12 • *Representation Theorems*

A sequence of positive integers, $a_1, a_2, \ldots, a_n, \ldots$, is **complete** with respect to the positive integers if and only if every positive integer m is the sum of a finite number of the members of the sequence, where each member is used at most once in any given **representation.** We state the following theorem without proof.

THEOREM I

The sequence defined by $a_n = 2^n$ $(n \geq 0)$ is complete.

For example, since

$$\begin{array}{llll} a_0 = 2^0 = 1, & a_2 = 2^2 = 4, & a_4 = 2^4 = 16, & \\ a_1 = 2^1 = 2, & a_3 = 2^3 = 8, & a_5 = 2^5 = 32, & \ldots, \end{array}$$

we can write, for example:

$$\begin{array}{lll} 1 = 1 & 5 = 4 + 1 & 9 = 8 + 1 \\ 2 = 2 & 6 = 4 + 2 & 10 = 8 + 2 \\ 3 = 2 + 1 & 7 = 4 + 2 + 1 & 11 = 8 + 2 + 1 \\ 4 = 4 & 8 = 8 & 12 = 8 + 4 \end{array}$$

It can be proved that each representation is unique, and you should recognize this as the basis for the binary system of numeration. For example:

1: 1	5: 101	9: 1001
2: 10	6: 110	10: 1010
3: 11	7: 111	11: 1011
4: 100	8: 1000	12: 1100

The ancient Egyptians used the principle described on the preceding page in their method of multiplication. For example, to find

$$243 \times 25,$$

they would write two columns — powers of 2 and the corresponding products:

1 ✓	243 ✓
2	486
4	972
8 ✓	1944 ✓
16 ✓	3888 ✓

Since $25 = 16 + 8 + 1$,

$$243 \times 25 = 3888 + 1944 + 243 = 6075,$$

as you can see by applying the distributive property.

We shall now prove an interesting theorem about the Fibonacci sequence.

THEOREM II

The Fibonacci sequence of numbers, where $a_n = F_n$ $(n \geq 1)$, is complete.

To discover a method of proof, let us look at the following table:

F_1	F_2	F_3	F_4		F_5			F_6	...
1	1	2	3	4	5	6	7	8	...

We observe that we can write:

$$1 = F_1 = F_2$$
$$2 = F_3 = F_2 + F_1$$
$$3 = F_4 = F_3 + F_2$$
$$4 = F_4 + F_2 = F_3 + F_2 + F_1$$
$$5 = F_5 = F_4 + F_3$$
$$6 = F_5 + F_2 = F_4 + F_3 + F_2$$
$$7 = F_5 + F_3 = F_4 + F_3 + F_2 + F_1$$
$$8 = F_6 = F_5 + F_4 = F_5 + F_3 + F_2$$

Thus, each positive integer from 1 through 8 can be represented in at least two ways as a sum of Fibonacci numbers where each is used at most once in any representation.

Proof of Theorem II. We know from (I_1) (page 52) that

$$F_n - 1 = F_1 + F_2 + F_3 + \cdots + F_{n-2},$$

and we observe in the list on page 70 that for $3 \leq n \leq 6$, each integer

$$m = 1, 2, 3, \ldots, F_n - 1$$

can be represented as a sum of some or all of the Fibonacci numbers $F_1, F_2, F_3, \ldots, F_{n-2}$. That is, for $n = 3$, $F_n = F_3 = 2$, $F_n - 1 = 2 - 1 = 1$, $F_{n-2} = F_1 = 1$, and we have

$$1 = F_1;$$

for $n = 4$, $F_n = F_4 = 3$, $F_n - 1 = 3 - 1 = 2$, $F_{n-2} = F_2 = 1$, and we have

$$1 = F_1, 2 = F_2 + F_1;$$

and so on. We shall use this as the basis of induction (recall page 54).

