2 • *Rabbits, Fibonacci Numbers, and Lucas Numbers*

Fibonacci introduced a problem in the *Liber Abaci* by a story that may be summarized as follows. Suppose that

(1) there is one pair of rabbits in an enclosure on the first day of January;
(2) this pair will produce another pair of rabbits on February first and on the first day of every month thereafter; and
(3) each new pair will mature for one month and then produce a new pair on the first day of the third month of its life and on the first day of every month thereafter.

The problem is to find the number of pairs of rabbits in the enclosure on the first day of the following January after the births have taken place on that day.

It will be helpful to make a chart to keep count of the pairs of rabbits. Let A denote an adult pair of rabbits and let B denote a "baby pair" of rabbits. Thus, on January first, we have only an A; on February first we have that A and a B; and on March first, we have the original A, a new B, and the former B, which has become an A:

Date	*Pairs*	*Number of A's*	*Number of B's*
January 1	A	1	0
February 1	A B	1	1
March 1	A B A	2	1

To continue the chart conveniently, we condense our notation as follows. To get the next line of symbols, in any line we replace each A by AB and each B by A. Thus, we have the representation shown in the table at the top of the next page.

Date	*Pairs*	*Number of A's*	*Number of B's*
March 1	ABA	2	1
April 1	ABAAB	3	2
May 1	ABAABABA	5	3
June 1	ABAABABAABAAB	8	5

We now see that the number of A's on July 1 will be the sum of the number of A's on June 1 and the number of B's born on that day (which become A's on July 1). The number of B's on July 1 is the same as the number of A's on June 1. We complete the table for the year:

	Month	*Number of A's*	*Number of B's*	*Total number of pairs*
1	January	1	0	1
	After births on first of			
2	February	1	1	2
3	March	2	1	3
4	April	3	2	5
5	May	5	3	8
6	June	8	5	13
7	July	13	8	21
8	August	21	13	34
9	September	34	21	55
10	October	55	34	89
11	November	89	55	144
12	December	144	89	233
13	January	233	144	377

Thus, we see that under the conditions of the problem, the number of pairs of rabbits in the enclosure one year later would be 377.

We can draw some conclusions by studying the table. It is clear that the number of A's on the following February 1 is 377. Of these, 376 were originally B's, descendants of the original A. Therefore, if we add all the numbers in the column headed "Number of B's," we have

$$S = 0 + 1 + 1 + 2 + 3 + 5 + 8 + 13 + 21 + 34 + 55 + 89 + 144 \\ = 376.$$

From this, we observe that the sum of the first 12 entries in the column headed "Number of A's" is one less than 377, which would be the 14th entry in that column. This is a specific instance of a general result which we shall establish later in this section.

Further examination of the table on page 3 reveals that each entry in the columns of numbers may be found in accordance with a pattern. For example, the entries in each line after the second may be found as the sum of the two preceding entries in that column. Those in line 3 are:

$$2 = 1 + 1 \qquad 1 = 0 + 1 \qquad 3 = 1 + 2$$

Those in line 4 are:

$$3 = 1 + 2 \qquad 2 = 1 + 1 \qquad 5 = 2 + 3$$

Can we describe this pattern by some kind of formula? Yes, as we shall now show.

In general, ordered sets of numbers such as those in the columns of the table on page 3 are called *sequences*. A sequence may be *finite* or *infinite*. An infinite sequence may be designated by symbols such as

$$u_1, u_2, u_3, \ldots, u_n, \ldots,$$

where the subscripts indicate the order of the *terms*, with n a positive integer. An example of a sequence is the *arithmetic progression*

$$\begin{array}{cccccc} u_1, & u_2 & u_3, & \ldots, & u_n, & \ldots \\ \downarrow & \downarrow & \downarrow & & \downarrow & \\ 2, & 5, & 8, & \ldots, & \overbrace{2 + (n - 1)3}, & \ldots \end{array}$$

where a formula for the nth term is

$$u_n = 2 + (n - 1)3.$$

Another way to specify this sequence would be to state the first term,

$$u_1 = 2,$$

and the formula

$$u_n = u_{n-1} + 3, \qquad n > 1.$$

Such a definition is said to be a *recursive definition*, and the formula is called a *recursion formula* or a *recurrence formula*. (The words "recursive," "recursion," and "recurrence" all come from a Latin verb meaning "to run back.")

