!}{m!(n-m)!}.$$

For example,

$$\binom{5}{3} = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}{(1 \cdot 2 \cdot 3)(1 \cdot 2)} = 10.$$

Compare this with the coefficient for $m = 3$ in the expansion of $(x + y)^5$

How are the Fibonacci numbers related to Pascal's triangle? Let us write Pascal's triangle in the following form:



We can now see that the sums along the rising diagonals are the Fibonacci numbers:

$$F_1 = 1, \quad F_2 = 1, \quad F_3 = 1 + 1, \quad F_4 = 1 + 2,$$
$$F_5 = 1 + 3 + 1, \quad F_6 = 1 + 4 + 3, \quad F_7 = 1 + 5 + 6 + 1, \quad \text{etc.}$$

It can be proved that in general,

$$F_{n+1} = \sum_{i=0}^{[n/2]} \binom{n-i}{i},$$

where $[x]$ denotes the greatest integer not greater than x (Section 6). For example,

$$F_5 = \sum_{i=0}^{2} \binom{4-i}{i} = \binom{4}{0} + \binom{3}{1} + \binom{2}{2} = 1 + 3 + 1 = 5,$$

$$F_8 = \sum_{i=0}^{3} \binom{7-i}{i} = \binom{7}{0} + \binom{6}{1} + \binom{5}{2} + \binom{4}{3}$$
$$= 1 + \frac{6!}{(1!)(5!)} + \frac{5!}{(2!)(3!)} + \frac{4!}{(3!)(1!)}$$
$$= 1 + 6 + 10 + 4 = 21.$$

Now suppose that we look at $\sum_{i=0}^{n}\binom{n}{i}F_i$, where $F_i = \dfrac{\alpha^i - \beta^i}{\alpha - \beta}$. Thus,

$$\sum_{i=0}^{n}\binom{n}{i}F_i = \frac{1}{\alpha-\beta}\left[\sum_{i=0}^{n}\binom{n}{i}\alpha^i - \sum_{i=0}^{n}\binom{n}{i}\beta^i\right]$$

Since from formula (A) on page 49 we have $\sum_{i=0}^{n}\binom{n}{i}x^i = (1+x)^n$ (see Exercise 1 below), we can write

$$\sum_{i=0}^{n}\binom{n}{i}F_i = \frac{1}{\alpha-\beta}[(1+\alpha)^n - (1+\beta)^n]$$

But, $1 + \alpha = \alpha^2$ and $1 + \beta = \beta^2$; therefore,

$$\sum_{i=0}^{n}\binom{n}{i}F_i = \frac{\alpha^{2n} - \beta^{2n}}{\alpha - \beta} = F_{2n}.$$

For example,

$$\sum_{i=0}^{5}\binom{5}{i}F_i = \binom{5}{0}F_0 + \binom{5}{1}F_1 + \binom{5}{2}F_2 + \binom{5}{3}F_3 + \binom{5}{4}F_4 + \binom{5}{5}F_5$$
$$= (1)(0) + (5)(1) + (10)(1) + (10)(2) + (5)(3) + (1)(5)$$
$$= 0 + 5 + 10 + 20 + 15 + 5 = 55 = F_{10}.$$

EXERCISES

Demonstrate the following:

1. $\sum_{i=0}^{n}\binom{n}{i}x^i = (1+x)^n$
2. $\sum_{i=0}^{n}\binom{n}{i} = 2^n$
3. $\sum_{i=0}^{n}\binom{n}{i}(-1)^i = 0,\quad n \geq 1$

4. Verify that the two formulas for binomial coefficients agree by using the factorial notation to demonstrate that

$$\binom{n}{0} = \binom{n}{n} = 1 \quad \text{and} \quad \binom{n}{m} = \binom{n-1}{m} + \binom{n-1}{m-1}.$$

5. Verify that $\sum_{i=0}^{4}\binom{4}{i}F_i = F_8$.

Using the Binet form show that

6. $\sum_{i=0}^{n}\binom{n}{i}F_{i+j} = F_{2n+j}$.
7. $\sum_{i=0}^{n}\binom{n}{i}L_{i+j} = L_{2n+j}$.
8. $\sum_{i=0}^{n}\binom{n}{i}(-1)^i F_{i+j} = (-1)^{j+1}F_{n-j}$
9. $\sum_{i=0}^{n}\binom{n}{i}(-1)^i F_{2i+j} = (-1)^n F_{n+j}$

10 • *Selected Identities Involving the Fibonacci and Lucas Numbers*

A great many identities have been discovered that reveal interesting patterns of relationship between the Fibonacci and Lucas numbers.

In Section 2 we developed our first Fibonacci identity:

$$(\mathrm{I}_1) \qquad F_1 + F_2 + F_3 + \cdots + F_n = F_{n+2} - 1, \quad n \geq 1$$

We shall now give several other proofs of this.

Derivation of (I_1) *from the definition.* Since for all positive integers n the Fibonacci recurrence formula

$$F_{n+2} = F_{n+1} + F_n, \quad n \geq 1,$$

holds, we may write the sequence of equations:

$$\begin{aligned} F_1 &= F_3 - F_2 \\ F_2 &= F_4 - F_3 \\ F_3 &= F_5 - F_4 \\ &\cdots\cdots\cdots \\ F_{n-1} &= F_{n+1} - F_n \\ F_n &= F_{n+2} - F_{n+1} \end{aligned}$$

By adding these term by term and noting the cancellations on the right, we find

$$\begin{aligned} F_1 + F_2 + F_3 + \cdots + F_n &= F_{n+2} - F_2 \\ &= F_{n+2} - 1. \end{aligned}$$

Derivation of (I_1) *using the Binet form.* Since

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta},$$

we have

$$F_1 + F_2 + \cdots + F_n = \frac{\alpha - \beta}{\alpha - \beta} + \frac{\alpha^2 - \beta^2}{\alpha - \beta} + \cdots + \frac{\alpha^n - \beta^n}{\alpha - \beta}$$
$$= \frac{(\alpha + \alpha^2 + \cdots + \alpha^n) - (\beta + \beta^2 + \cdots + \beta^n)}{\alpha - \beta},$$

or

$$F_1 + F_2 + \cdots + F_n$$
$$= \frac{1}{\alpha - \beta}[(\alpha + \alpha^2 + \cdots + \alpha^n) - (\beta + \beta^2 + \cdots + \beta^n)].$$

But

$$\alpha + \alpha^2 + \cdots + \alpha^n = (1 + \alpha + \alpha^2 + \cdots + \alpha^n) - 1,$$
$$\beta + \beta^2 + \cdots + \beta^n = (1 + \beta + \beta^2 + \cdots + \beta^n) - 1.$$

From the formula for the sum of a geometric progression (recalled on page 49), we see that

$$1 + \alpha + \alpha^2 + \cdots + \alpha^n = \frac{1 - \alpha^{n+1}}{1 - \alpha},$$
$$1 + \beta + \beta^2 + \cdots + \beta^n = \frac{1 - \beta^{n+1}}{1 - \beta}.$$

Therefore,

$$F_1 + F_2 + \cdots + F_n$$
$$= \frac{1}{\alpha - \beta}\left[\left(\frac{1 - \alpha^{n+1}}{1 - \alpha} - 1\right) - \left(\frac{1 - \beta^{n+1}}{1 - \beta} - 1\right)\right],$$

and since $1 - \alpha = \beta$ and $1 - \beta = \alpha$, we have

$$F_1 + F_2 + \cdots + F_n = \frac{\alpha(1 - \alpha^{n+1}) - \beta(1 - \beta^{n+1})}{\alpha\beta(\alpha - \beta)}.$$

Since $\alpha\beta = -1$, we have

$$F_1 + F_2 + \cdots + F_n = \frac{\alpha - \alpha^{n+2} - \beta + \beta^{n+2}}{(-1)(\alpha - \beta)}$$
$$= \frac{\alpha^{n+2} - \beta^{n+2}}{\alpha - \beta} - \frac{\alpha - \beta}{\alpha - \beta}$$
$$= F_{n+2} - 1.$$

Proof of (I_1) *by mathematical induction.* We wish to prove

$$P(n)\colon\ F_1 + F_2 + F_3 + \cdots + F_n = F_{n+2} - 1.$$

There are two parts to the proof:

1. The statement $P(1)$ is found true by trial.
   Here $P(1)$ is $F_1 = F_3 - 1$, which is true since $1 = 2 - 1$.
2. One proves: If $P(k)$ is true for some integer k ($k \geq 1$), then $P(k + 1)$ must also be true.
   In this case, we assume

   $$P(k)\colon\ F_1 + F_2 + \cdots + F_k = F_{k+2} - 1$$

   and must prove

   $$P(k+1)\colon\ F_1 + F_2 + \cdots + F_{k+1} = F_{k+3} - 1.$$

   By adding F_{k+1} to both sides of

   $$F_1 + F_2 + \cdots + F_k = F_{k+2} - 1,$$

   we have

   $$\begin{aligned} F_1 + F_2 + \cdots + F_k + F_{k+1} &= F_{k+2} + F_{k+1} - 1 \\ &= F_{k+3} - 1, \end{aligned}$$

   since $F_{k+3} = F_{k+2} + F_{k+1}$. Therefore $P(k + 1)$ is true, and the proof is complete by mathematical induction.

The first part of the proof, where $P(1)$ is verified by direct trial, is often called the *basis for induction.* Clearly, if no statements can be found to be true by trial, we have no basis for our attempted proof. The second part is often called the *inductive transition* or *inductive step.* The assumption that $P(k)$ is true for some integer k ($k \geq 1$) is the *inductive hypothesis.* If we can show that the inductive hypothesis is sufficient to prove that $P(k + 1)$ is true, then we have completed the inductive transition.

It is important that *both* part 1 and part 2 be satisfied.

The declaration, "The proof is complete by mathematical induction," following such a proof is a simple statement telling the reader what method has been used.

There is a corresponding identity for Lucas numbers:

(I_2) $$L_1 + L_2 + L_3 + \cdots + L_n = L_{n+2} - 3, \quad n \geq 1$$

This is to be proved in Exercise 2, page 60.

As our next identity, let us consider:

$$(I_3) \qquad F_1^2 + F_2^2 + F_3^2 + \cdots + F_n^2 = F_n F_{n+1}, \quad n \geq 1$$

We shall prove this by mathematical induction. Here we have

$$P(n)\colon F_1^2 + F_2^2 + F_3^2 + \cdots + F_n^2 = F_n F_{n+1}.$$

Then

$$P(1)\colon F_1^2 = F_1 F_2$$

is easily seen to be true, since $(1)^2 = (1)(1)$. Thus, we have completed the *basis for induction*.

Now we *suppose* that

$$P(k)\colon F_1^2 + F_2^2 + \cdots + F_k^2 = F_k F_{k+1}$$

is true (the *inductive hypothesis*), and we undertake to prove:

$$P(k+1)\colon F_1^2 + F_2^2 + \cdots + F_{k+1}^2 = F_{k+1} F_{k+2}$$

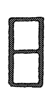
To do so, add F_{k+1}^2 to both sides of equality $P(k)$, obtaining

$$(F_1^2 + F_2^2 + \cdots + F_k^2) + F_{k+1}^2 = F_k F_{k+1} + F_{k+1}^2 = F_{k+1}(F_k + F_{k+1})$$
$$= F_{k+1} F_{k+2}.$$

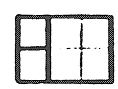
Therefore, we have shown that if $P(k)$ is true, then $P(k+1)$ is true, and we have completed the *inductive transition*. The proof is complete by mathematical induction.

Formula (I_3) may be pictured geometrically as follows:

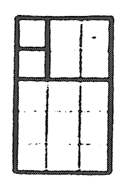
$$F_1^2 + F_2^2 = F_2 \times F_3$$
$$(1)^2 + (1)^2 = 1 \times 2$$



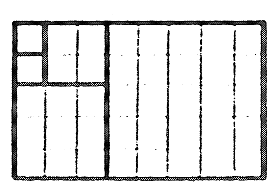
$$F_1^2 + F_2^2 + F_3^2 = F_3 \times F_4$$
$$(1)^2 + (1)^2 + (2)^2 = 2 \times 3$$



$$F_1^2 + F_2^2 + F_3^2 + F_4^2 = F_4 \times F_5$$
$$(1)^2 + (1)^2 + (2)^2 + (3)^2 = 3 \times 5$$



$$F_1^2 + F_2^2 + F_3^2 + F_4^2 + F_5^2 = F_5 \times F_6$$
$$(1)^2 + (1)^2 + (2)^2 + (3)^2 + (5)^2 = 5 \times 8$$



There is a corresponding identity for Lucas numbers:

$$(I_4) \qquad L_1^2 + L_2^2 + L_3^2 + \cdots + L_n^2 = L_n L_{n+1} - 2, \quad n \geq 1$$

This is to be proved in Exercise 5, page 60.

Now consider the following pair of identities (to be proved in Exercises 6 and 8, page 60):

$$(\mathrm{I}_5) \qquad F_1 + F_3 + F_5 + \cdots + F_{2n-1} = F_{2n}, \quad n \geq 1$$

$$(\mathrm{I}_6) \qquad F_2 + F_4 + F_6 + \cdots + F_{2n} = F_{2n+1} - 1, \quad n \geq 1$$

Verify by adding term by term that the sum of (I_5) and (I_6) is the same as (I_1) with n replaced by $2n$.