For the second part of the proof, we assume that every integer $m = 1, 2, 3, \ldots, F_k - 1$, $k \geq 3$, is representable using the Fibonacci numbers $F_1, F_2, \ldots, F_{k-2}$. We must prove that every integer $m = 1, 2, 3, \ldots, F_{k+1} - 1$ is representable using $F_1, F_2, \ldots, F_{k-1}$, $k \geq 3$. If we add F_{k-1} to each of the given representations, we shall have representations for

$$1 + F_{k-1}, \quad 2 + F_{k-1}, \quad \ldots, \quad F_k - 1 + F_{k-1},$$

where $F_k + F_{k-1} - 1 = F_{k+1} - 1$. We now have representations for the two sequences of consecutive positive integers

$$1, \quad 2, \quad 3, \quad \ldots, \quad F_k - 1$$

and

$$1 + F_{k-1}, \quad 2 + F_{k-1}, \quad 3 + F_{k-1}, \quad \ldots, \quad F_{k+1} - 1.$$

Are there any omissions between $F_k - 1$ and $1 + F_{k-1}$? No, since for $k = 3$,

$$F_k - 1 = F_3 - 1 = 1 \qquad \text{and} \qquad 1 + F_{k-1} = 1 + F_2 = 2;$$

for $k = 4$,

$$F_k - 1 = F_4 - 1 = 2 \qquad \text{and} \qquad 1 + F_{k-1} = 1 + F_3 = 3;$$

and for $k \geq 5$, since $F_k - F_{k-1} \geq 2$, we have

$$F_k - 1 \geq 1 + F_{k-1},$$

and so there is an overlap. In any case, every integer $m = 1, 2, 3, \ldots, F_{k+1} - 1$, $k \geq 3$, is representable using $F_1, F_2, \ldots, F_{k-1}$, which is what we set out to prove.

Thus, the proof is complete by mathematical induction.

THEOREM III

The Fibonacci number sequence, where $a_n = F_n$ $(n \geq 1)$, with an arbitrary F_n missing is complete.

Observe, for example, that if F_4 is omitted from the display on page 70, every integer from 1 through 8 can still be represented by a sum of other Fibonacci numbers.

Proof. From the proof of the previous theorem we note that we can properly represent any number $m = 1, 2, 3, \ldots, F_{n+1} - 1$ by using only the Fibonacci numbers $F_1, F_2, F_3, \ldots, F_{n-1}$, that is, without using F_n. Then F_{n+1} can represent itself, and when we add F_{n+1} to the representations for $m = 1, 2, 3, \ldots, F_{n+1} - 1$, we have representations for $m = 1, 2, 3, \ldots, 2F_{n+1} - 1$. Since $2F_{n+1} - 1 > F_{n+2} - 1$, we can easily proceed over the trouble spot.

THEOREM IV

The Fibonacci sequence of numbers, where $a_n = F_n$ $(n \geq 1)$, with any two arbitrary Fibonacci numbers F_p and F_n missing is incomplete.

Proof. Since

$$F_1 + F_2 + F_3 + \cdots + F_k = F_{k+2} - 1,$$

if $F_p < F_k$ is missing, then

$$F_1 + F_2 + F_3 + \cdots + F_{p-1} + F_{p+1} + \cdots + F_k = F_{k+2} - F_p - 1.$$

But with $F_n > F_p$ also missing, we can reach

$$\begin{aligned} F_1 + F_2 + \cdots + F_{p-1} + F_{p+1} + \cdots + F_{n-1} \\ = F_{n+1} - F_p - 1 < F_{n+1} - 1. \end{aligned}$$

Since $F_p \geq 1$, there is a number $F_{n+1} - 1$ without a proper representation. This happened because even if we used all of those *available* numbers less than F_n at once, we cannot reach $F_{n+1} - 1$ and any other Fibonacci numbers which are *available* are too large, since the Fibonacci numbers are a strictly increasing sequence for $n \geq 2$.

For example, suppose that we try to find a representation of 20 without using $F_5 = 5$ and $F_7 = 13$. Those available are then F_1, F_2, F_3, F_4, F_6. The maximum attainable is $1 + 1 + 2 + 3 + 8 = 15$. The next available Fibonacci number is 21 so that we cannot find representations for 16, 17, 18, 19, and 20.

