7 • *Divisibility Properties of the Fibonacci and Lucas Numbers*

Let us look at the first few Fibonacci numbers:

F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9	F_{10}	F_{11}	F_{12}	F_{13}	F_{14}	$F_{15}\ldots$
1	1	2	3	5	8	13	21	34	55	89	144	233	377	610 . . .

We observe the following:

1. Every third F_n is even; that is, $F_3 = 2$ divides $F_6 = 8$, $F_9 = 34$, $F_{12} = 144$, $F_{15} = 610, \ldots$.
2. $F_4 = 3$ divides $F_8 = 21$, $F_{12} = 144, \ldots$.
3. $F_5 = 5$ divides $F_{10} = 55$, $F_{15} = 610, \ldots$.
4. $F_6 = 8$ divides $F_{12} = 144, \ldots$.
5. $F_7 = 13$ divides $F_{14} = 377, \ldots$.

These examples suggest the following theorem.

THEOREM I

Every Fibonacci number F_k divides every Fibonacci number F_{nk} for $n = 1, 2, 3, \ldots$; or if r is divisible by s, then F_r is divisible by F_s.

First, consider, for example,

$$F_{10} = \frac{\alpha^{10} - \beta^{10}}{\alpha - \beta}.$$

The difference of equal powers of two numbers can be factored, often in various ways. Here we may divide $\alpha^{10} - \beta^{10}$ by $\alpha^2 - \beta^2$ or by $\alpha^5 - \beta^5$ as shown on the following page.

$$\begin{aligned} F_{10} &= \frac{\alpha^{10} - \beta^{10}}{\alpha - \beta} \\ &= \frac{\alpha^2 - \beta^2}{\alpha - \beta}(\alpha^8 + \alpha^6\beta^2 + \alpha^4\beta^4 + \alpha^2\beta^6 + \beta^8) \\ &= F_2[(\alpha^8 + \beta^8) + \alpha^2\beta^2(\alpha^4 + \beta^4) + \alpha^4\beta^4] \end{aligned}$$

Since $\alpha\beta = -1$, we have

(A) $$F_{10} = F_2(L_8 + L_4 + 1).$$

Also, we have

$$\begin{aligned} F_{10} &= \frac{\alpha^{10} - \beta^{10}}{\alpha - \beta} \\ &= \frac{\alpha^5 - \beta^5}{\alpha - \beta}(\alpha^5 + \beta^5), \text{ or} \end{aligned}$$

(B) $$F_{10} = F_5 L_5.$$

Thus, since $(L_8 + L_4 + 1)$ in equation (A) is an integer, F_2 is a factor of F_{10}; also F_5 is a factor of F_{10} as shown by equation (B).

In general, to see why F_{nk} is divisible by F_k, consider the factorization

(C) $$\begin{aligned} F_{nk} &= \frac{\alpha^{nk} - \beta^{nk}}{\alpha - \beta} \\ &= \frac{\alpha^k - \beta^k}{\alpha - \beta}(\alpha^{(n-1)k} + \alpha^{(n-2)k}\beta^k \\ &\qquad + \alpha^{(n-3)k}\beta^{2k} + \cdots + \alpha^k\beta^{(n-2)k} + \beta^{(n-1)k}) \end{aligned}$$

In the right-hand parenthesis, the first and last terms may be paired to form a Lucas number:

$$\alpha^{(n-1)k} + \beta^{(n-1)k} = L_{(n-1)k}$$

The second and next-to-last terms may be paired to form a product of $(-1)^k$ and a Lucas number:

$$\begin{aligned} \alpha^{(n-2)k}\beta^k + \alpha^k\beta^{(n-2)k} &= \alpha^k\beta^k(\alpha^{(n-3)k} + \beta^{(n-3)k}) \\ &= (\alpha\beta)^k L_{(n-3)k} = (-1)^k L_{(n-3)k} \end{aligned}$$

And so on. Notice that the number of terms in the right-hand parenthesis of (C) is n. If n is even, then the terms match up in symmetric pairs to make Lucas numbers, clearly adding up to an integer. See, for example, equation (B), where

$$F_{10} = F_{2\cdot 5} = F_5 L_5,$$

with $k = 5$ and $n = 2$, which is even. If n is odd, then the terms still match up in symmetric pairs except for the middle term, which is of the form $(\alpha\beta)^{\frac{(n-1)k}{2}}$, clearly an integer. Again the right-hand parenthetical expression in (C) is an integer. See, for example, equation (A) where

$$F_{10} = F_{5\cdot 2} = F_2(L_8 + L_4 + 1),$$

with $k = 2$ and $n = 5$, which is odd.