In Section 5 we found two interesting identities connecting the Fibonacci and Lucas numbers:

$$(\mathrm{I}_7) \qquad F_{2n} = F_n L_n, \quad n \geq 1$$

$$(\mathrm{I}_8) \qquad L_n = F_{n-1} + F_{n+1}, \quad n \geq 1$$

These were to be proved in Exercises 1 and 2, page 29. We also have:

$$(\mathrm{I}_9) \qquad F_n = \tfrac{1}{5}(L_{n-1} + L_{n+1}), \quad n \geq 1$$

This can be proved by applying (I_8) (see Exercise 12, page 61).

From (I_7) and (I_8) we can derive:

$$(\mathrm{I}_{10}) \qquad F_{2n} = F_{n+1}^2 - F_{n-1}^2, \quad n \geq 1$$

See Exercise 13, page 61.

By writing

$$F_{2n+1} = F_{2n+2} - F_{2n} = F_{2(n+1)} - F_{2n}$$

and applying (I_{10}), we can derive:

$$(\mathrm{I}_{11}) \qquad F_{2n+1} = F_{n+1}^2 + F_n^2, \quad n \geq 1$$

See Exercise 14, page 61.

Also in Section 5 we found this identity:

$$(\mathrm{I}_{12}) \qquad 5F_n^2 = L_n^2 - 4(-1)^n$$

This was to be proved in Exercise 10 on page 29.

Let us now look again at the Fibonacci sequence:

F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9	F_{10} ...
1	1	2	3	5	8	13	21	34	55 ...

Notice that:

$$F_1F_3 - F_2^2 = 1$$
$$F_2F_4 - F_3^2 = -1$$
$$F_3F_5 - F_4^2 = 1$$
$$F_4F_6 - F_5^2 = -1$$

and so on.

In general, we may write:

$$(I_{13}) \qquad F_{n-1}F_{n+1} - F_n^2 = (-1)^n, \quad n \geq 1$$

We shall prove this by mathematical induction, starting with $n = 1$.

$$P(1)\colon\ F_0F_2 - F_1^2 = 0 \cdot 1 - 1^2 = -1$$

We have established the *inductive basis*.

Assume for some integer $k \geq 1$ that

$$P(k)\colon\ F_{k+1}F_{k-1} - F_k^2 = (-1)^k.$$

Let us look for $P(k + 1)$:

$$\begin{aligned} F_{k+2}F_k - F_{k+1}^2 &= (F_{k+1} + F_k)F_k - F_{k+1}^2 \\ &= F_k^2 + F_{k+1}(F_k - F_{k+1}) \\ &= F_k^2 - F_{k+1}F_{k-1} \\ &= (-1)(F_{k+1}F_{k-1} - F_k^2) \\ &= (-1)(-1)^k = (-1)^{k+1} \end{aligned}$$

Thus,

$$P(k + 1)\colon\ F_{k+2}F_k - F_{k+1}^2 = (-1)^{k+1}$$

is true. The proof is complete by mathematical induction.

The mathematician Charles Lutwidge Dodgson, whose pen name was Lewis Carroll, liked to entertain his friends with puzzles. According to his nephew,* one of his favorites was this geometrical paradox. Let us cut the square on the left in Figure 18 as marked and reassemble the pieces to form the rectangle on the right.

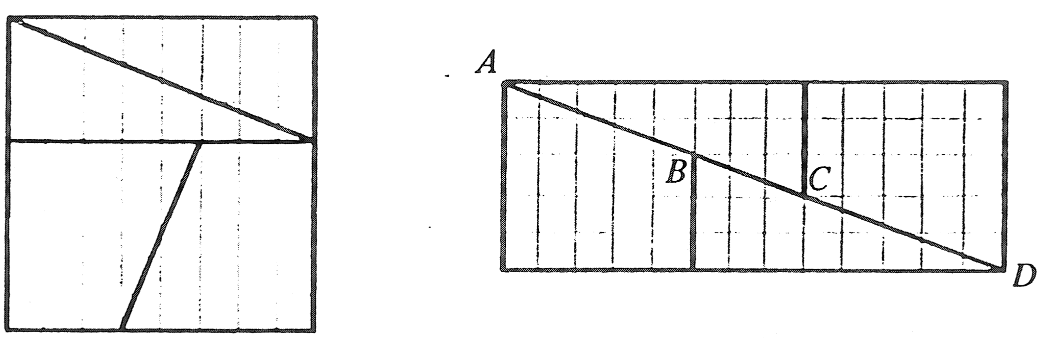


Figure 18

The area of the square is $8 \times 8 = 64$, while the area of the rectangle is $5 \times 13 = 65$. What happened?

* *Diversions and Digressions of Lewis Carroll*, edited by Stuart Dodgson Collingwood (New York: Dover Publications, Inc., 1961), pp. 316–317.

If you draw the diagrams pictured in Figure 18 making the square units rather large, you can discover what happened. The points A, B, C, and D, which in the small figure appear to lie on a straight line, actually are the vertices of a parallelogram, the area of which is the unexpectedly added unit. If you make the unit squares 1 inch on a side, the height of the added parallelogram will be nearly $\frac{1}{8}$ inch (see Exercise 15, page 61).

How is such a puzzle discovered? Have you noticed that the numbers 5, 8, and 13 are the consecutive Fibonacci numbers F_5, F_6, and F_7? Therefore from identity (I_{13}) we have

$$5 \cdot 13 - 8^2 = (-1)^6,$$

or

$$65 - 64 = 1.$$

Any triple of Fibonacci numbers such that the middle one has an even subscript (index) will produce this difference of 1. However, the larger the numbers used, the less noticeable is the added parallelogram.

In Figure 19 the situation is shown schematically with the parallelogram exaggerated:

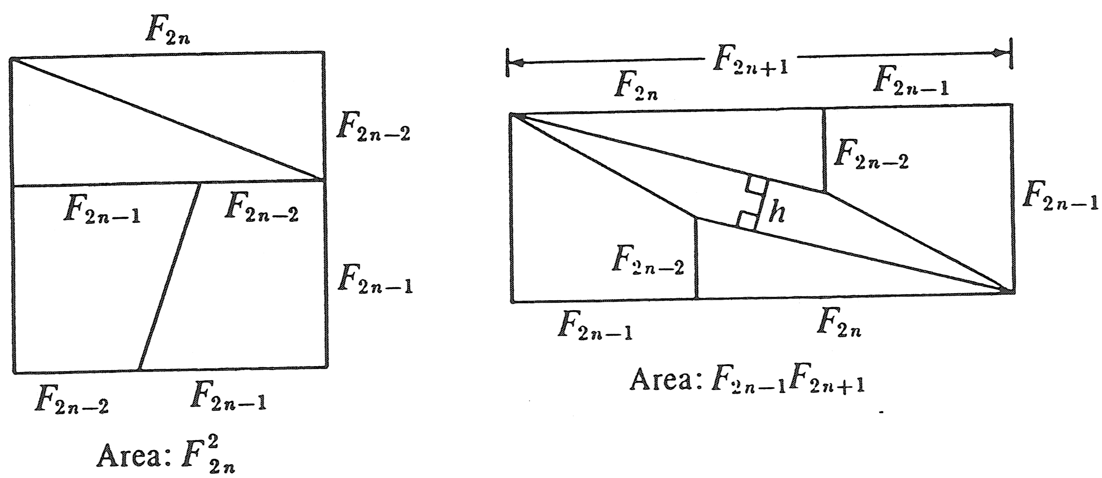


Figure 19

Since

$$F_{2n-1}F_{2n+1} - F_{2n}^2 = (-1)^{2n} = 1,$$

the area of the parallelogram is one square unit. The greatest width, the height h, of the parallelogram is easily found. Since the area equals the product of the height and the length of the base, we have

$$1 = h\sqrt{F_{2n}^2 + F_{2n-2}^2} \quad \text{(by the Pythagorean theorem)},$$

or

$$h = \frac{1}{\sqrt{F_{2n}^2 + F_{2n-2}^2}}.$$

A great many more identities have been discovered. The following additional list will indicate the various types.

(I_{14}) $$L_n = F_{n+2} - F_{n-2}$$

(I_{15}) $$L_{4n} + 2 = L_{2n}^2$$

(I_{16}) $$L_{4n} - 2 = 5F_{2n}^2$$

(I_{17}) $$L_{4n+2} + 2 = 5F_{2n+1}^2$$

(I_{18}) $$L_{4n+2} - 2 = L_{2n+1}^2$$

(I_{19}) $$F_{n-k}F_{n+k} - F_n^2 = (-1)^{n+k+1}F_k^2$$

(I_{20}) $$L_{n-k}L_{n+k} - L_n^2 = 5(-1)^{n+k}F_k^2$$

(I_{21}) $$F_{n+p} + F_{n-p} = F_nL_p, \quad p \text{ even}$$

(I_{22}) $$F_{n+p} + F_{n-p} = L_nF_p, \quad p \text{ odd}$$

(I_{23}) $$F_{n+p} - F_{n-p} = F_nL_p, \quad p \text{ odd}$$

(I_{24}) $$F_{n+p} - F_{n-p} = L_nF_p, \quad p \text{ even}$$

(I_{25}) $$F_{n+p}^2 - F_{n-p}^2 = F_{2n}F_{2p}$$

(I_{26}) $$F_{m+n+1} = F_{m+1}F_{n+1} + F_mF_n$$

(I_{27}) $$L_{m+n+1} = F_{m+1}L_{n+1} + F_mL_n$$

(I_{28}) $$F_nL_{n+k} - F_{n+k}L_n = 2(-1)^{n+1}F_k$$

(I_{29}) $$F_kF_{k+1}F_{k+3}F_{k+4} = F_{k+2}^4 - 1$$

(I_{30}) $$L_{2n}L_{2n+2} - 1 = 5F_{2n+1}^2$$

(I_{31}) $$L_nL_{n+1} - L_{2n+1} = (-1)^n$$

(I_{32}) $$5F_n = L_{n+2} - L_{n-2}$$

(I_{33}) $$L_n^2 + 4L_{n-1}L_{n+1} = 25F_n^2$$

(I_{34}) $$L_{n-1}L_{n+1} + F_{n-1}F_{n+1} = 6F_n^2$$

(I_{35}) $$F_n^2 + 4F_{n-1}F_{n+1} = L_n^2$$

(I_{36}) $$F_{n+1}^2 - 4F_nF_{n-1} = F_{n-2}^2$$

(I_{37}) $$F_{m-1}F_{m+1} - F_{m+2}F_{m-2} = 2(-1)^m$$

(I_{38}) $$L_nF_{m-n} + F_nL_{m-n} = 2F_m$$

(I_{39}) $$F_{2m}^2 = 5F_m^4 + 4(-1)^mF_m^2$$

(I_{40}) $$\sum_{k=1}^{n} kF_k = (n+1)F_{n+2} - F_{n+4} + 2$$

(I41) $$\sum_{k=0}^{2n} \binom{2n}{k} F_{2k+p} = 5^n F_{2n+p}$$

(I42) $$\sum_{k=0}^{2n+1} \binom{2n+1}{k} F_{2k+p} = 5^n L_{2n+p+1}$$

(I43) $$L_m L_n + L_{m+1} L_{n+1} = 5F_{m+n+1}$$

(I44) $$\left(\frac{L_p + \sqrt{5}\, F_p}{2}\right)^n = \frac{L_{np} + \sqrt{5}\, F_{np}}{2}$$

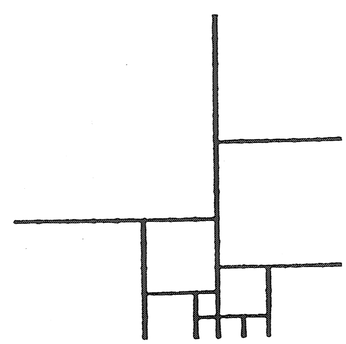
(I45) $$\sum_{k=0}^{2n+2} \binom{2n+2}{k} F_k^2 = 5^n L_{2n+2}$$

(I46) $$\sum_{k=0}^{2n+2} \binom{2n+2}{k} L_k^2 = 5^{n+1} L_{2n+2}$$

(I47) $$\sum_{k=0}^{2n+1} \binom{2n+1}{k} F_k^2 = 5^n F_{2n+1}$$

EXERCISES

1. Write (I1) using summation notation for the left-hand member.
2. Prove (I2) by mathematical induction.
3. Write (I3) using summation notation for the left-hand member.
4. The diagram at the right illustrates the statement
   $$F_7^2 = F_6^2 + 3F_5^2 + 2(F_4^2 + F_3^2 + F_2^2 + F_1^2).$$
   Use (I3) to prove that, in general,
   $$F_{n+1}^2 = F_n^2 + 3F_{n-1}^2 + 2(F_{n-2}^2 + \cdots + F_1^2).$$
   Show that the number of squares is $2n$.



5. Prove (I4) by mathematical induction.
6. Derive (I5) by using the sequence of equations $F_1 = F_2$, $F_3 = F_4 - F_2$, $F_5 = F_6 - F_4$, and so on.
7. Write (I5) using summation notation for the left-hand member.
8. Prove (I6) by mathematical induction.
9. Write (I6) using summation notation for the left-hand member.

10. Find and prove an identity for Lucas numbers corresponding to (I_5).

11. Find and prove an identity for Lucas numbers corresponding to (I_6).

12. Use (I_8) to prove (I_9).

13. Use (I_7) and (I_8) to prove (I_{10}).

14. Use (I_{10}) to prove (I_{11}).

15. Verify that if the small squares in Figure 18 are drawn 1 inch square on a side, the height h of the parallelogram is less than $\frac{1}{8}$ inch.

16. On graph paper, carefully draw a square with each side $F_8 = 21$ units, cut it as indicated in Figure 19, and assemble the pieces as a rectangle as shown there. (You will be able to surprise your friends with the result.)