We shall now consider the corresponding properties for the Lucas sequence of numbers. First, let us look at the following table:

L_0	L_1	L_2	L_3			L_4		...
2	1	3	4	5	6	7	8	...

$$1 = L_1$$
$$2 = L_0$$
$$3 = L_2 = L_1 + L_0$$
$$4 = L_3 = L_2 + L_1$$
$$5 = L_3 + L_1 = L_2 + L_0$$
$$6 = L_3 + L_0 = L_2 + L_1 + L_0$$
$$7 = L_4 = L_3 + L_2 = L_3 + L_1 + L_0$$
$$8 = L_4 + L_1 = L_3 + L_2 + L_1$$

We observe that each positive integer from 3 through 8 can be represented in at least two ways as a sum of Lucas numbers where each is used at most once in any representation.

THEOREM V

The Lucas number sequence, where $a_n = L_{n-1}$ $(n \geq 1)$, is complete.

We know from (I_2) (page 54) that

$$L_1 + L_2 + L_3 + \cdots + L_n = L_{n+2} - 3,$$

and so $$L_0 + L_1 + L_2 + L_3 + \cdots + L_n = L_{n+2} - 1.$$

Since $a_1 = L_0 = 2, a_2 = L_1 = 1, a_3 = L_2 = 3, \ldots, a_{n+2} = a_{n+1} + a_n$, the sequence a_n is a generalized Fibonacci sequence whose sum is

$$a_1 + a_2 + a_3 + \cdots + a_n = a_{n+2} - 1.$$

Thus, this theorem can be proved by mathematical induction as Theorem II was proved.

Is the Lucas number sequence complete if any single term is missing? Clearly, without $a_1 = L_0$ we have no representation for 2, and without $a_2 = L_1$ we have no representation for 1. It is as yet an unanswered question as to which Lucas numbers can be left out without destroying completeness. For example, we can see that $L_3 = 4$ can be left out:

$1 = 1$	$4 = 3 + 1$	$7 = 7$	$10 = 7 + 3$
$2 = 2$	$5 = 3 + 2$	$8 = 7 + 1$	$11 = 11$
$3 = 3$	$6 = 3 + 2 + 1$	$9 = 7 + 2$	$12 = 11 + 1$

It would seem that if any Lucas number L_k, $k > 1$, is omitted, the resulting sequence is still complete. One would expect that if any two Lucas numbers were missing, the resulting sequence would be incomplete.

We now ask: If each integer m is to be represented by the *least* number of Fibonacci numbers what are the conditions for this *minimal representation*?

Certainly, we would never need both F_1 and F_2 in any minimal representation, and so we choose to use F_2 as the 1.

If, for a given integer m, the Fibonacci numbers in any representation are arranged in order of size, let us direct our attention to the largest, say, F_n. Clearly, if the representation has more than one Fibonacci number, then it cannot contain both F_n and F_{n-1} and be a minimal representation because we could replace $F_n + F_{n-1}$ by F_{n+1} and thereby reduce the number used.

Now, if we have F_n but not F_{n-1}, then we could have F_{n-2} but not both F_{n-2} and F_{n-3} because then we could replace $F_{n-2} + F_{n-3}$ by F_{n-1} and next $F_{n-1} + F_n$ by F_{n+1}, thereby getting a double reduction. We can see, therefore, that any minimal representation cannot have two adjacent Fibonacci numbers.

These restrictions are precisely the conditions imposed by the following interesting theorem, which we state without proof.

THEOREM VI (E. Zeckendorf)*

Each positive integer m can be represented as the sum of distinct numbers in the sequence defined by $a_n = F_{n+1}$ $(n \geq 1)$ using no two consecutive Fibonacci numbers, and such a representation is unique.

From this we see that each positive integer has a minimal representation as described above, and such a representation is unique.

If in a representation of a positive integer m using the terms of the sequence

$$1, 2, 3, 5, 8, \ldots, F_{n+1}, \ldots$$

we desire to use the *maximum* number of these Fibonacci numbers, thus obtaining a *maximal representation*, we should replace F_k by $F_{k-1} + F_{k-2}$ whenever possible (avoiding repetitions, of course). This process results in the conditions described in the following theorem, which we state without proof.

