4 • *Some Geometry Related to the Golden Section*

We shall now consider several geometric problems and their solutions.

PROBLEM 1*

Suppose that we wish to remove from a rectangle, $ABCD$, three right triangles of equal area, $\triangle PAQ$, $\triangle QBC$, and $\triangle CDP$, as shown in Figure 4. How shall we locate points P and Q?

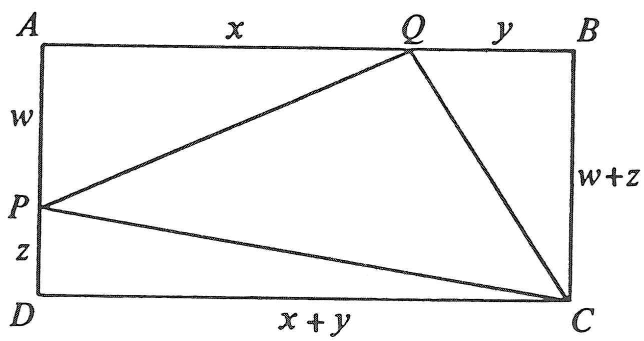


Figure 4

Solution. Let

$$AQ = x, \quad QB = y, \quad AP = w, \quad \text{and} \quad PD = z.$$

Then, since the areas of triangles PAQ, QBC, and CDP are to be equal, we have

$$\tfrac{1}{2}xw = \tfrac{1}{2}y(w + z) = \tfrac{1}{2}z(x + y),$$

or

$$xw = yw + yz = xz + yz.$$

* J. A. H. Hunter, "Triangle Inscribed in a Rectangle," *The Fibonacci Quarterly*, Vol. 1, No. 3 (October, 1963), page 66.

From

$$yw + yz = xz + yz$$

we have

$$yw = xz,$$

or

$$\frac{w}{z} = \frac{x}{y}.$$

Also, from

$$xw = y(w + z)$$

we have

$$\frac{x}{y} = \frac{w + z}{w} = 1 + \frac{z}{w} = 1 + \frac{1}{\frac{w}{z}}.$$

Since $\frac{w}{z} = \frac{x}{y}$, we have

$$\frac{x}{y} = 1 + \frac{1}{\frac{x}{y}},$$

or

$$\left(\frac{x}{y}\right)^2 - \frac{x}{y} - 1 = 0.$$

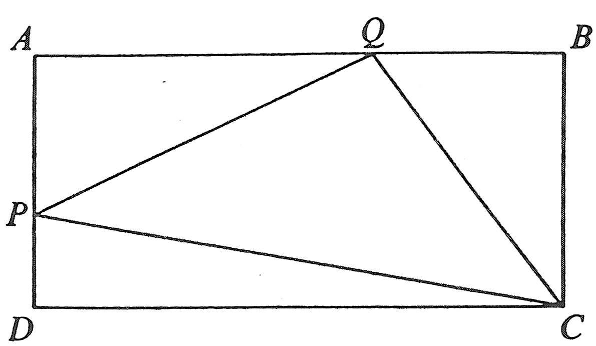
Again choosing the positive root, we have

$$\frac{x}{y} = \alpha = \frac{1 + \sqrt{5}}{2}.$$

But also

(A) $$\frac{w}{z} = \frac{x}{y} = \alpha,$$

and so the points P and Q must divide sides $\overline{AD}$ and $\overline{AB}$, respectively, in the Golden Section. Thus:



$$\frac{AP}{PD} = \frac{AQ}{QB} = \alpha$$

PROBLEM 2*

If the rectangle $ABCD$ in Problem 1 had been a Golden Rectangle, then the additional condition

(B) $$\frac{x+y}{w+z} = \alpha$$

would be imposed. Show that in this case $\triangle PQC$ would be an isosceles right triangle with the right angle at vertex Q.