17. Recall the definition of a generalized Fibonacci sequence on page 7. Show by mathematical induction that if $H_1 = p$, $H_2 = q$, and $H_{n+2} = H_{n+1} + H_n$, $n \geq 1$, then $H_{n+2} = qF_{n+1} + pF_n$.

18. Using (I_{26}) on page 59, show by mathematical induction that F_n divides F_{nk}, $k > 0$.
Hint: Let $m + 1 = nk$. Then (I_{26}) becomes $F_{(k+1)n} = F_{kn}F_{n+1} + F_{kn-1}F_n$.

11 • *Two-by-Two Matrices Related to the Fibonacci Numbers*

A two-by-two *matrix* is represented by a symbol such as

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where a, b, c, and d represent any real numbers. These numbers are called *elements*. In this section, all matrices mentioned are to be considered to be two-by-two matrices.

Two matrices,

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} e & f \\ g & h \end{pmatrix},$$

are *equal* if and only if their corresponding elements are equal. That is,

$$A = B$$

if and only if

$$a = e, \quad b = f, \quad c = g, \quad \text{and} \quad d = h.$$

The matrix which is the *sum* of two matrices A and B is defined to be

$$A + B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} a + e & b + f \\ c + g & d + h \end{pmatrix}.$$

The zero matrix Z is defined as

$$Z = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

It can be shown (Exercise 1, page 68) that

$$A + Z = Z + A = A,$$

and so Z is the *additive identity element* for the system of two-by-two matrices.

Also, it can be shown (Exercise 2, page 68) that, in general,

$$A + B = B + A.$$

Thus, *matrix addition is commutative.* It can also be shown (Exercise 3, page 68) that *matrix addition is associative.*

We define the *negative*, or *additive inverse, of A* to be

$$-A = \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}$$

and the *difference* of A and B as

$$A - B = A + (-B).$$

You can verify (Exercise 4, page 68) that

$$A + (-A) = Z.$$

When we are dealing with matrices, we refer to the real numbers as *scalars.* The product of a scalar s and a matrix A is defined to be

$$sA = s\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} sa & sb \\ sc & sd \end{pmatrix}.$$

It can be shown (Exercise 5, page 68) that

$$(sr)A = s(rA).$$

The matrix which is the *product* of two matrices A and B is defined to be

$$AB = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix}.$$

The identity matrix I is defined as

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

It can be shown (Exercise 7, page 68) that

$$AI = IA = A,$$

and so I is the *multiplicative identity element* for this system. It can also be shown that *matrix multiplication is associative.*

Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 1 \\ 0 & 1 \end{pmatrix}$; then:

$$AB = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 \cdot 3 + 1 \cdot 0 & 1 \cdot 1 + 1 \cdot 1 \\ 1 \cdot 3 + 0 \cdot 0 & 1 \cdot 1 + 0 \cdot 1 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 3 & 1 \end{pmatrix}$$

$$BA = \begin{pmatrix} 3 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 3 \cdot 1 + 1 \cdot 1 & 3 \cdot 1 + 1 \cdot 0 \\ 0 \cdot 1 + 1 \cdot 1 & 0 \cdot 1 + 1 \cdot 0 \end{pmatrix} = \begin{pmatrix} 4 & 3 \\ 1 & 0 \end{pmatrix}$$

Therefore, in this case $AB \neq BA$. This demonstrates that *matrix multiplication is not always commutative.*

If $A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, then we have

$$AB = \begin{pmatrix} 1 \cdot 1 + (-1)1 & 1 \cdot 1 + (-1)1 \\ (-1)1 + 1 \cdot 1 & (-1)1 + 1 \cdot 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = Z.$$

Here we note that the product AB is the zero matrix although neither A nor B is the zero matrix. In elementary mathematics, if the product of two numbers is zero, then at least one of the numbers has to be zero. For matrices (which are not numbers) this rule does not apply.

If A, B, and C are matrices, it can be shown (Exercise 8, page 68) that *matrix multiplication is distributive over matrix addition.* That is,

$$A(B + C) = AB + AC \quad \text{and} \quad (B + C)A = BA + CA.$$

Since matrix multiplication is not always commutative, these two distributive laws do not necessarily say the same thing.

Associated with each matrix A is a number, called the *determinant* of matrix A, and denoted by det A, which is defined as

$$\det A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

The matrix A is said to be *nonsingular* if det $A \neq 0$; otherwise, matrix A is *singular*.

We now prove a simple theorem.

THEOREM I

The determinant, det AB, of the product of two matrices, A and B, is the product of the determinants, det A and det B. That is,

$$\det AB = (\det A)(\det B).$$

Proof. Using the matrices A and B shown at the beginning of this section, we have

$$\det A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc, \qquad \det B = \begin{vmatrix} e & f \\ g & h \end{vmatrix} = eh - fg,$$

$$\begin{aligned} \det AB &= \begin{vmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{vmatrix} \\ &= (ae + bg)(cf + dh) - (af + bh)(ce + dg) \\ &= acef + adeh + bcfg + bdgh - (acef + adfg + bceh + bdgh) \\ &= adeh + bcfg - adfg - bceh \\ &= (ad - bc)(eh - fg) \\ &= (\det A)(\det B). \end{aligned}$$

Before looking at powers of a two-by-two matrix, let us recall how powers were defined for real numbers. If a is a nonzero real number, then one can define all integral powers of a as follows:

$$a^0 = 1, \qquad a^1 = a, \qquad a^{n+1} = a^n a^1,$$

$$\text{and} \quad a^{-n} = \frac{1}{a^n} \quad \text{for positive integral } n$$

The definition $a^{n+1} = a^n a^1$ is called an *inductive definition.* We shall also use an inductive definition for powers of matrices. For any nonzero matrix A,

$$A^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I, \qquad A^1 = A, \qquad \text{and} \qquad A^{n+1} = A^n A^1$$

for positive integral n.

We shall now consider a particular matrix

$$Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad \text{with } \det Q = -1,$$

and we shall prove the following theorem.

THEOREM II

For $n \geq 1$, the nth power of Q is given by

$$Q^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}.$$

Proof. We shall use mathematical induction. Clearly, for $n = 1$,

$$Q^1 = \begin{pmatrix} F_2 & F_1 \\ F_1 & F_0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

Assume that $Q^k = \begin{pmatrix} F_{k+1} & F_k \\ F_k & F_{k-1} \end{pmatrix}$; then

$$\begin{aligned} Q^{k+1} = Q^k Q^1 &= \begin{pmatrix} F_{k+1} & F_k \\ F_k & F_{k-1} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} F_{k+1} + F_k & F_{k+1} \\ F_k + F_{k-1} & F_k \end{pmatrix} \\ &= \begin{pmatrix} F_{k+2} & F_{k+1} \\ F_{k+1} & F_k \end{pmatrix}. \end{aligned}$$

The proof is complete by mathematical induction.

THEOREM III

$$\det Q^n = (-1)^n, \qquad n \geq 1.$$

Proof. The proof is left to you (Exercise 9, page 68).

From Theorem II and Theorem III we have

$$\det Q^n = \begin{vmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{vmatrix} = F_{n-1}F_{n+1} - F_n^2 = (-1)^n, \quad n \geq 1.$$

Thus, we have still another derivation of (I_{13}) of Section 10.

Of much technical interest in the algebra of matrices is the characteristic polynomial. For a matrix A, the *characteristic polynomial* is defined to be

$$\begin{aligned} P(x) = \det(A - xI) &= \left|\begin{pmatrix} a & b \\ c & d \end{pmatrix} - x\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right| \\ &= \left|\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} -x & 0 \\ 0 & -x \end{pmatrix}\right| \\ &= \begin{vmatrix} a - x & b \\ c & d - x \end{vmatrix} \\ &= x^2 - (a + d)x + (ad - bc). \end{aligned}$$

The equation $P(x) = 0$, or

$$x^2 - (a + d)x + (ad - bc) = 0,$$

is called the *characteristic equation* of matrix A. Note that the constant term is the determinant of matrix A and the linear term has coefficient $-(a + d)$, which is the negative of $(a + d)$. The sum $(a + d)$ of the elements on the diagonal of matrix A is called the *trace* of matrix A. The roots of the characteristic equation are called the *characteristic roots* of matrix A.

For Q, the characteristic polynomial is

$$\begin{aligned} \det(Q - xI) &= \left|\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}\right| \\ &= \begin{vmatrix} 1 - x & 1 \\ 1 & -x \end{vmatrix} = x^2 - x - 1, \end{aligned}$$

and the characteristic equation is

$$x^2 - x - 1 = 0,$$

which we have called the Fibonacci quadratic equation (page 11). The characteristic roots of Q are then, of course,

$$\alpha = \frac{1 + \sqrt{5}}{2} \qquad \text{and} \qquad \beta = \frac{1 - \sqrt{5}}{2}.$$

Now what are the characteristic equation and the characteristic roots of Q^n?

$$\det(Q^n - xI) = \begin{vmatrix} F_{n+1} - x & F_n \\ F_n & F_{n-1} - x \end{vmatrix}$$
$$= x^2 - (F_{n+1} + F_{n-1})x + (F_{n+1}F_{n-1} - F_n^2)$$

But $F_{n+1} + F_{n-1} = L_n$ [from (I_8) of Section 10] and $F_{n+1}F_{n-1} - F_n^2 = (-1)^n$ [(I_{13})]. Thus,

$$\det(Q^n - xI) = x^2 - L_n x + (-1)^n,$$

and the characteristic equation is

$$x^2 - L_n x + (-1)^n = 0.$$

The characteristic roots are given by (using the quadratic formula)

$$x = \frac{L_n \pm \sqrt{L_n^2 - 4(-1)^n}}{2}.$$

We now recall from (I_{12}) in Section 10 that $L_n^2 - 4(-1)^n = 5F_n^2$; thus, the roots are

$$\frac{L_n + \sqrt{5}\,F_n}{2} \quad \text{and} \quad \frac{L_n - \sqrt{5}\,F_n}{2}.$$

However, we recall that these are the expressions that we found for α^n and β^n on page 27, and so we have the following theorem.

THEOREM IV

The characteristic roots of matrix Q^n are the nth powers of the characteristic roots of Q, and the trace of Q^n is L_n.

You can easily verify (Exercise 10, page 68) that

$$Q^2 = Q + I,$$

or

$$Q^2 - Q - I = Z.$$

Thus, Q may be said to satisfy its characteristic equation

$$x^2 - x - 1 = 0.$$

This is an instance of the following theorem, which we state without proof.

THEOREM V (Hamilton-Cayley* Theorem)

Every square matrix satisfies its own characteristic equation.

* Sir W. R. Hamilton (1805–1865), Irish mathematician; Arthur Cayley (1821–1895), English mathematician.

EXERCISES

1. Using

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad Z = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

and the definition of addition, show that

$$A + Z = Z + A = A.$$

2. Using

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$$

and the definition of addition, show that

$$A + B = B + A.$$

3. Using

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}, \quad \text{and} \quad C = \begin{pmatrix} i & j \\ k & l \end{pmatrix}$$

and the definition of addition, show that

$$(A + B) + C = A + (B + C).$$

4. Show that $A + (-A) = Z$.

5. Using the definition of scalar multiplication, show that $(sr)A = s(rA)$.

6. Show that $-A = (-1)(A)$.

7. Using

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and the definition of multiplication, show that

$$AI = IA = A.$$

8. Using A, B, and C as given in Exercise 3, show that

$$A(B + C) = AB + AC \quad \text{and} \quad (B + C)A = BA + CA.$$

9. Prove that $\det(Q^n) = (-1)^n$ for $Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, $n \geq 1$.

10. Show that $Q^2 = Q + I$, or $Q^2 - Q - I = Z$.

12 • *Representation Theorems*

A sequence of positive integers, $a_1, a_2, \ldots, a_n, \ldots$, is **complete** with respect to the positive integers if and only if every positive integer m is the sum of a finite number of the members of the sequence, where each member is used at most once in any given **representation.** We state the following theorem without proof.

THEOREM I

> The sequence defined by $a_n = 2^n$ $(n \geq 0)$ is complete.

For example, since

$$a_0 = 2^0 = 1, \quad a_2 = 2^2 = 4, \quad a_4 = 2^4 = 16,$$
$$a_1 = 2^1 = 2, \quad a_3 = 2^3 = 8, \quad a_5 = 2^5 = 32, \quad \ldots,$$

we can write, for example:

$1 = 1$	$5 = 4 + 1$	$9 = 8 + 1$
$2 = 2$	$6 = 4 + 2$	$10 = 8 + 2$
$3 = 2 + 1$	$7 = 4 + 2 + 1$	$11 = 8 + 2 + 1$
$4 = 4$	$8 = 8$	$12 = 8 + 4$

It can be proved that each representation is unique, and you should recognize this as the basis for the binary system of numeration. For example:

1: 1	5: 101	9: 1001
2: 10	6: 110	10: 1010
3: 11	7: 111	11: 1011
4: 100	8: 1000	12: 1100

The ancient Egyptians used the principle described on the preceding page in their method of multiplication. For example, to find

$$243 \times 25,$$

they would write two columns — powers of 2 and the corresponding products:

1 ✓	243 ✓
2	486
4	972
8 ✓	1944 ✓
16 ✓	3888 ✓

Since $25 = 16 + 8 + 1$,

$$243 \times 25 = 3888 + 1944 + 243 = 6075,$$

as you can see by applying the distributive property.

We shall now prove an interesting theorem about the Fibonacci sequence.

THEOREM II

The Fibonacci sequence of numbers, where $a_n = F_n$ $(n \geq 1)$, is complete.