THEOREM VII†

Each positive integer m can be represented as the sum of distinct numbers in the sequence defined by $a_n = F_{n+1}$ $(n \geq 1)$ with the condition that whenever F_k $(k \geq 4)$ is used, at least one of each pair F_q, F_{q-1} $(3 \leq q < k)$ must be used, and such a representation is unique.

* John L. Brown, Jr., "Zeckendorf's Theorem and Some Applications," *The Fibonacci Quarterly*, Vol. 2, No. 3 (October, 1964), pages 163–168.

† John L. Brown, Jr., "A New Characteristic of the Fibonacci Numbers," *The Fibonacci Quarterly*, Vol. 3, No. 1 (February, 1965), pages 1–8.

From Theorem VII we see that each positive integer has a maximal representation as described on page 74, and such a representation is unique.

It turns out that all positive integers have both a unique minimum and a unique maximum representation (while some have only one representation, which satisfies both conditions) from the sequence:

$$\begin{matrix} F_2 & F_3 & F_4 & F_5 & F_6 & F_7 & F_8 & \dots \\ 1 & 2 & 3 & 5 & 8 & 13 & 21 & \dots \end{matrix}$$

We have the following representations (using each term in the sequence at most once) for 1–21:

Minimal	*Same*	*Maximal*
$1 =$	F_2	
$2 =$	F_3	
$3 = F_4$		$F_3 + F_2$
$4 =$	$F_4 + F_2$	
$5 = F_5$		$F_4 + F_3$
$6 = F_5 + F_2$		$F_4 + F_3 + F_2$
$7 =$	$F_5 + F_3$	
$8 = F_6$		$F_5 + F_3 + F_2$
$9 = F_6 + F_2$		$F_5 + F_4 + F_2$
$10 = F_6 + F_3$		$F_5 + F_4 + F_3$
$11 = F_6 + F_4$		$F_5 + F_4 + F_3 + F_2$
$12 =$	$F_6 + F_4 + F_2$	
$13 = F_7$		$F_6 + F_4 + F_3$
$14 = F_7 + F_2$		$F_6 + F_4 + F_3 + F_2$
$15 = F_7 + F_3$		$F_6 + F_5 + F_3$
$16 = F_7 + F_4$		$F_6 + F_5 + F_3 + F_2$
$17 = F_7 + F_4 + F_2$		$F_6 + F_5 + F_4 + F_2$
$18 = F_7 + F_5$		$F_6 + F_5 + F_4 + F_3$
$19 = F_7 + F_5 + F_2$		$F_6 + F_5 + F_4 + F_3 + F_2$
$20 =$	$F_7 + F_5 + F_3$	
$21 = F_8$		$F_7 + F_5 + F_3 + F_2$

Notice that each integer $F_k - 1$ $(k \geq 3)$ has a single representation satisfying both minimal and maximal conditions, and *these are the only positive integers for which this is true.**

* David A. Klarner, "Representations of N as a Sum of Distinct Elements from Special Sequences," *The Fibonacci Quarterly*, Vol. 4, No. 4 (December, 1966), pages 289–306.

A standard puzzle problem is to determine the number of separate weights needed to weigh any given integral number of pounds, supposing that the thing to be weighed is placed on one side and the weights are to be placed on the other side of the balance system. You can see that the solution is related to the idea of completeness of a sequence of positive integers as defined at the beginning of this section.

For example, any integral number of pounds from 1 to 31 can be weighed with one 1-pound weight, one 2-pound weight, one 4-pound weight, one 8-pound weight, and one 16-pound weight, because of Theorem I.

Now suppose, as a fairy tale, that Fibonacci was a professional weigher with weights $F_1, F_2, F_3, F_4, \ldots, F_n, \ldots$ pounds, traveling from place to place weighing things for people. He had to be able to weigh any positive integral number of pounds.

Theorem II tells us that Fibonacci could always do his job if he had all of his weights with him.