Solution. From (A) on page 15 we have

$$x = \alpha y \quad \text{and} \quad w = \alpha z,$$

and so, from (B) above, we have

$$\alpha = \frac{x+y}{w+z} = \frac{\alpha y + y}{\alpha z + z} = \frac{(\alpha+1)y}{(\alpha+1)z}, \quad \text{or} \quad y = \alpha z.$$

But we had $w = \alpha z$, and so

$$w = y; \quad \text{that is,} \quad \overline{AP} \cong \overline{QB}.$$

Since we were given originally in Problem 1 that

$$\tfrac{1}{2}xw = \tfrac{1}{2}y(w+z),$$

the fact that $w = y$ implies that

$$x = w + z, \quad \text{that is,} \quad \overline{AQ} \cong \overline{BC}.$$

Therefore, right triangles PAQ and QBC are congruent. Thus,

$$\overline{PQ} \cong \overline{QC} \quad \text{and} \quad \angle AQP \cong \angle BCQ.$$

Moreover,

$$m^\circ\angle BCQ + m^\circ\angle CQB = 90.$$

Since

$$m^\circ\angle AQP + m^\circ\angle PQC + m^\circ\angle CQB = 180^\circ$$

and

$$m^\circ\angle AQP = m^\circ\angle BCQ,$$

we have

$$m^\circ\angle PQC = 90.$$

Therefore, since in $\triangle PQC$ we now have $\overline{PQ} \cong \overline{QC}$ and $m^\circ\angle PQC = 90$, $\triangle PQC$ is an isosceles right triangle, as pictured in Figure 5.

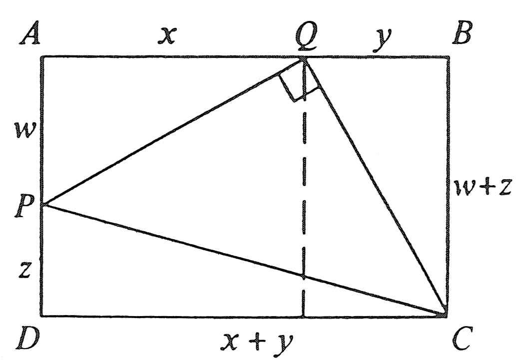


Figure 5

* H. E. Huntley, "Fibonacci Geometry," *The Fibonacci Quarterly*, Vol. 2, No. 2 (April, 1964), page 104.

PROBLEM 3

Do two triangles exist which have measures of five of their six parts (three angles and three sides) equal and yet are *not* congruent? Your first impulsive answer may be a resounding No! However, we propose to show that this is indeed possible.

Solution. Clearly, if among the five parts are three sides, then the triangles must be congruent. The only possibility then is to have the three angles of one triangle congruent to the three angles of the other triangle (thus, the triangles are similar) and two sides of one triangle congruent to two sides of the other. (Notice that it is *not* specified that these sides be corresponding sides.) One example of such a pair of triangles (in this case with integral sides) is shown in Figure 6, where

$$\frac{27}{18} = \frac{18}{12} = \frac{12}{8}.$$

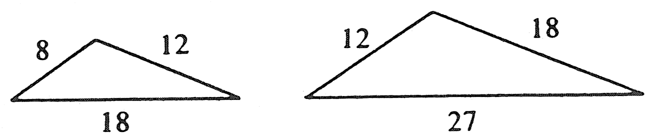


Figure 6

Let us see how to find pairs of triangles having just five parts congruent. First, we know that the triangles must be similar; that is, the measures of the sides must be related as shown in Figure 7, where r is the ratio of similarity:

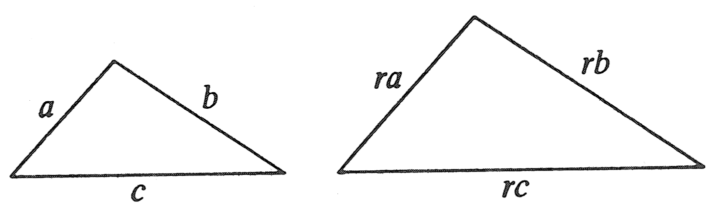


Figure 7

The additional conditions are

$$b = ra \qquad \text{and} \qquad c = rb = r^2a.$$

Thus, the measures of the sides of the two triangles will be

$$a,\ ra,\ r^2a \qquad \text{and} \qquad ra,\ r^2a,\ r^3a,$$

as shown in Figure 8:

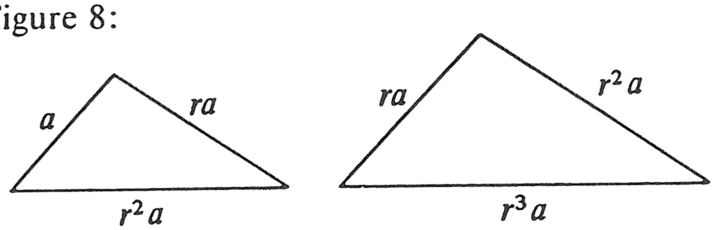


Figure 8

If $a = 8$ and $r = \frac{3}{2}$, you will find the measures of the sides of the triangles shown in Figure 6. Try other values to find other triangles having this property.