To discover a method of proof, let us look at the following table:

F_1	F_2	F_3	F_4		F_5			F_6	. . .
1	1	2	3	4	5	6	7	8	. . .

We observe that we can write:

$$1 = F_1 = F_2$$
$$2 = F_3 = F_2 + F_1$$
$$3 = F_4 = F_3 + F_2$$
$$4 = F_4 + F_2 = F_3 + F_2 + F_1$$
$$5 = F_5 = F_4 + F_3$$
$$6 = F_5 + F_2 = F_4 + F_3 + F_2$$
$$7 = F_5 + F_3 = F_4 + F_3 + F_2 + F_1$$
$$8 = F_6 = F_5 + F_4 = F_5 + F_3 + F_2$$

Thus, each positive integer from 1 through 8 can be represented in at least two ways as a sum of Fibonacci numbers where each is used at most once in any representation.

Proof of Theorem II. We know from (I_1) (page 52) that

$$F_n - 1 = F_1 + F_2 + F_3 + \cdots + F_{n-2},$$

and we observe in the list on page 70 that for $3 \leq n \leq 6$, each integer

$$m = 1, 2, 3, \ldots, F_n - 1$$

can be represented as a sum of some or all of the Fibonacci numbers F_1, F_2, F_3, . . . , F_{n-2}. That is, for $n = 3$, $F_n = F_3 = 2$, $F_n - 1 = 2 - 1 = 1$, $F_{n-2} = F_1 = 1$, and we have

$$1 = F_1;$$

for $n = 4$, $F_n = F_4 = 3$, $F_n - 1 = 3 - 1 = 2$, $F_{n-2} = F_2 = 1$, and we have

$$1 = F_1, 2 = F_2 + F_1;$$

and so on. We shall use this as the basis of induction (recall page 54).

For the second part of the proof, we assume that every integer $m = 1$, 2, 3, . . . , $F_k - 1$, $k \geq 3$, is representable using the Fibonacci numbers F_1, F_2, . . . , F_{k-2}. We must prove that every integer $m = 1, 2, 3, \ldots$, $F_{k+1} - 1$ is representable using F_1, F_2, . . . , F_{k-1}, $k \geq 3$. If we add F_{k-1} to each of the given representations, we shall have representations for

$$1 + F_{k-1}, \quad 2 + F_{k-1}, \quad \ldots, \quad F_k - 1 + F_{k-1},$$

where $F_k + F_{k-1} - 1 = F_{k+1} - 1$. We now have representations for the two sequences of consecutive positive integers

$$1, \quad 2, \quad 3, \quad \ldots, \quad F_k - 1$$

and

$$1 + F_{k-1}, \quad 2 + F_{k-1}, \quad 3 + F_{k-1}, \quad \ldots, \quad F_{k+1} - 1.$$

Are there any omissions between $F_k - 1$ and $1 + F_{k-1}$? No, since for $k = 3$,

$$F_k - 1 = F_3 - 1 = 1 \qquad \text{and} \qquad 1 + F_{k-1} = 1 + F_2 = 2;$$

for $k = 4$,

$$F_k - 1 = F_4 - 1 = 2 \qquad \text{and} \qquad 1 + F_{k-1} = 1 + F_3 = 3;$$

and for $k \geq 5$, since $F_k - F_{k-1} \geq 2$, we have

$$F_k - 1 \geq 1 + F_{k-1},$$

and so there is an overlap. In any case, every integer $m = 1, 2, 3, \ldots$, $F_{k+1} - 1$, $k \geq 3$, is representable using F_1, F_2, . . . , F_{k-1}, which is what we set out to prove.

Thus, the proof is complete by mathematical induction.

THEOREM III

The Fibonacci number sequence, where $a_n = F_n$ $(n \geq 1)$, with an arbitrary F_n missing is complete.

Observe, for example, that if F_4 is omitted from the display on page 70, every integer from 1 through 8 can still be represented by a sum of other Fibonacci numbers.

Proof. From the proof of the previous theorem we note that we can properly represent any number $m = 1, 2, 3, \ldots, F_{n+1} - 1$ by using only the Fibonacci numbers $F_1, F_2, F_3, \ldots, F_{n-1}$, that is, without using F_n. Then F_{n+1} can represent itself, and when we add F_{n+1} to the representations for $m = 1, 2, 3, \ldots, F_{n+1} - 1$, we have representations for $m = 1, 2, 3, \ldots, 2F_{n+1} - 1$. Since $2F_{n+1} - 1 > F_{n+2} - 1$, we can easily proceed over the trouble spot.

THEOREM IV

The Fibonacci sequence of numbers, where $a_n = F_n$ $(n \geq 1)$, with any two arbitrary Fibonacci numbers F_p and F_n missing is incomplete.

Proof. Since

$$F_1 + F_2 + F_3 + \cdots + F_k = F_{k+2} - 1,$$

if $F_p < F_k$ is missing, then

$$F_1 + F_2 + F_3 + \cdots + F_{p-1} + F_{p+1} + \cdots + F_k = F_{k+2} - F_p - 1.$$

But with $F_n > F_p$ also missing, we can reach

$$F_1 + F_2 + \cdots + F_{p-1} + F_{p+1} + \cdots + F_{n-1}$$
$$= F_{n+1} - F_p - 1 < F_{n+1} - 1.$$

Since $F_p \geq 1$, there is a number $F_{n+1} - 1$ without a proper representation. This happened because even if we used all of those *available* numbers less than F_n at once, we cannot reach $F_{n+1} - 1$ and any other Fibonacci numbers which are *available* are too large, since the Fibonacci numbers are a strictly increasing sequence for $n \geq 2$.

For example, suppose that we try to find a representation of 20 without using $F_5 = 5$ and $F_7 = 13$. Those available are then F_1, F_2, F_3, F_4, F_6. The maximum attainable is $1 + 1 + 2 + 3 + 8 = 15$. The next available Fibonacci number is 21 so that we cannot find representations for 16, 17, 18, 19, and 20.

We shall now consider the corresponding properties for the Lucas sequence of numbers. First, let us look at the following table:

L_0	L_1	L_2	L_3			L_4		...
2	1	3	4	5	6	7	8	...

$$1 = L_1$$
$$2 = L_0$$
$$3 = L_2 = L_1 + L_0$$
$$4 = L_3 = L_2 + L_1$$
$$5 = L_3 + L_1 = L_2 + L_0$$
$$6 = L_3 + L_0 = L_2 + L_1 + L_0$$
$$7 = L_4 = L_3 + L_2 = L_3 + L_1 + L_0$$
$$8 = L_4 + L_1 = L_3 + L_2 + L_1$$

We observe that each positive integer from 3 through 8 can be represented in at least two ways as a sum of Lucas numbers where each is used at most once in any representation.

THEOREM V

The Lucas number sequence, where $a_n = L_{n-1}$ $(n \geq 1)$, is complete.

We know from (I_2) (page 54) that

$$L_1 + L_2 + L_3 + \cdots + L_n = L_{n+2} - 3,$$

and so $$L_0 + L_1 + L_2 + L_3 + \cdots + L_n = L_{n+2} - 1.$$

Since $a_1 = L_0 = 2, a_2 = L_1 = 1, a_3 = L_2 = 3, \ldots, a_{n+2} = a_{n+1} + a_n$, the sequence a_n is a generalized Fibonacci sequence whose sum is

$$a_1 + a_2 + a_3 + \cdots + a_n = a_{n+2} - 1.$$

Thus, this theorem can be proved by mathematical induction as Theorem II was proved.

Is the Lucas number sequence complete if any single term is missing? Clearly, without $a_1 = L_0$ we have no representation for 2, and without $a_2 = L_1$ we have no representation for 1. It is as yet an unanswered question as to which Lucas numbers can be left out without destroying completeness. For example, we can see that $L_3 = 4$ can be left out:

$1 = 1$	$4 = 3 + 1$	$7 = 7$	$10 = 7 + 3$
$2 = 2$	$5 = 3 + 2$	$8 = 7 + 1$	$11 = 11$
$3 = 3$	$6 = 3 + 2 + 1$	$9 = 7 + 2$	$12 = 11 + 1$

It would seem that if any Lucas number L_k, $k > 1$, is omitted, the resulting sequence is still complete. One would expect that if any two Lucas numbers were missing, the resulting sequence would be incomplete.

We now ask: If each integer m is to be represented by the *least* number of Fibonacci numbers what are the conditions for this *minimal representation*?

Certainly, we would never need both F_1 and F_2 in any minimal representation, and so we choose to use F_2 as the 1.

If, for a given integer m, the Fibonacci numbers in any representation are arranged in order of size, let us direct our attention to the largest, say, F_n. Clearly, if the representation has more than one Fibonacci number, then it cannot contain both F_n and F_{n-1} and be a minimal representation because we could replace $F_n + F_{n-1}$ by F_{n+1} and thereby reduce the number used.

Now, if we have F_n but not F_{n-1}, then we could have F_{n-2} but not both F_{n-2} and F_{n-3} because then we could replace $F_{n-2} + F_{n-3}$ by F_{n-1} and next $F_{n-1} + F_n$ by F_{n+1}, thereby getting a double reduction. We can see, therefore, that any minimal representation cannot have two adjacent Fibonacci numbers.

These restrictions are precisely the conditions imposed by the following interesting theorem, which we state without proof.

THEOREM VI (E. Zeckendorf)*

> Each positive integer m can be represented as the sum of distinct numbers in the sequence defined by $a_n = F_{n+1}$ $(n \geq 1)$ using no two consecutive Fibonacci numbers, and such a representation is unique.

From this we see that each positive integer has a minimal representation as described above, and such a representation is unique.

If in a representation of a positive integer m using the terms of the sequence

$$1, 2, 3, 5, 8, \ldots, F_{n+1}, \ldots$$

we desire to use the *maximum* number of these Fibonacci numbers, thus obtaining a *maximal representation*, we should replace F_k by $F_{k-1} + F_{k-2}$ whenever possible (avoiding repetitions, of course). This process results in the conditions described in the following theorem, which we state without proof.

THEOREM VII†

> Each positive integer m can be represented as the sum of distinct numbers in the sequence defined by $a_n = F_{n+1}$ $(n \geq 1)$ with the condition that whenever F_k $(k \geq 4)$ is used, at least one of each pair F_q, F_{q-1} $(3 \leq q < k)$ must be used, and such a representation is unique.

* John L. Brown, Jr., "Zeckendorf's Theorem and Some Applications," *The Fibonacci Quarterly*, Vol. 2, No. 3 (October, 1964), pages 163–168.

† John L. Brown, Jr., "A New Characteristic of the Fibonacci Numbers," *The Fibonacci Quarterly*, Vol. 3, No. 1 (February, 1965), pages 1–8.

From Theorem VII we see that each positive integer has a maximal representation as described on page 74, and such a representation is unique.

It turns out that all positive integers have both a unique minimum and a unique maximum representation (while some have only one representation, which satisfies both conditions) from the sequence:

F_2	F_3	F_4	F_5	F_6	F_7	F_8	...
1	2	3	5	8	13	21	...

We have the following representations (using each term in the sequence at most once) for 1–21:

Minimal	*Same*	*Maximal*
$1 =$	F_2	
$2 =$	F_3	
$3 = F_4$		$F_3 + F_2$
$4 =$	$F_4 + F_2$	
$5 = F_5$		$F_4 + F_3$
$6 = F_5 + F_2$		$F_4 + F_3 + F_2$
$7 =$	$F_5 + F_3$	
$8 = F_6$		$F_5 + F_3 + F_2$
$9 = F_6 + F_2$		$F_5 + F_4 + F_2$
$10 = F_6 + F_3$		$F_5 + F_4 + F_3$
$11 = F_6 + F_4$		$F_5 + F_4 + F_3 + F_2$
$12 =$	$F_6 + F_4 + F_2$	
$13 = F_7$		$F_6 + F_4 + F_3$
$14 = F_7 + F_2$		$F_6 + F_4 + F_3 + F_2$
$15 = F_7 + F_3$		$F_6 + F_5 + F_3$
$16 = F_7 + F_4$		$F_6 + F_5 + F_3 + F_2$
$17 = F_7 + F_4 + F_2$		$F_6 + F_5 + F_4 + F_2$
$18 = F_7 + F_5$		$F_6 + F_5 + F_4 + F_3$
$19 = F_7 + F_5 + F_2$		$F_6 + F_5 + F_4 + F_3 + F_2$
$20 =$	$F_7 + F_5 + F_3$	
$21 = F_8$		$F_7 + F_5 + F_3 + F_2$

Notice that each integer $F_k - 1$ $(k \geq 3)$ has a single representation satisfying both minimal and maximal conditions, and *these are the only positive integers for which this is true.**

* David A. Klarner, "Representations of N as a Sum of Distinct Elements from Special Sequences," *The Fibonacci Quarterly*, Vol. 4, No. 4 (December, 1966), pages 289–306.

A standard puzzle problem is to determine the number of separate weights needed to weigh any given integral number of pounds, supposing that the thing to be weighed is placed on one side and the weights are to be placed on the other side of the balance system. You can see that the solution is related to the idea of completeness of a sequence of positive integers as defined at the beginning of this section.

For example, any integral number of pounds from 1 to 31 can be weighed with one 1-pound weight, one 2-pound weight, one 4-pound weight, one 8-pound weight, and one 16-pound weight, because of Theorem I.

Now suppose, as a fairy tale, that Fibonacci was a professional weigher with weights $F_1, F_2, F_3, F_4, \ldots, F_n, \ldots$ pounds, traveling from place to place weighing things for people. He had to be able to weigh any positive integral number of pounds.

Theorem II tells us that Fibonacci could always do his job if he had all of his weights with him.