We can use an extension of this idea to specify the sequences in the columns of the table on page 3. For example, to specify the sequence in the column headed "Number of A's," we state the first two terms,

$$u_1 = 1, \qquad u_2 = 1,$$

and the recursive, or recurrence, formula

$$\text{(R)} \qquad u_n = u_{n-1} + u_{n-2}, \qquad n > 2.$$

This gives the sequence

$$1, 1, 2, 3, 5, 8, 13, \ldots$$

as we wished. For the column headed "Number of B's," we have $u_1 = 0$, $u_2 = 1$, and the same recurrence formula, yielding the sequence

$$0, 1, 1, 2, 3, 5, 8, 13, \ldots.$$

For the column headed "Total number of pairs," we have $u_1 = 1$, $u_2 = 2$, and the sequence

$$1, 2, 3, 5, 8, 13, \ldots.$$

Because of its source in Fibonacci's rabbit problem, the sequence

$$\mathbf{1, 1, 2, 3, 5, 8, 13, \ldots}$$

is called the **Fibonacci sequence**, and its terms are called **Fibonacci numbers.** We shall denote the nth Fibonacci number by F_n; thus,

$$F_1 = 1, \quad F_2 = 1, \quad F_3 = 2, \quad F_4 = 3, \quad F_5 = 5, \quad F_6 = 8, \ldots.$$

Moreover, we may write these alternative forms:

$$F_1 = F_2 = 1, \qquad F_n = F_{n-1} + F_{n-2}, \qquad n > 2,$$

or

$$F_1 = F_2 = 1, \qquad F_{n+1} = F_n + F_{n-1}, \qquad n > 1,$$

or

$$F_1 = F_2 = 1, \qquad F_{n+2} = F_{n+1} + F_n, \qquad n \geq 1.$$

We can now give a more formal discussion of the Fibonacci rabbit problem. For all positive integral n, we define for the first day of the nth month:

A_n = number of A's (adult pairs of rabbits)
B_n = number of B's (baby pairs of rabbits)
T_n = total number of pairs of rabbits $= A_n + B_n$

Only the A's on the first day of the nth month will produce B's on the first day of the $(n + 1)$st month. Thus,

$$B_{n+1} = A_n, \qquad n \geq 1.$$

In making up the table on page 3, we observed that the number of A's on the first day of the $(n + 2)$nd month is the sum of the number of A's on the first day of the $(n + 1)$st month and the number of B's born on that day. Thus,

$$A_{n+2} = A_{n+1} + B_{n+1},$$

and since $B_{n+1} = A_n$, we have

$$A_{n+2} = A_{n+1} + A_n, \qquad n \geq 1.$$

We also observe from the table that $A_1 = 1$ and $A_2 = 1$. Thus, the sequence

$$A_1, A_2, A_3, \ldots,$$

is the Fibonacci sequence, and

$$A_n = F_n, \qquad n \geq 1.$$

Since $B_{n+1} = A_n$ for $n \geq 1$, we have

$$B_n = A_{n-1} = F_{n-1} \text{ for } n \geq 2.$$

If we now let $n = 1$ in this last formula, we have

$$B_1 = F_0.$$

If we let $n = 1$ in the formula $F_{n+1} = F_n + F_{n-1}$, we have

$$F_2 = F_1 + F_0$$

or

$$F_0 = F_2 - F_1 = 1 - 1 = 0,$$

which checks with $B_1 = 0$ in the table. Thus, *we have now defined F_n for $n = 0$.*

Finally, the total number of pairs on the first day of the nth month is

$$T_n = A_n + B_n = F_n + F_{n-1} = F_{n+1}.$$

We can now establish the following result, already suggested by the specific instance shown at the bottom of page 3:

The sum of the first n Fibonacci numbers is one less than the $(n + 2)$nd Fibonacci number.