PROBLEM 4

If $a = 4$ and $r = 2$ in Problem 3, then the measures of the sides of the triangles would be

$$4, \quad 8, \quad 16 \qquad \text{and} \qquad 8, \quad 16, \quad 32.$$

Can there be triangles with sides of these measures? No, because

$$4 + 8 < 16 \qquad \text{and} \qquad 8 + 16 < 32.$$

What restrictions must be placed on the values of r, assuming for the present that $r > 1$?

Solution. If a, ra, and r^2a are to be the measures of the sides of a triangle, then the sum of each two must be greater than the third. Thus, we have three inequalities:

$$\begin{array}{lll} \text{(i)} \quad a + ra > r^2a & \text{or, since } a > 0, & 1 + r > r^2 \\ \text{(ii)} \quad ra + r^2a > a & \text{or, since } a > 0, & r + r^2 > 1 \\ \text{(iii)} \quad r^2a + a > ra & \text{or, since } a > 0, & r^2 + 1 > r \end{array}$$

We are looking for the solution set of these three inequalities.

On the assumption that $r > 1$, we have

$$r^2 > r > 1,$$

and inequalities (ii) and (iii) hold. Thus, we need to consider inequality (i), which we shall write as

$$r^2 < r + 1, \qquad \text{or} \qquad r^2 - r - 1 < 0.$$

Recall that

$$x^2 - x - 1 = 0$$

is the Fibonacci quadratic equation with roots

$$\alpha = \frac{1 + \sqrt{5}}{2} \qquad \text{and} \qquad \beta = \frac{1 - \sqrt{5}}{2}.$$

Therefore, we can write

$$r^2 - r - 1 = \left(r - \frac{1 + \sqrt{5}}{2}\right)\left(r - \frac{1 - \sqrt{5}}{2}\right).$$

For the second factor, we have

$$r - \frac{1 - \sqrt{5}}{2} = r - \frac{1}{2} + \frac{\sqrt{5}}{2},$$

and this is positive for $r > 1$.

Therefore, to have

$$r^2 - r - 1 < 0$$

we must have

$$r - \frac{1+\sqrt{5}}{2} < 0, \qquad \text{or} \qquad r < \frac{1+\sqrt{5}}{2},$$

that is, $r < \alpha$, and so if $r > 1$, we must have

$$1 < r < \alpha$$

in order to have a pair of triangles with just five parts congruent.

PROBLEM 5

Can a pair of right triangles have just five parts congruent?

Solution. Suppose that $r > 1$. Then r^2a is the measure of the longest side. If the triangles are to be right triangles, we have by the Pythagorean theorem that

$$(r^2a)^2 = a^2 + (ra)^2.$$

Thus,

$$r^4a^2 - r^2a^2 - a^2 = 0$$

or, since $a \neq 0$,

$$r^4 - r^2 - 1 = 0.$$

Therefore,

$$r^2 = \alpha,$$

and so the positive value of r in this case is $\sqrt{\alpha}$.

How can we construct such a pair of right triangles? Recall that in a right triangle, the altitude from the vertex of the right angle to the hypotenuse separates the given triangle into two triangles that are similar to each other and also to the given triangle. That is, in Figure 9, where angle C is a right angle,

$$\triangle ACD \sim \triangle CBD \sim \triangle ABC.$$

Notice that $\triangle ACD$ and $\triangle CBD$ are similar and have one side, $\overline{CD}$, in common. If we had $\overline{AD}$ congruent to $\overline{BC}$, we would have two right triangles that have just five parts congruent. These are shown as

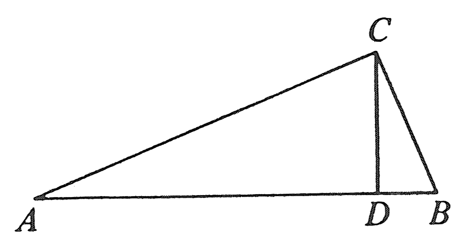


Figure 9

$$\triangle A'C'D' \qquad \text{and} \qquad \triangle C'B'D'$$

in Figure 10 on the next page.