Theorem III tells us that Fibonacci could still do his job if he somehow lost one of his Fibonacci weights.

Theorem IV tells us that Fibonacci could not do his job if he somehow lost any two of his Fibonacci weights.

Theorem VI (Zeckendorf) states that if Fibonacci did not use his F_1 weight and lined up the rest of his weights in order of size, then any job he might be given would have *just one* solution if he did *not* choose any two weights which were adjacent in the lineup.

Theorem VII states that if Fibonacci did not use his F_1 weight and lined up the rest of his weights in order of size, then any job he might be given would have *just one* solution if whenever he used his F_k weight ($k \geq 4$), he also used at least one of each pair of weights, F_q and F_{q-1} ($3 \leq q < k$).

Theorems for Lucas numbers corresponding to Theorems VI and VII on page 74 are the following.

THEOREM VIII

Each positive integer m can be represented as the sum of distinct numbers in the sequence $a_n = L_{n-1}$ ($n \geq 1$) with the conditions that no two consecutive Lucas numbers are used in the same representation and that L_0 and L_2 are not both used in the same representation, and such a representation is unique.

THEOREM IX

Each positive integer m can be represented as the sum of distinct numbers in the sequence $a_n = L_{n-1}$ $(n \geq 1)$ with the conditions that whenever L_k $(k \geq 2)$ is used, at least one of each pair L_q, L_{q-1} $(1 \leq q < k)$ must be used and L_1 and L_3 are not both to be used unless L_0 or L_2 is also in the same representation, and each such representation is unique.

From the sequence

L_1	L_0	L_2	L_3	L_4	L_5	L_6	...
1	2	3	4	7	11	18	...

we have the following representations (using each term in the sequence at most once) for 1–18:

Theorem VIII	*Same*	*Theorem IX*
$1 =$	L_1	
$2 =$	L_0	
$3 = L_2$		$L_0 + L_1$
$4 = L_3$		$L_1 + L_2$
$5 = L_3 + L_1$		$L_2 + L_0$
$6 =$	$L_3 + L_0$	
$7 = L_4$		$L_3 + L_1 + L_0$
$8 = L_4 + L_1$		$L_3 + L_2 + L_1$
$9 = L_4 + L_0$		$L_3 + L_2 + L_0$
$10 = L_4 + L_2$		$L_3 + L_2 + L_1 + L_0$
$11 = L_5$		$L_4 + L_2 + L_1$
$12 = L_5 + L_1$		$L_4 + L_2 + L_0$
$13 = L_5 + L_0$		$L_4 + L_3 + L_0$
$14 = L_5 + L_2$		$L_4 + L_3 + L_1 + L_0$
$15 = L_5 + L_3$		$L_4 + L_3 + L_2 + L_1$
$16 = L_5 + L_3 + L_1$		$L_4 + L_3 + L_2 + L_0$
$17 = L_5 + L_3 + L_0$		$L_4 + L_3 + L_2 + L_1 + L_0$
$18 = L_6$		$L_5 + L_3 + L_1 + L_0$

Notice that the representation according to Theorem VIII is the minimal representation and the representation according to Theorem IX is the maximal representation.

EXERCISES

1. Find a representation of each integer 1, 2, 3, . . . , 20 using Fibonacci numbers with distinct subscripts, including $F_1 = 1$ but omitting $F_5 = 5$.

2. Find the first integer not representable in terms of Fibonacci numbers with distinct subscripts if you are denied use of Fibonacci numbers $F_4 = 3$ and $F_6 = 8$.

3. Express 27 as the sum of three distinct Fibonacci numbers. In how many ways can you do this?

4. Express 1966 in Zeckendorf form. The available Fibonacci numbers are: 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, and 1597.

5. Find the minimal and maximal representations of 32 using distinct terms of the sequence $F_2, F_3, \ldots$.

6. Find the minimal and maximal representations of 32 using distinct terms of the sequence $L_1, L_0, L_2, L_3, \ldots$.

13 • *Fibonacci Numbers in Nature*

Fibonacci numbers appear in nature in a number of unexpected ways.

What do you notice about the number of petals (or petal-like parts) in the flowers pictured in Figure 20?

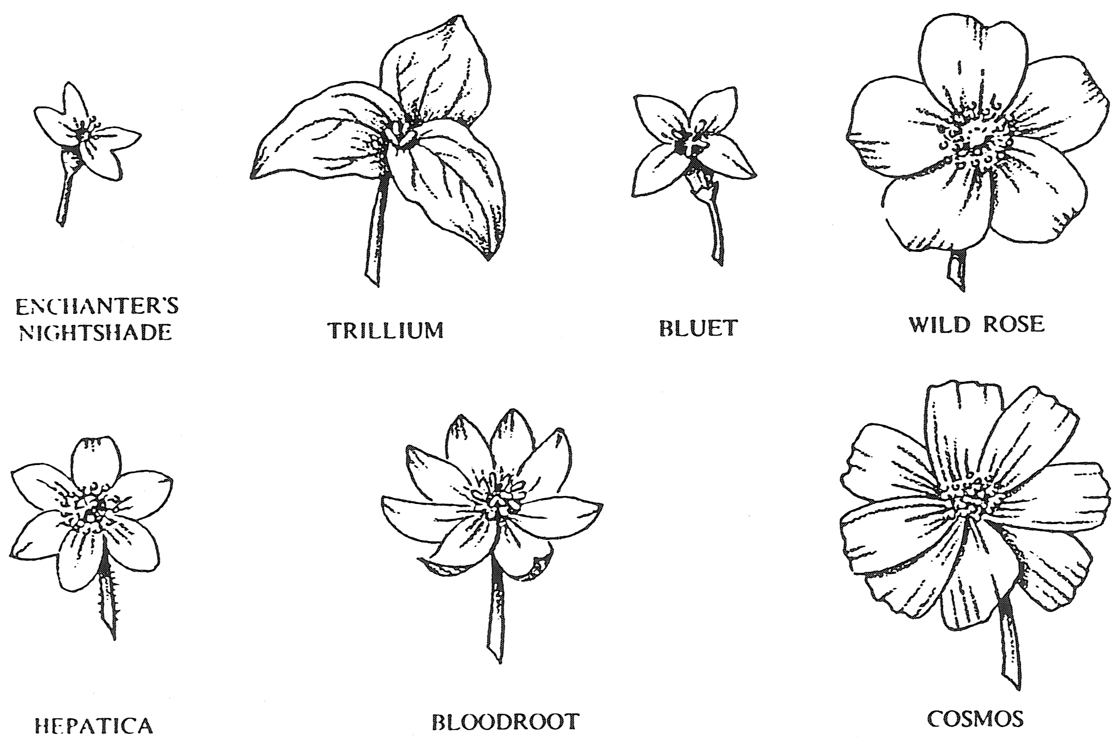


Figure 20

The enchanter's nightshade has 2 petals (even though they are deeply cleft and appear to be four). Thus, the flowers above have 2, 3, 5, or 8 (Fibonacci numbers), or 4 (a Lucas number), or 6 petals or petal-like parts. Others with 5 petals or parts are buttercups and columbine, for example. A day lily blossom has 2 sets of 3 petal-like parts, alternating. An iris blossom has 3 standards (vertical parts) alternating with 3 falls. It is interesting to observe that so many such numbers occurring in nature are Fibonacci numbers.

The numbers of petal-like parts of a flower, such as an aster, cosmos, daisy, or gaillardia, in the composite family is consistently a Fibonacci number or very close to one. A flower of a single variety of gaillardia may have 13 of these parts. Frank Land* reports finding 21, 34, 55, and 89 "petals" on daisies and other members of the composite family.

You can also find Fibonacci numbers in the arrangement of leaves (or twigs) on a stem. Select one leaf as a starting point and then count up the stem until you reach a leaf that is directly above your starting point. The number of leaves is usually a Fibonacci number. Then suppose that a string were wound around the stem following the leaves. The number of turns taken around the stem between the starting point and the leaf directly above it is also usually a Fibonacci number. (The turns may be clockwise or counterclockwise.) The result is often stated as a ratio:

$$\frac{\text{number of turns}}{\text{number of leaves}}$$

Some simplified examples are pictured in Figure 21:

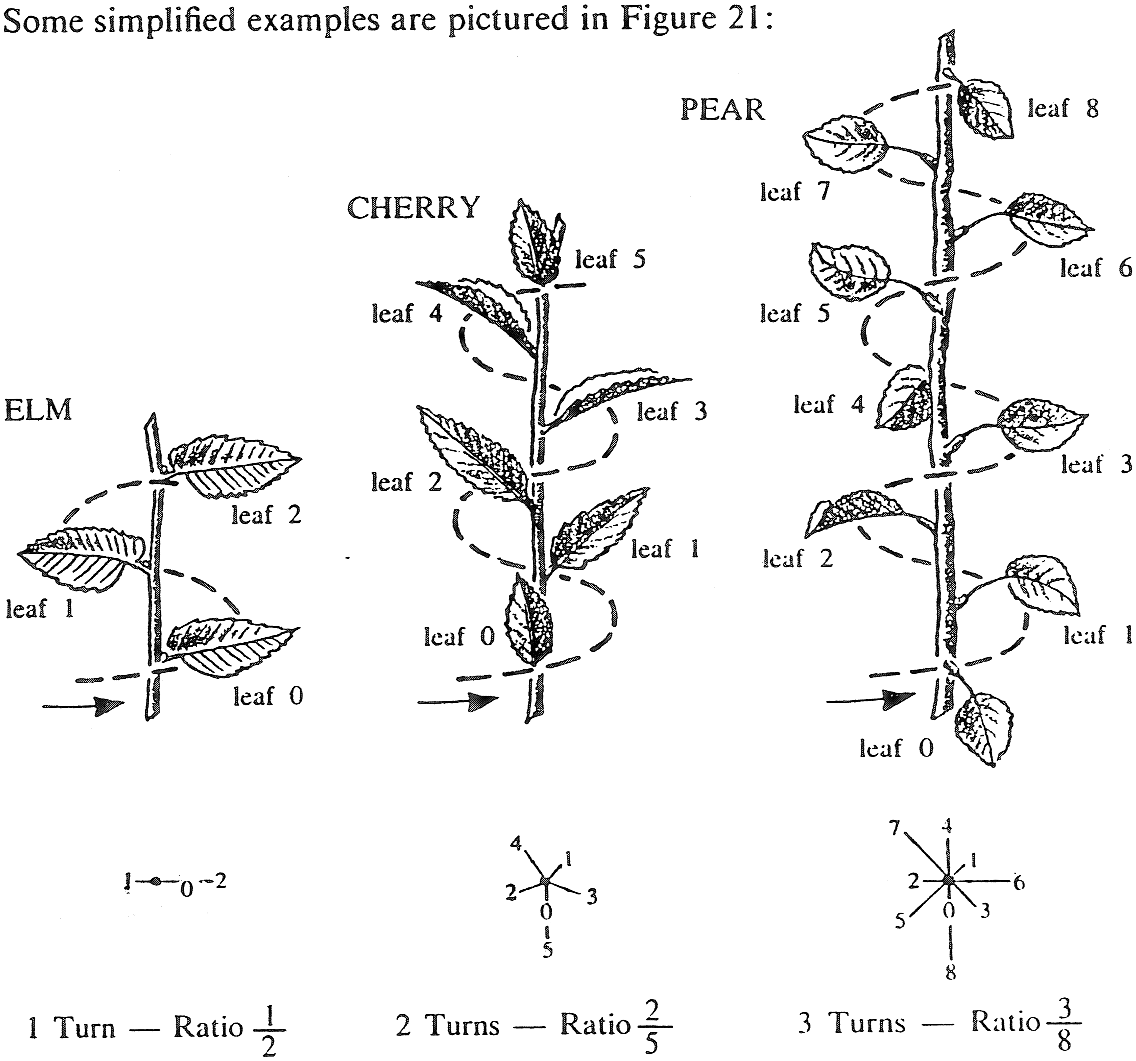
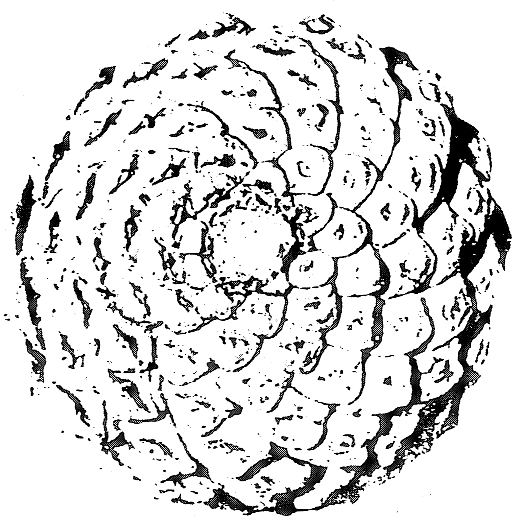


Figure 21

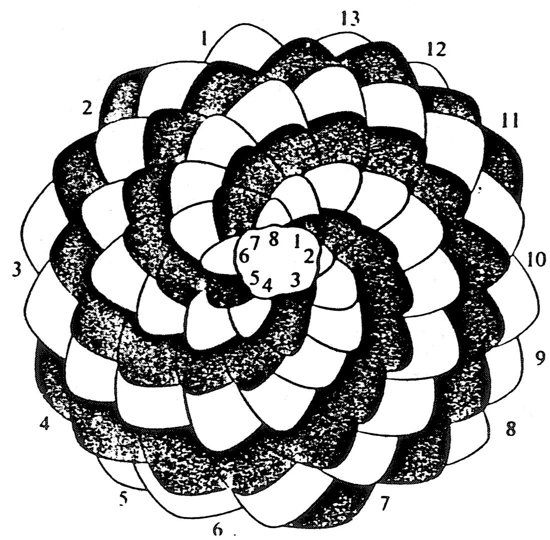
* Frank Land, *The Language of Mathematics* (London: John Murray, 1960), Chapter 13.