Theorem III tells us that Fibonacci could still do his job if he somehow lost one of his Fibonacci weights.

Theorem IV tells us that Fibonacci could not do his job if he somehow lost any two of his Fibonacci weights.

Theorem VI (Zeckendorf) states that if Fibonacci did not use his F_1 weight and lined up the rest of his weights in order of size, then any job he might be given would have *just one* solution if he did *not* choose any two weights which were adjacent in the lineup.

Theorem VII states that if Fibonacci did not use his F_1 weight and lined up the rest of his weights in order of size, then any job he might be given would have *just one* solution if whenever he used his F_k weight ($k \geq 4$), he also used at least one of each pair of weights, F_q and F_{q-1} ($3 \leq q < k$).

Theorems for Lucas numbers corresponding to Theorems VI and VII on page 74 are the following.

THEOREM VIII

Each positive integer m can be represented as the sum of distinct numbers in the sequence $a_n = L_{n-1}$ ($n \geq 1$) with the conditions that no two consecutive Lucas numbers are used in the same representation and that L_0 and L_2 are not both used in the same representation, and such a representation is unique.

THEOREM IX

Each positive integer m can be represented as the sum of distinct numbers in the sequence $a_n = L_{n-1}$ $(n \geq 1)$ with the conditions that whenever L_k $(k \geq 2)$ is used, at least one of each pair L_q, L_{q-1} $(1 \leq q < k)$ must be used and L_1 and L_3 are not both to be used unless L_0 or L_2 is also in the same representation, and each such representation is unique.

From the sequence

$$\begin{array}{cccccccc} L_1 & L_0 & L_2 & L_3 & L_4 & L_5 & L_6 & \ldots \\ 1 & 2 & 3 & 4 & 7 & 11 & 18 & \ldots \end{array}$$

we have the following representations (using each term in the sequence at most once) for 1–18:

Theorem VIII	*Same*	*Theorem IX*
$1 =$	L_1	
$2 =$	L_0	
$3 = L_2$		$L_0 + L_1$
$4 = L_3$		$L_1 + L_2$
$5 = L_3 + L_1$		$L_2 + L_0$
$6 =$	$L_3 + L_0$	
$7 = L_4$		$L_3 + L_1 + L_0$
$8 = L_4 + L_1$		$L_3 + L_2 + L_1$
$9 = L_4 + L_0$		$L_3 + L_2 + L_0$
$10 = L_4 + L_2$		$L_3 + L_2 + L_1 + L_0$
$11 = L_5$		$L_4 + L_2 + L_1$
$12 = L_5 + L_1$		$L_4 + L_2 + L_0$
$13 = L_5 + L_0$		$L_4 + L_3 + L_0$
$14 = L_5 + L_2$		$L_4 + L_3 + L_1 + L_0$
$15 = L_5 + L_3$		$L_4 + L_3 + L_2 + L_1$
$16 = L_5 + L_3 + L_1$		$L_4 + L_3 + L_2 + L_0$
$17 = L_5 + L_3 + L_0$		$L_4 + L_3 + L_2 + L_1 + L_0$
$18 = L_6$		$L_5 + L_3 + L_1 + L_0$

Notice that the representation according to Theorem VIII is the minimal representation and the representation according to Theorem IX is the maximal representation.

EXERCISES

1. Find a representation of each integer 1, 2, 3, . . . , 20 using Fibonacci numbers with distinct subscripts, including $F_1 = 1$ but omitting $F_5 = 5$.

2. Find the first integer not representable in terms of Fibonacci numbers with distinct subscripts if you are denied use of Fibonacci numbers $F_4 = 3$ and $F_6 = 8$.

3. Express 27 as the sum of three distinct Fibonacci numbers. In how many ways can you do this?

4. Express 1966 in Zeckendorf form. The available Fibonacci numbers are: 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, and 1597.

5. Find the minimal and maximal representations of 32 using distinct terms of the sequence $F_2, F_3, \ldots$.

6. Find the minimal and maximal representations of 32 using distinct terms of the sequence $L_1, L_0, L_2, L_3, \ldots$.