If we use the letters shown in Figure 10 to represent the measures of the sides, we have

$$\frac{x+y}{x} = \frac{z}{w} = \frac{x}{y}.$$

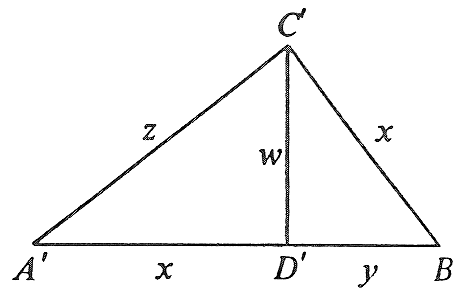


Figure 10

From

$$\frac{x}{y} = \frac{x+y}{x}$$

we have

$$\left(\frac{x}{y}\right)^2 - \frac{x}{y} - 1 = 0, \qquad \text{and so} \qquad \frac{x}{y} = \alpha.$$

Thus, the point D' must divide $\overline{A'B'}$ in the Golden Section.

Let us find the ratio of similarity for $\triangle A'C'D'$ and $\triangle C'B'D'$, that is, the value of

$$r = \frac{w}{y} = \frac{x}{w} = \frac{z}{x}.$$

Since $x = \alpha y$, we have

$$\frac{w}{y} = \frac{\alpha y}{w}, \qquad \text{or} \qquad \frac{w^2}{y^2} = \alpha.$$

Therefore, since $r > 0$,

$$r = \frac{w}{y} = \sqrt{\alpha},$$

as predicted by our computation on page 19.

Some of the possible shapes for triangles having just five parts congruent are shown in Figure 11. Those sketched are right and oblique triangles. In order for such triangles to have only acute angles, we must have

$$1 < r < \sqrt{\alpha}, \qquad \sqrt{\alpha} \doteq 1.27.$$

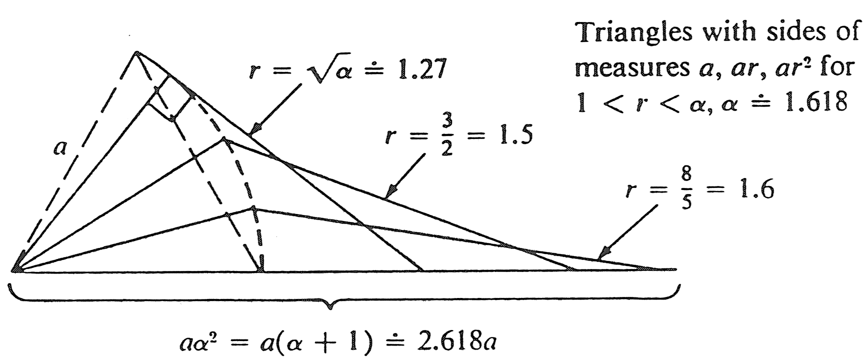


Figure 11

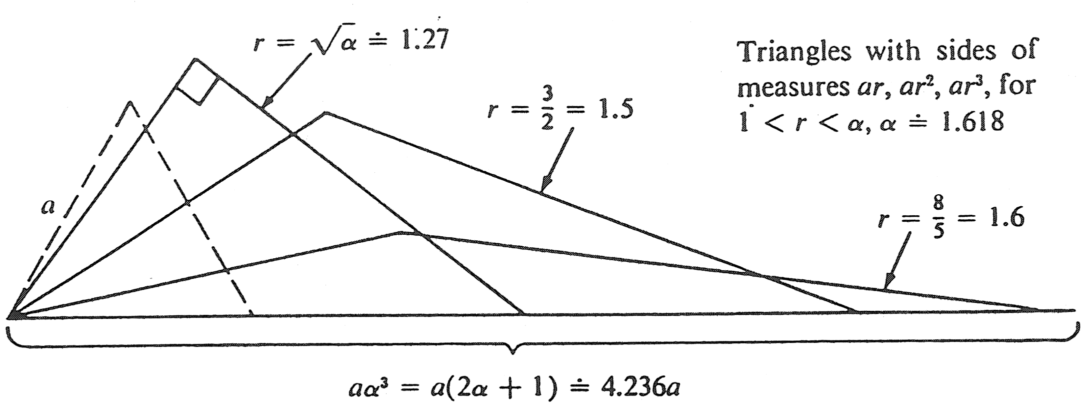


Figure 11 (continued)