Symbolically:

$$F_1 + F_2 + \cdots + F_n = F_{n+2} - 1, \qquad n \geq 1.$$

We remember that $F_{n+2} = A_{n+2}$ and that A_{n+2} is the number of A's (adult pairs of rabbits) in the enclosure on the first day of the $(n + 2)$nd month.

Originally, we had only one A. Where did the extra A's come from? Each of the extra A's was first a B.

How many more A's do we now have? The number of extra A's is

$$A_{n+2} - 1.$$

Now, one month after being born, each B became an A. If we add the number of B's from the first day of the first month to the first day of the $(n + 1)$st month, the sum is the number of A's other than the original pair that we have on the first day of the $(n + 2)$nd month. Thus,

$$B_1 + B_2 + B_3 + \cdots + B_{n+1} = A_{n+2} - 1.$$

But, remembering that $B_1 = 0$, $B_n = F_{n-1}$, and $A_{n+2} = F_{n+2}$, we have

$$F_1 + F_2 + \cdots + F_n = F_{n+2} - 1, \qquad n \geq 1;$$

as we wished to show.

This formula is an example of a *Fibonacci number identity*. We shall prove this identity again later in three different ways (Section 10).

Many different sequences may be specified by using formula (R) on page 4 and choosing different numbers for the first two terms. For example, if we take $u_1 = 1$ and $u_2 = 3$, we have

$$\mathbf{1, 3, 4, 7, 11, 18, 29, 47, \ldots,}$$

which we shall call the **Lucas sequence,** in honor of the nineteenth-century French mathematician E. Lucas. Lucas did much work in recurrent sequences and gave the Fibonacci sequence its name. The terms of the Lucas sequence are called **Lucas numbers,** and we shall denote the nth Lucas number by L_n. The Lucas numbers are closely related to the Fibonacci numbers, as we shall show in this booklet.

In general, if we take the first two terms of a sequence defined by (R) as arbitrary integers p and q, that is, $u_1 = p$ and $u_2 = q$, then we have

$$p, \quad q, \quad p + q, \quad p + 2q, \quad 2p + 3q, \quad 3p + 5q, \quad \ldots,$$

which is called a **generalized Fibonacci sequence.** We shall denote the nth term of this sequence by H_n. It may be shown by mathematical induction (see Exercise 17, Section 10) that this generalized Fibonacci sequence is related to the Fibonacci sequence by the formula

$$H_{n+2} = H_2 F_{n+1} + H_1 F_n, \qquad n \geq 0,\ F_0 = 0,$$

or, expressed in terms of the starting values, p and q,

$$H_{n+2} = qF_{n+1} + pF_n.$$

EXERCISES

1. Compute the first 20 Fibonacci numbers.
2. Compute the first 20 Lucas numbers.
3. Study the results of Exercises 1 and 2, looking for any possible relationships or number patterns.
4. If $H_1 = 1$, $H_2 = 4$, and $H_{n+2} = H_{n+1} + H_n$, $n \geq 1$, compute the first 20 terms of this generalized Fibonacci sequence.

5. Verify that:

   a. $L_5 = F_6 + F_4$

   b. $F_9 = F_5^2 + F_4^2$

   c. $L_7 + L_9 = 5F_8$

   d. $H_{20} = (4)F_{19} + (1)F_{18}$ in Exercise 4.

6. Verify that:

   a. $F_8 = L_4F_4$

   b. $\dfrac{F_{10}}{F_5}$ is an integer.

   c. $\dfrac{F_{12}}{F_4}$ is an integer.

   d. $F_7F_9 - F_8^2 = 1$

   e. $L_3L_5 - L_4^2 = -5$

7. Verify that $F_1 + F_2 + F_3 + F_4 + F_5 + F_6 = F_8 - 1$.

8. Verify that $F_1 + F_2 + F_3 + F_4 + F_5 + F_6 + F_7 + F_8 + F_9 + F_{10} = 11F_7$.

9. Show that:

   a. When F_{13} is divided by F_8, the remainder is F_3.

   b. When F_{15} is divided by F_8, the remainder is F_1.