The beech tree has a ratio of $\frac{1}{3}$, and the ratio for the pussy willow is $\frac{5}{13}$! Such arrangement of leaves on a stem is called *phyllotaxis* (from Greek *phyllon* meaning *leaf* and *taxis* meaning *arrangement*).

Perhaps the best known examples of the appearance of Fibonacci numbers (and occasionally Lucas numbers) in nature are the numbers of spirals in the seed patterns of sunflowers, the scale patterns of pine cones, and so on. A pine cone is pictured in Figure 22, and the two sets of spirals are pictured in the diagram beside it.



The seed-bearing scales of a pine cone grow outward in a spiral pattern from the point where it is attached to the branch.



13 strips spiral to the left.
8 strips spiral to the right.

Figure 22

The author has raised a sunflower that had 89 spirals to the right and 55 spirals to the left and also one, eighteen inches across, that had 144 spirals to the right, 89 spirals to the left, and 55 shallow spirals to the right. An excellent photograph of the head of a daisy is shown on page 93 of the volume on *Mathematics* in the *Life Science Library*.* The diagram accompanying it shows clearly its 34 spirals to the left and 21 to the right.

A particularly interesting specimen of sunflower was given by the author to Brother Alfred Brousseau of St. Mary's College, California. After he had patiently counted the spirals, he found 123 spirals to the right, 76 spirals to the left, and a shallow set of 47 spirals to the right — a Lucas number sunflower! Brother Alfred Brousseau has also collected and classified by their Fibonacci spiral patterns cones from all but one of the twenty species of California pines. He reported his results in an article, "On the Trail of the California Pine," in *The Fibonacci Quarterly*, Vol. 6, No. 1 (February, 1968), pp. 69–76.

* David Bergamini and the Editors of *Life*, *Mathematics*, Life Science Library (New York: Time Incorporated, 1963).

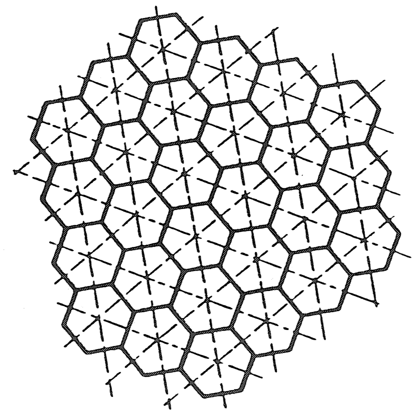


Figure 23

The spiral curves described on page 81 are called *parastichies* (from Greek *para* meaning *beside* and *stichos* meaning *row*). Parastichies on pineapples are especially interesting. Since pineapple scales are roughly hexagonal in shape, three sets of spirals can be found, as pictured in Figure 23. Parastichies on pineapples are described on pages 21–22 of *Mathematical Diversions* by J. A. H. Hunter and Joseph S. Madachy (Princeton, N. J.: D. Van Nostrand Co., Inc., 1963).

Other references for discussions of Fibonacci numbers in nature are:

E. J. Karchmar, "Phyllotaxis," *The Fibonacci Quarterly*, Vol. 3, No. 1 (February, 1965), pp. 64–66.

Sr. Mary de Sales McNabb, "Phyllotaxis," *The Fibonacci Quarterly*, Vol. 1, No. 4 (December, 1963), pp. 57–60.

A List of the First 100 Fibonacci Numbers and the First 100 Lucas Numbers

F_{1}	1	L_{1}	1
F_{2}	1	L_{2}	3
F_{3}	2	L_{3}	4
F_{4}	3	L_{4}	7
F_{5}	5	L_{5}	11
F_{6}	8	L_{6}	18
F_{7}	13	L_{7}	29
F_{8}	21	L_{8}	47
F_{9}	34	L_{9}	76
F_{10}	55	L_{10}	123
F_{11}	89	L_{11}	199
F_{12}	144	L_{12}	322
F_{13}	233	L_{13}	521
F_{14}	377	L_{14}	843
F_{15}	610	L_{15}	1364
F_{16}	987	L_{16}	2207
F_{17}	1597	L_{17}	3571
F_{18}	2584	L_{18}	5778
F_{19}	4181	L_{19}	9349
F_{20}	6765	L_{20}	15127
F_{21}	10946	L_{21}	24476
F_{22}	17711	L_{22}	39603
F_{23}	28657	L_{23}	64079
F_{24}	46368	L_{24}	103682
F_{25}	75025	L_{25}	167761
F_{26}	121393	L_{26}	271443
F_{27}	196418	L_{27}	439204
F_{28}	317811	L_{28}	710647
F_{29}	514229	L_{29}	1149851
F_{30}	832040	L_{30}	1860498
F_{31}	1346269	L_{31}	3010349
F_{32}	2178309	L_{32}	4870847
F_{33}	3524578	L_{33}	7881196
F_{34}	5702887	L_{34}	12752043
F_{35}	9227465	L_{35}	20633239
F_{36}	14930352	L_{36}	33385282
F_{37}	24157817	L_{37}	54018521
F_{38}	39088169	L_{38}	87403803
F_{39}	63245986	L_{39}	141422324
F_{40}	102334155	L_{40}	228826127
F_{41}	165580141	L_{41}	370248451
F_{42}	267914296	L_{42}	599074578
F_{43}	433494437	L_{43}	969323029
F_{44}	701408733	L_{44}	1568397607
F_{45}	1134903170	L_{45}	2537720636

F_{46}	1836311903	L_{46}	4106118243
F_{47}	2971215073	L_{47}	6643838879
F_{48}	4807526976	L_{48}	1 0749957122
F_{49}	7778742049	L_{49}	1 7393796001
F_{50}	1 2586269025	L_{50}	2 8143753123
F_{51}	2 0365011074	L_{51}	4 5537549124
F_{52}	3 2951280099	L_{52}	7 3681302247
F_{53}	5 3316291173	L_{53}	11 9218851371
F_{54}	8 6267571272	L_{54}	19 2900153618
F_{55}	13 9583862445	L_{55}	31 2119004989
F_{56}	22 5851433717	L_{56}	50 5019158607
F_{57}	36 5435296162	L_{57}	81 7138163596
F_{58}	59 1286729879	L_{58}	132 2157322203
F_{59}	95 6722026041	L_{59}	213 9295485799
F_{60}	154 8008755920	L_{60}	346 1452808002
F_{61}	250 4730781961	L_{61}	560 0748293801
F_{62}	405 2739537881	L_{62}	906 2201101803
F_{63}	655 7470319842	L_{63}	1466 2949395604
F_{64}	1061 0209857723	L_{64}	2372 5150497407
F_{65}	1716 7680177565	L_{65}	3838 8099893011
F_{66}	2777 7890035288	L_{66}	6211 3250390418
F_{67}	4494 5570212853	L_{67}	10050 1350283429
F_{68}	7272 3460248141	L_{68}	16261 4600673847
F_{69}	11766 9030460994	L_{69}	26311 5950957276
F_{70}	19039 2490709135	L_{70}	42573 0551631123
F_{71}	30806 1521170129	L_{71}	68884 6502588399
F_{72}	49845 4011879264	L_{72}	111457 7054219522
F_{73}	80651 5533049393	L_{73}	180342 3556807921
F_{74}	130496 9544928657	L_{74}	291800 0611027443
F_{75}	211148 5077978050	L_{75}	472142 4167835364
F_{76}	341645 4622906707	L_{76}	763942 4778862807
F_{77}	552793 9700884757	L_{77}	1236084 8946698171
F_{78}	894439 4323791464	L_{78}	2000027 3725560978
F_{79}	1447233 4024676221	L_{79}	3236112 2672259149
F_{80}	2341672 8348467685	L_{80}	5236139 6397820127
F_{81}	3788906 2373143906	L_{81}	8472251 9070079276
F_{82}	6130579 0721611591	L_{82}	13708391 5467899403
F_{83}	9919485 3094755497	L_{83}	22180643 4537978679
F_{84}	16050064 3816367088	L_{84}	35889035 0005878082
F_{85}	25969549 6911122585	L_{85}	58069678 4543856761
F_{86}	42019614 0727489673	L_{86}	93958713 4549734843
F_{87}	67989163 7638612258	L_{87}	152028391 9093591604
F_{88}	110008777 8366101931	L_{88}	245987105 3643326447
F_{89}	177997941 6004714189	L_{89}	398015497 2736918051
F_{90}	288006719 4370816120	L_{90}	644002602 6380244498
F_{91}	466004661 0375530309	L_{91}	1042018099 9117162549
F_{92}	754011380 4746346429	L_{92}	1686020702 5497407047
F_{93}	1220016041 5121876738	L_{93}	2728038802 4614569596
F_{94}	1974027421 9868223167	L_{94}	4414059505 0111976643
F_{95}	3194043463 4990099905	L_{95}	7142098307 4726546239
F_{96}	5168070885 4858323072	L_{96}	1 1556157812 4838522882
F_{97}	8362114348 9848422977	L_{97}	1 8698256119 9565069121
F_{98}	1 3530185234 4706746049	L_{98}	3 0254413932 4403592003
F_{99}	2 1892299583 4555169026	L_{99}	4 8952670052 3968661124
F_{100}	3 5422484817 9261915075	L_{100}	7 9207083984 8372253127

Partial Solutions

Section 2

1–2. See list on page 83. **3.** Several will be developed later in the text.
4. 1, 4, 5, 9, 14, . . . , 7375, 11,933, 19,308

Section 3

1. Using the quadratic formula with $a = 1$, $b = -1$, and $c = -1$, you have

$$x = \frac{1 \pm \sqrt{1 - 4(1)(-1)}}{2} = \frac{1 \pm \sqrt{5}}{2}; \; \alpha = \frac{1 + \sqrt{5}}{2}, \; \beta = \frac{1 - \sqrt{5}}{2}.$$

2. $\alpha \doteq \dfrac{1 + 2.236}{2} \doteq 1.618$; $\beta \doteq \dfrac{1 - 2.236}{2} \doteq -.618$

3. a. $\alpha + \beta = \dfrac{1 + \sqrt{5}}{2} + \dfrac{1 - \sqrt{5}}{2} = 1$

b. $\alpha - \beta = \dfrac{1 + \sqrt{5}}{2} - \dfrac{1 - \sqrt{5}}{2} = \sqrt{5}$

c. $\alpha\beta = \left(\dfrac{1 + \sqrt{5}}{2}\right)\left(\dfrac{1 - \sqrt{5}}{2}\right) = \dfrac{(1)^2 - (\sqrt{5})^2}{4} = \dfrac{1 - 5}{4} = -1$

4. $1 + \dfrac{1}{\alpha} = 1 + \dfrac{1}{\dfrac{1 + \sqrt{5}}{2}} = 1 + \dfrac{2(1 - \sqrt{5})}{(1 + \sqrt{5})(1 - \sqrt{5})} = \dfrac{1 + \sqrt{5}}{2} = \alpha$

5. a. $\alpha^3 = \alpha(\alpha^2) = \alpha(\alpha + 1) = \alpha^2 + \alpha = (\alpha + 1) + \alpha = 2\alpha + 1$
b. $\alpha^4 = \alpha(\alpha^3) = \alpha(2\alpha + 1) = 2\alpha^2 + \alpha = 2(\alpha + 1) + \alpha = 3\alpha + 2$
c. $\alpha^5 = \alpha(\alpha^4) = \alpha(3\alpha + 2) = 3\alpha^2 + 2\alpha = 3(\alpha + 1) + 2\alpha = 5\alpha + 3$
Note: It can be proved by mathematical induction (reviewed on page 54) that, in general, $\alpha^n = F_n\alpha + F_{n-1}$, $n \geq 1$.

6. $\alpha^3 - \dfrac{1}{\alpha^3} = 2\alpha + 1 - \dfrac{1}{2\alpha + 1} = 2 + \sqrt{5} - \dfrac{1}{2 + \sqrt{5}}\left(\dfrac{2 - \sqrt{5}}{2 - \sqrt{5}}\right)$

$= 2 + \sqrt{5} + 2 - \sqrt{5} = 4 = L_3$

7. Since $\alpha\beta = -1$, $\alpha^{-1} = -\beta$; $\dfrac{\alpha^4 - \alpha^{-4}}{\sqrt{5}} = \dfrac{\alpha^4 - \beta^4}{\alpha - \beta} = F_4$.

Section 4

1. In any triangle, we must have $a + b > c$, $b + c > a$, $c + a > b$. For any three consecutive Fibonacci numbers, $F_n + F_{n+1} = F_{n+2}$, and so there can be no triangle with sides having measures F_n, F_{n+1}, F_{n+2}. In general, consider Fibonacci numbers F_p, F_q, F_r, where $F_p \leq F_{q-1}$ and $F_{q+1} \leq F_r$. Since $F_{q-1} + F_q = F_{q+1}$ and $F_p \leq F_{q-1}$, we have $F_p + F_q \leq F_{q+1}$, and since $F_{q+1} \leq F_r$, we have $F_p + F_q \leq F_r$. Therefore, there can be no triangle with sides having measures F_p, F_q, and F_r.

2. The measures of the sides of the triangles must be (page 17) a, ra, r^2a and ra, r^2a, r^3a with $r \neq 1$ (page 21). Therefore, it is impossible to have $a = ra$ or $a = r^2a$ or $ra = r^2a$, and so on, and so neither triangle can be isosceles.