Up to now we have restricted ourselves to $r > 1$. If $r > 1$ is the ratio of similarity of the larger triangle to the smaller triangle, then

$$r' = \frac{1}{r}$$

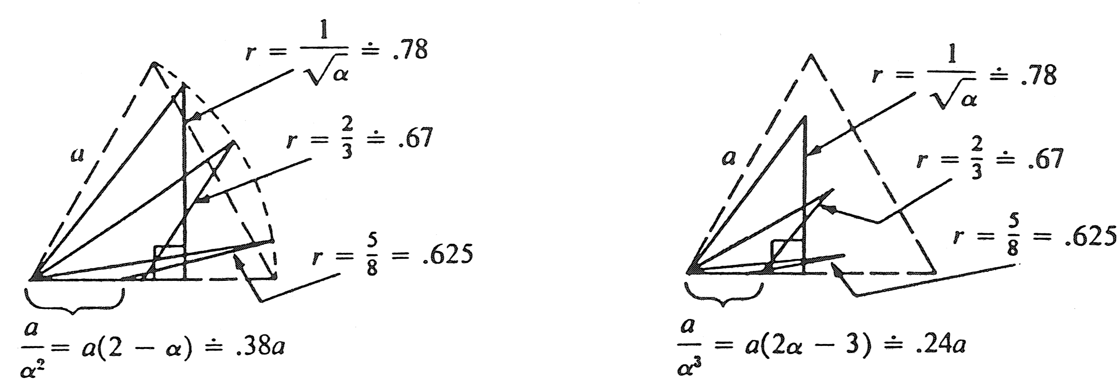
is the ratio of similarity of the smaller triangle to the larger triangle. Thus, in general, we may have

$$\frac{1}{\alpha} < r < 1 \qquad \text{as well as} \qquad 1 < r < \alpha,$$

that is, approximately

$$.618 < r < 1 \qquad \text{or} \qquad 1 < r < 1.618.$$

The triangles pictured in Figure 12 have ratios that are the reciprocals of those for the triangles in Figure 11. Notice that each pair of triangles in Figure 11 is similar to the pair in Figure 12 that has the reciprocal ratio (compare Exercise 6 on page 25). (If r were 1, then the triangles would be congruent equilateral triangles.)



Triangles with sides of measures a, ar, ar^2 for $\frac{1}{\alpha} < r < 1, \frac{1}{\alpha} \doteq .618$

Triangles with sides of measures ar, ar^2, ar^3 for $\frac{1}{\alpha} < r < 1, \frac{1}{\alpha} \doteq .618$

Figure 12

In Section 3 we defined a Golden Rectangle. We shall now define a Golden Triangle. In Figure 10, the ratio of the area of $\triangle A'B'C'$ to the area of $\triangle A'C'D'$ can be found as follows:

$$\text{Area } \triangle A'B'C' = \tfrac{1}{2}w(x+y)$$
$$\text{Area } \triangle A'C'D' = \tfrac{1}{2}wx$$
$$\frac{\text{Area } \triangle A'B'C'}{\text{Area } \triangle A'C'D'} = \frac{x+y}{x} = \frac{\alpha y + y}{\alpha y} = \frac{\alpha + 1}{\alpha} = \frac{\alpha^2}{\alpha} = \alpha$$

A triangle that has this property is called a **Golden Triangle**; that is, a **Golden Triangle** is one such that when a triangle similar to it is removed from it, the ratio of the area of the Golden Triangle to the area of the *remaining triangle* is α. That is, in Figure 10 when $\triangle C'B'D'$ is removed from $\triangle A'B'C'$,

$$\frac{\text{Area } \triangle A'B'C'}{\text{Area } \triangle A'C'D'} = \alpha.$$

We also note that

$$\frac{\text{Area } \triangle A'C'D'}{\text{Area } \triangle C'B'D'} = \frac{\frac{1}{2}wx}{\frac{1}{2}wy} = \frac{x}{y} = \alpha,$$

and

$$\frac{\text{Area } \triangle A'B'C'}{\text{Area } \triangle C'B'D'} = \frac{\text{Area } \triangle A'B'C'}{\text{Area } \triangle A'C'D'} \cdot \frac{\text{Area } \triangle A'C'D'}{\text{Area } \triangle C'B'D'} = \alpha \cdot \alpha = \alpha^2.$$

PROBLEM 6

Show that an isosceles triangle with vertex angle measuring 36° is a Golden Triangle.

