3 • *The Golden Section and the Fibonacci Quadratic Equation*

Suppose that we are given a line segment $\overline{AB}$, and that we are to find a point C on it (between A and B) such that the length of the greater part is the mean proportional between the length of the whole segment and the length of the lesser part; that is, in Figure 1,

$$\frac{AB}{AC} = \frac{AC}{CB},$$

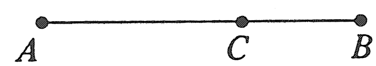

Figure 1

where $AB \neq 0$, $AC \neq 0$, and $CB \neq 0$.

We first find a positive numerical value for the ratio $\dfrac{AB}{AC}$. For convenience, let

$$x = \frac{AB}{AC} \qquad (x > 0).$$

Then

$$x = \frac{AB}{AC} = \frac{AC + CB}{AC} = 1 + \frac{CB}{AC} = 1 + \frac{1}{\dfrac{AC}{CB}} = 1 + \frac{1}{\dfrac{AB}{AC}} = 1 + \frac{1}{x}.$$

From

$$x = 1 + \frac{1}{x}$$

we obtain, by multiplying both members of the equation by x,

$$x^2 = x + 1, \text{ or}$$

(F) $$x^2 - x - 1 = 0.$$

The roots of this quadratic equation are (as you can verify, see Exercise 1, page 13)

$$\alpha = \frac{1 + \sqrt{5}}{2} \qquad \text{and} \qquad \beta = \frac{1 - \sqrt{5}}{2}.$$

(α is the Greek letter *alpha*, and β is the Greek letter *beta*.) You can verify by computation that $\alpha > 0$ and $\beta < 0$; $\alpha \doteq 1.618$ and $\beta \doteq -.618$ (Exercise 2). Thus, we take the positive root, α, as the value of the desired ratio:

$$\frac{AB}{AC} = \frac{1+\sqrt{5}}{2}$$

We can now use this numerical value to devise a method for locating C on $\overline{AB}$. Draw $\overline{BD}$ perpendicular to $\overline{AB}$ at B, but half its length. Draw $\overline{AD}$. Make $\overline{DE}$ the same length as $\overline{BD}$, and $\overline{AC}$ the same length as $\overline{AE}$. Then

$$AB = 2BD, \qquad ED = BD$$

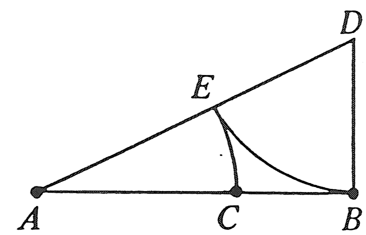

Figure 2

and, by the Pythagorean theorem,

$$AD = \sqrt{5}\,BD;$$

hence:

$$AC = AE = AD - ED = (\sqrt{5} - 1)BD$$

$$\frac{AB}{AC} = \frac{2BD}{(\sqrt{5}-1)BD} = \frac{2(\sqrt{5}+1)}{5-1} = \frac{\sqrt{5}+1}{2}$$

This computation verifies that the construction does indeed locate C on $\overline{AB}$ such that

$$\frac{AB}{AC} = \frac{1+\sqrt{5}}{2}.$$

Since α is a root of equation (F) on page 9, we have

$$\alpha^2 = \alpha + 1.$$

Multiplying both members of this equation by α^n (n can be any integer) yields

(A) $$\alpha^{n+2} = \alpha^{n+1} + \alpha^n.$$

If we let $u_n = \alpha^n, n \geq 1$, then $u_1 = \alpha$ and $u_2 = \alpha^2$, and we have the sequence

$$\alpha, \quad \alpha^2 = \alpha + 1, \quad \alpha^3 = \alpha^2 + \alpha, \quad \ldots,$$

which satisfies the recursive formula (R) on page 4. Similarly, we have

(B) $$\beta^{n+2} = \beta^{n+1} + \beta^n,$$

and the sequence

$$\beta, \quad \beta^2 = \beta + 1, \quad \beta^3 = \beta^2 + \beta, \quad \ldots$$

also satisfies (R).