3. a. $y = (x - a)(x - b),\ a < b$
If $a < x < b$, then $x - a > 0$ and $x - b < 0$, and so $y < 0$.
If $x \leq a$, then $x - a \leq 0$ and $x - b < 0$, and so $y \geq 0$.
If $x \geq b$, then $x - a > 0$ and $x - b \geq 0$, and so $y \geq 0$.

b. If $a = b$, then $y = (x - a)^2$; $(x - a)^2 \geq 0$ for all x.

4. a. $r^2 - r - 1 = 0$ has roots α and β, $\beta < \alpha$. Thus, $r^2 - r - 1 = (r - \alpha)(r - \beta)$, and from Ex. 3a, $r^2 - r - 1 < 0$ for $\beta < r < \alpha$.

b. $r^2 + r - 1 = 0$ has roots $-\alpha$ and $-\beta$, $-\alpha < -\beta$. Thus, $r^2 + r - 1 = [r - (-\alpha)][r - (-\beta)]$, and from Ex. 3a, $r^2 + r - 1 > 0$ when $r < -\alpha$ (both factors negative) or $r > -\beta$ (both factors positive). (Of course, in the application in Prob. 4, $r > 0$.)

c. Since $-\beta = \dfrac{\sqrt{5} - 1}{2}$ and $\alpha = \dfrac{1 + \sqrt{5}}{2}$, $-\beta < \alpha$. Since $\beta < -\beta < \alpha$,
$$\{r:\ \beta < r < \alpha\} \cap \{r:\ r < -\alpha \ \text{ or } \ r > -\beta\} = \{r:\ -\beta < r < \alpha\}.$$
Since $\alpha\beta = -1$, we have $-\beta = \dfrac{1}{\alpha}$, and $\dfrac{1}{\alpha} < r < \alpha$.

5. From Ex. 3b, $(r - 1)^2 \geq 0$ for all r. Thus, $r^2 - 2r + 1 \geq 0$, or $r^2 + 1 \geq 2r$, for all r. If $r > 0$, $2r > r$, and so $r^2 + 1 > r$ for $r > 0$.

6. If $r > 1$, $a < ar < ar^2$ and $\dfrac{a}{r^2} < \dfrac{a}{r} < a$. The sides can be paired by measures:
$$a \leftrightarrow \frac{a}{r^2},\quad ar \leftrightarrow \frac{a}{r},\quad ar^2 \leftrightarrow a.\quad \frac{\frac{a}{r^2}}{a} = \frac{\frac{a}{r}}{ar} = \frac{a}{ar^2} = \frac{1}{r^2},$$ the ratio of similarity.

7. Since $CG = DB = s + t$ and $CD = GB = s$, $\dfrac{CG}{GB} = \dfrac{s + t}{s} = \dfrac{s}{t} = \alpha$, and so $DCGB$ is a Golden Rectangle.

8. Since $\triangle AFB \sim \triangle FEB$, $\dfrac{AF}{FB} = \dfrac{FE}{EB} = \dfrac{s}{t} = \alpha$.

9. a. Since $\dfrac{s}{t} = \alpha$,
$$\frac{s^2 + 2st}{s^2 + t^2} = \frac{\frac{s^2}{t^2} + 2\left(\frac{s}{t}\right)}{\frac{s^2}{t^2} + 1} = \frac{\alpha^2 + 2\alpha}{\alpha^2 + 1} = \frac{\alpha(\alpha + 2)}{\alpha + 2} = \alpha.$$

b. $$\frac{t^2 - 2st}{s^2 + t^2} = \frac{s^2 + t^2}{s^2 + t^2} - \frac{s^2 + 2st}{s^2 + t^2} = 1 - \alpha = \beta$$

c. $$\frac{s^2 + 4st - t^2}{s^2 + t^2} = \frac{s^2 + 2st}{s^2 + t^2} - \frac{t^2 - 2st}{s^2 + t^2} = \alpha - \beta = \sqrt{5}$$

Section 5

1. $$F_{2n} = \frac{\alpha^{2n} - \beta^{2n}}{\alpha - \beta} = \left(\frac{\alpha^n - \beta^n}{\alpha - \beta}\right)(\alpha^n + \beta^n) = F_n L_n,\ n \geq 1$$

2. $$F_{n-1} + F_{n+1} = \frac{(\alpha^{n-1} - \beta^{n-1}) + (\alpha^{n+1} - \beta^{n+1})}{\alpha - \beta}$$
$$= \frac{\alpha^{n+1} + \alpha^{n-1} - \beta^{n+1} - \beta^{n-1}}{\alpha - \beta} = \frac{\alpha^{n+1} - (\alpha\beta)\alpha^{n-1} + (\alpha\beta)\beta^{n-1} - \beta^{n+1}}{\alpha - \beta}$$
$$= \frac{\alpha^n(\alpha - \beta) + \beta^n(\alpha - \beta)}{\alpha - \beta} = \alpha^n + \beta^n = L_n,\ \text{since } \alpha\beta = -1.$$

3. $L_{-n} = \alpha^{-n} + \beta^{-n}$, $n > 0$. Since $\alpha\beta = -1$, we have $\alpha^{-n} = (-1)^n\beta^n$ and $\beta^{-n} = (-1)^n\alpha^n$. Thus, $L_{-n} = (-1)^n(\alpha^n + \beta^n) = (-1)^n L_n$.

4. $F_n^2 = \left(\dfrac{\alpha^n - \beta^n}{\sqrt{5}}\right)^2$. Thus, $5F_n^2 = (\alpha^n - \beta^n)^2 = \alpha^{2n} - 2\alpha^n\beta^n + \beta^{2n}$
$$= \alpha^{2n} + \beta^{2n} - 2(-1)^n = L_{2n} - 2(-1)^n.$$

5. $L_n^2 = (\alpha^n + \beta^n)^2 = \alpha^{2n} + 2\alpha^n\beta^n + \beta^{2n} = \alpha^{2n} + \beta^{2n} + 2(-1)^n = L_{2n} + 2(-1)^n$.

6. $F_{n+1}L_n - L_{n+1}F_n$
$$= \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}(\alpha^n + \beta^n) - (\alpha^{n+1} + \beta^{n+1})\frac{\alpha^n - \beta^n}{\alpha - \beta}$$
$$= \frac{\alpha^{2n+1} - \beta^{n+1}\alpha^n + \alpha^{n+1}\beta^n - \beta^{2n+1} - \alpha^{2n+1} - \beta^{n+1}\alpha^n + \alpha^{n+1}\beta^n + \beta^{2n+1}}{\alpha - \beta}$$
$$= \frac{-\beta^{n+1}\alpha^n + \alpha^{n+1}\beta^n - \beta^{n+1}\alpha^n + \alpha^{n+1}\beta^n}{\alpha - \beta} = \frac{2(\alpha^{n+1}\beta^n - \alpha^n\beta^{n+1})}{\alpha - \beta}$$
$$= \frac{2(\alpha^n\beta^n)(\alpha - \beta)}{\alpha - \beta} = 2(-1)^n$$

7. $\dfrac{F_{n+1}}{F_n} - \dfrac{L_{n+1}}{L_n} = \dfrac{F_{n+1}L_n - L_{n+1}F_n}{F_nL_n} = \dfrac{2(-1)^n}{F_{2n}}$ by Ex. 6 and 1.

8. If $n < 0$, then $-n > 0$. For $-n > 0$, $F_{-(-n)} = (-1)^{-n+1}F_{-n}$,

or $F_{-n} = \dfrac{F_n}{(-1)^{-n+1}} = \dfrac{(-1)^{n+1}F_n}{(-1)^2} = (-1)^{n+1}F_n$.

9. If $n < 0$, then $-n > 0$. For $-n > 0$, $L_{-(-n)} = (-1)^{-n}L_{-n}$,

or $L_{-n} = \dfrac{L_n}{(-1)^{-n}} = (-1)^n L_n$.

10. From Ex. 4, $5F_n^2 = L_{2n} - 2(-1)^n$. From Ex. 5, $L_{2n} = L_n^2 - 2(-1)^n$. Thus, $5F_n^2 = L_n^2 - 4(-1)^n$.

11. $L_n = F_{n+1} + F_{n-1} = (F_{n+2} - F_n) + (F_n - F_{n-2}) = F_{n+2} - F_{n-2}$

Section 6

1. $F_{12} \doteq \dfrac{\alpha^{12}}{\sqrt{5}}$. $\log\dfrac{\alpha^{12}}{\sqrt{5}} \doteq 12(0.20898) - 0.3494 \doteq 2.5078 - 0.3494 \doteq 2.1584$.

Thus, $F_{12} = 144$.

2. $L_{12} \doteq \alpha^{12}$. $\log \alpha^{12} \doteq 2.5078$. Thus, $L_{12} = 322$.

3. $F_{34} \doteq \dfrac{\alpha^{34}}{\sqrt{5}} \doteq 570 \times 10^4$

4. $L_{33} \doteq \alpha^{34} \doteq 788 \times 10^4$

5. If $F_n = 1597$, then $F_{n+1} = \left[\dfrac{1597 + 1 + \sqrt{12752045}}{2}\right] = \left[\dfrac{1598 + 3571}{2}\right]$
$$= \left[\frac{5169}{2}\right] = [2584.5] = 2584.$$

6. If $L_n = 2207$, then $L_{n+1} = \left[\dfrac{2207 + 1 + \sqrt{24354245}}{2}\right] = \left[\dfrac{2208 + 4935}{2}\right]$
$$= \left[\frac{7143}{2}\right] = [3571.5] = 3571.$$

Section 7

1. $F_7 = 13$; $F_{14} = 13(29)$; $F_{21} = 13(842)$; $F_{28} = 13(24{,}447)$
2. $F_{10} = 55$; $F_{20} = 55(123)$; $F_{30} = 55(15{,}128)$; $F_{40} = 55(1{,}860{,}621)$
3. $F_{24} = 46{,}368 = 2(23{,}184) = 3(15{,}456) = 8(5796) = 21(2208) = 144(322)$
4. $F_{30} = 832{,}040 = 2(416{,}020) = 5(166{,}408) = 8(104{,}005)$ $= 55(15{,}128) = 610(1364)$
5. $L_4 = 7$; $F_8 = 7(3)$; $F_{16} = 7(141)$
6. $L_7 = 29$; $F_{14} = 29(13)$; $F_{28} = 29(10{,}959)$
7. $L_4 = 7$; $L_{12} = 7(46)$; $L_{20} = 7(2161)$
8. $L_5 = 11$; $L_{15} = 11(124)$; $L_{25} = 11(15{,}251)$
9. $F_{12} = 144 = 2(76 - 4) = 2(4)(18) = 3(47 + 1) = 8(18)$
10. $F_{18} = 2584 = 2(1364 - 76 + 4) = 8(322 + 1) = 34(76)$
11. $(F_{16}, F_{24}) = F_{(16,\,24)} = F_8 = 21$
12. $(F_{24}, F_{36}) = F_{(24,\,36)} = F_{12} = 144$

Section 8

1. Since 13 is F_7, the entry point is 7. The remainders on dividing by 13 are $R_0 = 0$, $R_1 = 1, \ldots,$ $R_7 = 0$, $R_8 = 8, \ldots,$ $R_{14} = 0$, $R_{15} = 12, \ldots,$ $R_{21} = 0$, $R_{22} = 5, \ldots,$ $R_{28} = 0$, $R_{29} = 1, \ldots,$ and so $K_{13} = 28$.
2. The remainders on dividing by 13 are $R_0 = 2$, $R_1 = 1, \ldots,$ $R_{19} = 2$, $R_{20} = 8, \ldots,$ $R_{23} = 2$, $R_{24} = 7, \ldots,$ $R_{28} = 2$, $R_{29} = 1, \ldots,$ and so $K_{13} = 28$. No remainder is 0, and so 13 does not divide any Lucas number.
3. The remainders on dividing by 10 are $R_0 = 0$, $R_1 = 1, \ldots, R_{15} = 0$, $R_{16} = 7$, $\ldots,$ $R_{30} = 0$, $R_{31} = 9, \ldots,$ $R_{45} = 0$, $R_{46} = 3, \ldots,$ $R_{60} = 0$, $R_{61} = 1$, $\ldots,$ and so $K_{10} = 60$. The entry point is 15.
4. The remainders on dividing by 6 are $R_0 = 2$, $R_1 = 1, \ldots,$ $R_6 = 0, \ldots,$ $R_{15} = 2$, $R_{16} = 5, \ldots,$ $R_{18} = 0, \ldots,$ $R_{21} = 2$, $R_{22} = 3, \ldots,$ $R_{24} = 2$, $R_{25} = 1, \ldots,$ and so $k_6 = 24$. The entry point is 6.
5. The remainders on dividing by 10 are $R_0 = 2$, $R_1 = 1, \ldots,$ $R_{12} = 2$, $R_{13} = 1, \ldots,$ and so $K_{10} = 12$. No remainder is 0, and so 10 has no entry point in the Lucas numbers. (Also, since no Lucas number is divisible by 5 (page 45), no Lucas number is divisible by 10.)