Solution. The base angles measure 72° each. If one of these base angles is bisected (see Figure 13), two isosceles triangles are formed. One triangle, $\triangle CDB$, is similar to the given triangle, $\triangle ABC$, while the other, $\triangle ACD$, is not.

In $\triangle ABC$ and $\triangle CDB$,

$$\frac{x+y}{x} = \frac{x}{y},$$

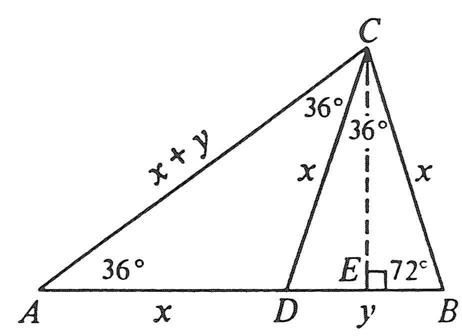


Figure 13

and again we have

$$\left(\frac{x}{y}\right)^2 - \frac{x}{y} - 1 = 0$$

and the positive result is

$$\frac{x}{y} = \alpha.$$

To find the areas of $\triangle ABC$ and $\triangle ADC$, draw altitude $\overline{CE}$. Then:

$$CE = \sqrt{x^2 - \frac{y^2}{4}} \qquad (CE \neq 0)$$

$$\text{Area } \triangle ABC = \tfrac{1}{2}(x + y)(CE)$$

$$\text{Area } \triangle ADC = \tfrac{1}{2}x(CE)$$

$$\frac{\text{Area } \triangle ABC}{\text{Area } \triangle ADC} = \frac{x + y}{x} = \alpha.$$

Thus, $\triangle ABC$ is a Golden Triangle.

We also note that

$$\frac{\text{Area } \triangle ADC}{\text{Area } \triangle CDB} = \frac{x}{y} = \alpha$$

and

$$\frac{\text{Area } \triangle ABC}{\text{Area } \triangle CDB} = \frac{x + y}{y} = \frac{x + y}{x} \cdot \frac{x}{y} = \alpha^2.$$

Since the central angle of a regular decagon is 36° (see Figure 14), we know from Problem 6 that the ratio of the radius r to the measure s of the side of an inscribed decagon is α.

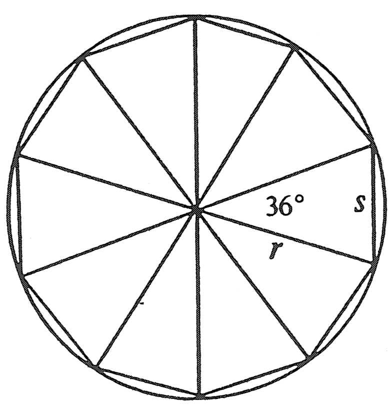


Figure 14

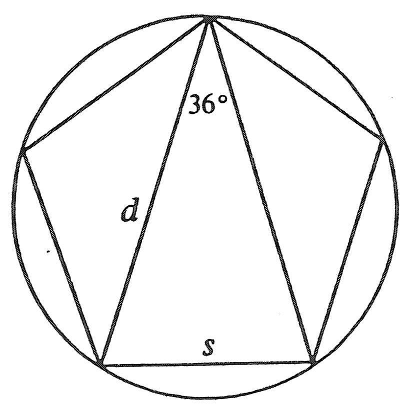


Figure 15

Also, in a regular inscribed pentagon, the angle between two adjacent diagonals at one vertex is 36° (see Figure 15), and so the ratio of the measure d of a diagonal to the measure s of a side is also α.

Notice that in Figure 13, $\dfrac{AB}{BC} = \alpha$ and that in Figure 10, $\dfrac{A'B'}{B'C'} = \alpha$. Thus, in each case the ratio of the measure of the longest side to the measure of the shortest side is α.

The Golden Triangle appears on pages 61–62 of Tobias Dantzig's *The Bequest of the Greeks* (New York: Charles Scribner's Sons, 1955) and also on page 42 of N. N. Vorobyov's *The Fibonacci Numbers* (Boston: D. C. Heath and Company, 1963). Also, see the article, "Golden Triangles, Rectangles, and Cuboids," by Marjorie Bicknell and Verner E. Hoggatt, Jr., in *The Fibonacci Quarterly*, Vol. 7, No. 1 (February, 1969), pages 73–91.

PROBLEM 7

Inscribe a square in a semicircle.