You can easily verify (Exercise 3) that

$$\alpha + \beta = 1 \quad \text{and} \quad \alpha - \beta = \sqrt{5}.$$

If we now subtract the members of equation (B) from the members of equation (A) and divide each member of the resulting equation by $\alpha - \beta\ (=\sqrt{5} \neq 0)$, we find

$$\frac{\alpha^{n+2} - \beta^{n+2}}{\alpha - \beta} = \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} + \frac{\alpha^n - \beta^n}{\alpha - \beta}.$$

If we now let $u_n = \dfrac{\alpha^n - \beta^n}{\alpha - \beta}$, $n \geq 1$, then we have

$$u_{n+2} = u_{n+1} + u_n$$

and

$$u_1 = \frac{\alpha - \beta}{\alpha - \beta} = 1,$$

$$u_2 = \frac{\alpha^2 - \beta^2}{\alpha - \beta} = \frac{(\alpha - \beta)(\alpha + \beta)}{\alpha - \beta} = \frac{(\sqrt{5})(1)}{\sqrt{5}} = 1.$$

Thus, this sequence u_n is precisely the Fibonacci sequence defined in Section 2, and so

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \qquad n = 1, 2, 3, \ldots.$$

This is called the **Binet form** for the Fibonacci numbers after the French mathematician Jacques-Phillipe-Marie Binet (1786–1856).

Because of the relationship of the roots, α and β, of the equation (F),

$$x^2 - x - 1 = 0,$$

to the Fibonacci numbers, we shall call equation (F) the **Fibonacci quadratic equation.**

We shall call the positive root of (F),

$$\alpha = \frac{1 + \sqrt{5}}{2},$$

the **Golden Section.** [This is often represented by ϕ (Greek letter *phi*) or by some other symbol, but we shall continue to use α in this booklet.]

The point C in Figures 1 and 2, dividing $\overline{AB}$ such that

$$\frac{AB}{AC} = \alpha = \frac{1 + \sqrt{5}}{2},$$

is said to *divide* $\overline{AB}$ *in the Golden Section.*

Suppose that the rectangle $ABCD$ in Figure 3 is such that if the square $AEFD$ is removed from the rectangle, the lengths of the sides of the remaining rectangle, $BCFE$, have the same ratio as the lengths of the sides of the rectangle $ABCD$. That is,

$$\frac{BC}{EB} = \frac{AB}{DA}.$$

Then if $DA = AE = BC = x$ and $EB = y$, we have

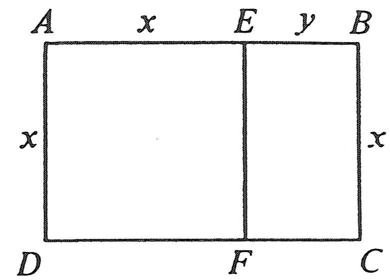

Figure 3

$$\frac{x}{y} = \frac{x+y}{x}, \qquad \text{or} \qquad \frac{x}{y} = 1 + \frac{y}{x}.$$

Multiplying both members of

$$\frac{x}{y} = 1 + \frac{y}{x}$$

by $\dfrac{x}{y}$, we find

$$\left(\frac{x}{y}\right)^2 = \frac{x}{y} + 1,$$

or

$$\left(\frac{x}{y}\right)^2 - \frac{x}{y} - 1 = 0,$$

which is in the form of equation (F), the variable now being $\left(\dfrac{x}{y}\right)$. Since x and y are positive, we seek the positive value of $\dfrac{x}{y}$. Thus,

$$\frac{x}{y} = \alpha = \frac{1+\sqrt{5}}{2}.$$

That is, the ratio of the length to the width for rectangle $BCFE$ (and also for rectangle $ABCD$) is the number α, the Golden Section. Such a rectangle is called a **Golden Rectangle.**

The proportions of the Golden Rectangle appear often throughout classical Greek art and architecture. As the German psychologists Gustav Theodor Fechner (1801–1887) and Wilhelm Max Wundt (1832–1920) have shown in a series of psychological experiments, most people do unconsciously favor "golden dimensions" when selecting pictures, cards, mirrors, wrapped parcels, and other rectangular objects. For some reason not fully known by either artists or psychologists, the Golden Rectangle holds great aesthetic appeal.

EXERCISES

1. Solve the Fibonacci quadratic equation,

$$x^2 - x - 1 = 0,$$

and verify that the roots are $\alpha = \frac{1 + \sqrt{5}}{2}$ and $\beta = \frac{1 - \sqrt{5}}{2}$ as stated in the text.

2. Verify that $\alpha > 0$ and $\beta < 0$, using $\sqrt{5} \doteq 2.236$. ($\doteq$ means "is approximately equal to.")

3. Verify that:

   a. $\alpha + \beta = 1$ b. $\alpha - \beta = \sqrt{5}$ c. $\alpha\beta = -1$

4. Verify that $\alpha = 1 + \frac{1}{\alpha}$.

5. Using $\alpha^2 = \alpha + 1$, verify that:

   a. $\alpha^3 = 2\alpha + 1$ b. $\alpha^4 = 3\alpha + 2$ c. $\alpha^5 = 5\alpha + 3$

6. Verify that $L_3 = \alpha^3 - \frac{1}{\alpha^3}$.

7. Verify that $F_4 = \frac{\alpha^4 - \alpha^{-4}}{\sqrt{5}}$.