Section 9

From (A) on page 49:

1. With $x = 1$ and $y = x$, $\displaystyle\sum_{i=0}^{n} \binom{n}{i} x^i = (1 + x)^n.$
2. With $x = 1, y = 1$, $\displaystyle\sum_{i=0}^{n} \binom{n}{i} = 2^n.$
3. With $x = 1, y = -1$, $\displaystyle\sum_{i=0}^{n} \binom{n}{i} (-1)^i = 0.$
4. $\displaystyle\binom{n}{0} = \frac{n!}{0!(n-0)!} = 1;\quad \binom{n}{n} = \frac{n!}{n!(n-n)!} = 1.\quad \binom{n}{m} = \frac{n!}{m!(n-m)!};$

$$\binom{n-1}{m} + \binom{n-1}{m-1} = \frac{(n-1)!}{m!(n-1-m)!} + \frac{(n-1)!}{(m-1)!(n-m)!}$$
$$= \frac{(n-m)(n-1)! + m(n-1)!}{m!(n-m)!} = \frac{n!}{m!(n-m)!} = \binom{n}{m}$$

5. $\displaystyle\sum_{i=0}^{4} \binom{4}{i} F_i = \binom{4}{0} F_0 + \binom{4}{1} F_1 + \binom{4}{2} F_2 + \binom{4}{3} F_3 + \binom{4}{4} F_4$

$$= F_0 + 4F_1 + 6F_2 + 4F_3 + F_4 = 21 = F_8$$

6. $\sum_{i=0}^{n}\binom{n}{i}F_{i+j} = \frac{1}{\alpha-\beta}\left[\sum_{i=0}^{n}\binom{n}{i}\alpha^{i+j} - \sum_{i=0}^{n}\binom{n}{i}\beta^{i+j}\right]$

$= \frac{1}{\alpha-\beta}\left[\alpha^j(1+\alpha)^n - \beta^j(1+\beta)^n\right] = \frac{1}{\alpha-\beta}[\alpha^{2n+j} - \beta^{2n+j}]$

$= F_{2n+j}$, since $1+\alpha = \alpha^2$ and $1+\beta = \beta^2$.

7. $\sum_{i=0}^{n}\binom{n}{i}L_{i+j} = \sum_{i=0}^{n}\binom{n}{i}\alpha^{i+j} + \sum_{i=0}^{n}\binom{n}{i}\beta^{i+j}$

$= \alpha^j(1+\alpha)^n + \beta^j(1+\beta)^n = \alpha^{2n+j} + \beta^{2n+j} = L_{2n+j}$

8. $\sum_{i=0}^{n}\binom{n}{i}(-1)^iF_{i+j} = \frac{1}{\alpha-\beta}\left[\sum_{i=0}^{n}\binom{n}{i}(-1)^i\alpha^{i+j} - \sum_{i=0}^{n}\binom{n}{i}(-1)^i\beta^{i+j}\right]$

$= \frac{1}{\alpha-\beta}\left[\alpha^j\sum_{i=0}^{n}\binom{n}{i}(-\alpha)^i - \beta^j\sum_{i=0}^{n}\binom{n}{i}(-\beta)^i\right]$

$= \frac{1}{\alpha-\beta}[\alpha^j(1-\alpha)^n - \beta^j(1-\beta)^n] = \frac{\alpha^j\beta^n - \beta^j\alpha^n}{\alpha-\beta}$

$= \frac{(\alpha\beta)^j(\beta^{n-j} - \alpha^{n-j})}{\alpha-\beta} = (-1)^{j+1}F_{n-j}$

9. $\sum_{i=0}^{n}\binom{n}{i}(-1)^iF_{2i+j} = \frac{1}{\alpha-\beta}\left[\alpha^j\sum_{i=0}^{n}\binom{n}{i}(-1)^i\alpha^{2i} - \beta^j\sum_{i=0}^{n}\binom{n}{i}(-1)^i\beta^{2i}\right].$

But $(-1^i)(\alpha^2)^i = (-\alpha^2)^i$ and $(-1^i)(\beta^2)^i = (-\beta^2)^i$.

Thus, $\sum_{i=0}^{n}\binom{n}{i}(-1)^iF_{2i+j} = \frac{1}{\alpha-\beta}[\alpha^j(1-\alpha^2)^n - \beta^j(1-\beta^2)^n]$

$= \frac{1}{\alpha-\beta}[\alpha^j(-\alpha)^n - \beta^j(-\beta)^n] = (-1)^n\left(\frac{\alpha^{n+j} - \beta^{n+j}}{\alpha-\beta}\right) = (-1)^nF_{n+j}.$

Section 10

1. $\sum_{i=1}^{n}F_i = F_{n+2} - 1,\ n \geq 1$

3. $\sum_{i=1}^{n}F_i^2 = F_nF_{n+1},\ n \geq 1$

4. $F_{n+1} = F_n + F_{n-1};\ F_{n+1}^2 = F_n^2 + 2F_nF_{n-1} + F_{n-1}^2.$
From (I_3), $F_nF_{n-1} = F_1^2 + \cdots + F_{n-1}^2.$
Thus, $F_{n+1}^2 = F_n^2 + 3F_{n-1}^2 + 2(F_{n-2}^2 + \cdots + F_1^2).$
The number of squares is $1 + 3 + 2(n-2) = 2n$.

7. $\sum_{i=1}^{n}F_{2i-1} = F_{2n},\ n \geq 1$

9. $\sum_{i=1}^{n}F_{2i} = F_{2n+1} - 1,\ n \geq 1$

10. $L_1 = L_2 - L_0$
$L_3 = L_4 - L_2$
$L_5 = L_6 - L_4$
......
$L_{2n-3} = L_{2n-2} - L_{2n-4}$
$L_{2n-1} = L_{2n} - L_{2n-2}$

$\sum_{i=1}^{n}L_{2i-1} = L_{2n} - 2$

11. $L_2 = L_3 - L_1$
$L_4 = L_5 - L_3$
$L_6 = L_7 - L_5$
......
$L_{2n-2} = L_{2n-1} - L_{2n-3}$
$L_{2n} = L_{2n+1} - L_{2n-1}$

$\sum_{i=1}^{n}L_{2i} = L_{2n+1} - 1$

12. $L_n = F_{n-1} + F_{n+1}$ (I8); $L_{n-1} = F_{n-2} + F_n$; $L_{n+1} = F_n + F_{n+2}$.

$$\tfrac{1}{5}(L_{n-1} + L_{n+1}) = \tfrac{1}{5}(F_{n-2} + 2F_n + F_{n+2})$$
$$= \tfrac{1}{5}[(F_n - F_{n-1}) + 2F_n + (F_n + F_{n+1})]$$
$$= \tfrac{1}{5}(4F_n + F_n) = F_n$$

13. $F_{2n} = F_nL_n$ (I7); $L_n = F_{n-1} + F_{n+1}$ (I8).

$$F_{2n} = F_n(F_{n-1} + F_{n+1}) = (F_{n+1} - F_{n-1})(F_{n+1} + F_{n-1})$$
$$= F_{n+1}^2 - F_{n-1}^2$$

14. $F_{2n} = F_{n+1}^2 - F_{n-1}^2$ (I10).

$$F_{2n+1} = F_{2(n+1)} - F_{2n} = F_{n+2}^2 - F_n^2 - F_{n+1}^2 + F_{n-1}^2$$
$$= (F_{n+1} + F_n)^2 - F_n^2 - F_{n+1}^2 + (F_{n+1} - F_n)^2 = F_{n+1}^2 + F_n^2$$

15. $h = \dfrac{1}{\sqrt{8^2 + 3^2}} < \dfrac{1}{\sqrt{8^2}}$, or $\dfrac{1}{8}$, since $8^2 + 3^2 > 8^2$.

17. To prove: $H_{n+2} = qF_{n+1} + pF_n$

$H_1 = p$; $H_2 = q$; $H_{n+2} = H_{n+1} + H_n$, $n \geq 1$

$n = 1$: $H_3 = H_2 + H_1 = q + p = qF_2 + pF_1$

$n = 2$: $H_4 = H_3 + H_2 = qF_2 + pF_1 + q = qF_3 + pF_2$

Thus, we have a basis for induction.

We assume (inductive hypothesis): $P(k-1)$: $H_{k+1} = qF_k + pF_{k-1}$

and $P(k)$: $H_{k+2} = qF_{k+1} + pF_k$

Adding, we have: $P(k+1)$: $H_{k+3} = qF_{k+2} + pF_{k+1}$

The proof is complete by mathematical induction.

Note: Observe the variation from the usual pattern of mathematical induction. The inductive basis has two (consecutive) validations, and the inductive hypothesis has two (consecutive) assumptions.

18. $F_{(k+1)n} = F_{kn}F_{n+1} + F_{kn-1}F_n$

To prove: F_n divides F_{nk}, $k > 0$.

$k = 1$: F_n divides F_n. Thus, we have a basis for induction.

We assume (inductive hypothesis): $P(p)$: F_n divides F_{np}.

But $F_{(p+1)n} = (F_{pn})F_{n+1} + F_{pn-1}(F_n)$.

Since F_n divides F_n and is assumed to divide F_{pn}, F_n must divide $F_{(p+1)n}$, and the proof is complete by mathematical induction.

Section 11

1. $$A + Z = \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a+0 & b+0 \\ c+0 & d+0 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = A$$

$$Z + A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0+a & 0+b \\ 0+c & 0+d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = A$$

2. $$A + B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix}$$
$$= \begin{pmatrix} e+a & f+b \\ g+c & h+d \end{pmatrix} = \begin{pmatrix} e & f \\ g & h \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix} = B + A$$

3. $$(A + B) + C = \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix}\right] + \begin{pmatrix} i & j \\ k & l \end{pmatrix}$$
$$= \begin{pmatrix} (a+e)+i & (e+f)+j \\ (c+g)+k & (d+h)+l \end{pmatrix} = \begin{pmatrix} a+(e+i) & e+(f+j) \\ c+(g+k) & d+(h+l) \end{pmatrix}$$
$$= \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \left[\begin{pmatrix} e & f \\ g & h \end{pmatrix} + \begin{pmatrix} i & j \\ k & l \end{pmatrix}\right] = A + (B + C)$$

4. $A + (-A) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix} = \begin{pmatrix} a-a & b-b \\ c-c & d-d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = Z$

5. $(sr)A = \begin{pmatrix} (sr)a & (sr)b \\ (sr)c & (sr)d \end{pmatrix} = \begin{pmatrix} s(ra) & s(rb) \\ s(rc) & s(rd) \end{pmatrix} = s(rA)$

6. $-A = \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix} = \begin{pmatrix} (-1)a & (-1)b \\ (-1)c & (-1)d \end{pmatrix} = (-1)(A)$

7. $AI = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a(1)+b(0) & a(0)+b(1) \\ c(1)+d(0) & c(0)+d(1) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = A$

$IA = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1(a)+0(c) & 1(b)+0(d) \\ 0(a)+1(c) & 0(b)+1(d) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = A$

8. $A(B+C) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e+i & f+j \\ g+k & h+l \end{pmatrix}$

$= \begin{pmatrix} a(e+i)+b(g+k) & a(f+j)+b(h+l) \\ c(e+i)+d(g+k) & c(f+j)+d(h+l) \end{pmatrix}$

$= \begin{pmatrix} (ae+bg)+(ai+bk) & (af+bh)+(aj+bl) \\ (ce+dg)+(ci+dk) & (cf+dh)+(cj+dl) \end{pmatrix} = AB + AC$

Similarly, $(B+C)A = BA + CA$.

9. To prove: $\det(Q^n) = (-1)^n$, $n \geq 1$.
$\det Q = -1$; $\det Q^2 = \det((Q)(Q)) = \det Q \det Q = (-1)^2$
Thus, we have a basis for induction.
We assume (inductive hypothesis): $P(k)$: $\det Q^k = (-1)^k$
We wish to prove $P(k+1)$: $\det Q^{k+1} = (-1)^{k+1}$.
$\det Q^{k+1} = \det((Q)(Q^k)) = \det Q \det Q^k = (-1)(-1)^k = (-1)^{k+1}$
The proof is complete by mathematical induction.

10. $Q^2 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$; $Q + I = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = Q^2$

Section 12

1.

$1 = F_2$	$11 = F_6 + F_4$
$2 = F_3$	$12 = F_6 + F_4 + F_2$
$3 = F_4$	$13 = F_7$
$4 = F_4 + F_2$	$14 = F_7 + F_2$
$5 = F_4 + F_3$	$15 = F_7 + F_3$
$6 = F_4 + F_3 + F_2$	$16 = F_7 + F_4$
$7 = F_4 + F_3 + F_2 + F_1$	$17 = F_7 + F_4 + F_2$
$8 = F_6$	$18 = F_7 + F_4 + F_3$
$9 = F_6 + F_2$	$19 = F_7 + F_4 + F_3 + F_2$
$10 = F_6 + F_3$	$20 = F_7 + F_4 + F_3 + F_2 + F_1$

2.

$1 = F_2$	$4 = F_3 + F_2 + F_1$	$7 = F_5 + F_3$
$2 = F_3$	$5 = F_5$	$8 = F_5 + F_3 + F_1$
$3 = F_3 + F_2$	$6 = F_5 + F_2$	$9 = F_5 + F_3 + F_2 + F_1$

There is no possible representation for 10 with F_4 and F_6 missing. The next available one is $F_7 = 13$, which is too large, and 9 is the sum of all the smaller available Fibonacci numbers.

3. There is only one representation of 27 using distinct Fibonacci numbers: $27 = 21 + 5 + 1$. This may be expressed as $27 = F_8 + F_5 + F_2$.

4. $1966 = 1597 + 233 + 89 + 34 + 13 = F_{17} + F_{13} + F_{11} + F_9 + F_7$

5. $32 = 21 + 8 + 3 = F_8 + F_6 + F_4$ (minimal, or Zeckendorf)
$= 13 + 8 + 5 + 3 + 2 + 1 = F_7 + F_6 + F_5 + F_4 + F_3 + F_2$ (max.)

6. $32 = 29 + 3 = L_7 + L_2$ (minimal)
$= 18 + 7 + 4 + 2 + 1 = L_6 + L_4 + L_3 + L_0 + L_1$ (maximal)

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