Solution. Figure 16 shows the completed construction where $AMNB$ is a square. The construction makes use of the fact that right triangles

$$OAM, \quad ODC, \quad OEF, \quad \text{and} \quad OBN$$

are similar.

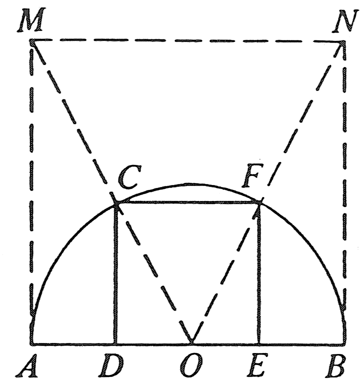


Figure 16

Now consider Figure 17, in which $\overline{AF}$ and $\overline{FB}$ have been drawn, forming similar right triangles ABF, AFE, and FBE. Thus,

$$\frac{t+s}{s} = \frac{s}{t},$$

from which we obtain

$$\left(\frac{s}{t}\right)^2 - \frac{s}{t} - 1 = 0,$$

and so the positive result is

$$\frac{s}{t} = \alpha.$$

Thus, point E divides $\overline{DB}$ in the Golden Section.

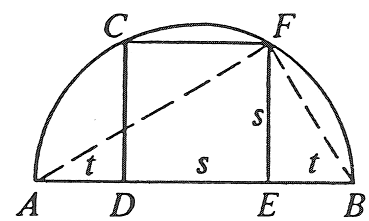


Figure 17

Two articles by Marvin Holt, which give further excellent material on the Golden Section and geometry, are "Mystery Puzzler and Phi" in *The Fibonacci Quarterly*, Vol. 3, No. 2 (April, 1965), pages 135–138, and "The Golden Section" in the *Pentagon*, Spring, 1964, pages 80–104.

The Golden Cuboid is discussed in an article of that title by H. Huntley in *The Fibonacci Quarterly*, Vol. 2, No. 3 (October, 1964), page 184.

Another interesting reference is *Patterns in Space* by Colonel R. S. Beard (available from Brother Alfred Brousseau, St. Mary's College, California 94575). This book describes many appearances of the Golden Section in variations of the regular solids.

EXERCISES

1. Show that there can be no triangle having three distinct Fibonacci numbers as measures of its sides.

2. Show that a pair of triangles which have the measures of five parts equal, but which are not congruent, cannot be isosceles.

3. Show that

   a. if $a < b$, then $y = (x - a)(x - b)$ is negative for $a < x < b$ and non-negative otherwise;

   b. if $a = b$, then $y = (x - a)^2 \geq 0$ for all x.

4. Using the results of Exercise 3a, show that, in general,

   a. inequality (i) on page 18 holds when

   $$r^2 - r - 1 < 0, \text{ that is, for } \beta < r < \alpha.$$

   b. inequality (ii) on page 18 holds when

   $$r^2 + r - 1 > 0, \text{ that is, for } r < -\alpha \text{ or } r > -\beta.$$

   c. inequalities (i) and (ii) both hold when

   $$-\beta < r < \alpha, \qquad \text{or} \qquad \frac{1}{\alpha} < r < \alpha,$$

   since $\alpha\beta = -1$ (Exercise 3c, page 13).

5. Using the result of Exercise 3b, show that inequality (iii) on page 18 holds for $r > 0$, that is, that

   $$r^2 + 1 \geq 2r > r \quad \text{for} \quad r > 0.$$

6. The measures of the sides of one triangle are a, ar, and ar^2 and those of a second triangle are a, $\frac{a}{r}$, and $\frac{a}{r^2}$. By suitably pairing the sides, show that the triangles are similar and find the ratio of similarity.

7. In Figure 16 extend $\overline{CF}$ to meet $\overline{BN}$ in point G. Show that rectangle $DCGB$ is a Golden Rectangle.

8. In Figure 17 show that $\dfrac{AF}{FB} = \alpha$.

9. In Figure 17 show that:

   a. $\dfrac{s^2 + 2st}{s^2 + t^2} = \alpha$ b. $\dfrac{t^2 - 2st}{s^2 + t^2} = \beta$ c. $\dfrac{s^2 + 4st - t^2}{s^2 + t^2} = \sqrt{5}$

   Hint: Recall that $\alpha^2 = \alpha + 1$, $\alpha + \beta = 1$, $\alpha - \beta = \sqrt{5}$.