Another interesting theorem is given in terms of the greatest common divisor of two positive integers. The greatest common divisor of two positive integers a and b is denoted by

$$(a, b) = d,$$

meaning that d is the greatest positive integer dividing both a and b. For example,

$$(14, 2) = 2, \quad (24, 15) = 3, \quad \text{and} \quad (6765, 610) = 5.$$

This theorem can now be stated as:

THEOREM II

$$(F_m, F_n) = F_{(m, n)}.$$

This means that the greatest common divisor of two Fibonacci numbers is a *Fibonacci number* whose subscript (index) is the greatest common divisor of the subscripts (indices) of the other two Fibonacci numbers. Thus,

$$(F_{15}, F_{14}) = (610, 377) = F_{(15, 14)} = F_1 = 1;$$

$$(F_9, F_6) = (34, 8) = F_{(9, 6)} = F_3 = 2;$$

$$(F_{12}, F_6) = (144, 8) = F_{(12, 6)} = F_6 = 8. \quad \text{Here } F_6 \text{ divides } F_{12}.$$

Theorem II can be proved by using the Euclidean Algorithm* or as the solution to a Diophantine equation.†

Theorem I and Theorem II may be combined as:

THEOREM III

F_n is divisible by F_m *if and only if* n is divisible by m.

* N. N. Vorobyov, *Fibonacci Numbers* (Boston: D. C. Heath and Co., 1963), pp. 22–24.

† Glenn Michael, "A New Proof for an Old Property," *The Fibonacci Quarterly*, Vol. 2, No. 1 (February, 1964), pp. 57–58.

Let us now look at the first few Lucas numbers:

L_1	L_2	L_3	L_4	L_5	L_6	L_7	L_8	L_9	L_{10}	L_{11}	L_{12}	L_{13}	L_{14}	L_{15} ...
1	3	4	7	11	18	29	47	76	123	199	322	521	843	1364...

We immediately notice that every third L_n is even, as was the case for the Fibonacci numbers.

We next state without proof two interesting theorems.

THEOREM IV (L. Carlitz)*

$$L_n \text{ divides } F_m \textit{ if and only if } m = 2kn, \qquad n > 1.$$

For example, $L_3 = 4$ divides $F_6 = 8$, $F_{12} = 144$, and so on.

THEOREM V (L. Carlitz)*

$$L_n \text{ divides } L_m \textit{ if and only if } m = (2k - 1)n, \qquad n > 1.$$

For example, $L_3 = 4$ divides $L_9 = 76$, $L_{15} = 1364$, and so on.

It is easy to see that two consecutive Fibonacci numbers have no common factor greater than one. If F_{n+2} and F_{n+1} had a common factor d, then $F_n = F_{n+2} - F_{n+1}$ would also be divisible by d. Thus, we could progress down to $F_2 = 1$, which d would have to divide. Since d is a positive integer, d must be 1. A similar argument applies to Lucas numbers. Thus, we have:

THEOREM VI

$$(F_{n+2}, F_{n+1}) = 1.$$

THEOREM VII

$$(L_{n+2}, L_{n+1}) = 1.$$

EXERCISES

Refer to the list of Fibonacci and Lucas numbers on page 83.

1. Verify that F_7 divides F_{14}, F_{21}, and F_{28}.
2. Verify that F_{10} divides F_{20}, F_{30}, and F_{40}.

* L. Carlitz, "A Note on Fibonacci Numbers," *The Fibonacci Quarterly*, Vol. 2, No. 1 (February, 1964), pages 15–28.

3. Verify that F_{24} is divisible by F_3, F_4, F_6, F_8, and F_{12}.

4. Verify that F_{30} is divisible by F_3, F_5, F_6, F_{10}, and F_{15}.

5. Verify that L_4 divides F_8 and F_{16}.

6. Verify that L_7 divides F_{14} and F_{28}.

7. Verify that L_4 divides L_{12} and L_{20}.

8. Verify that L_5 divides L_{15} and L_{25}.

9. Verify that

$$\begin{aligned} F_{12} &= F_3(L_9 - L_3) = F_3L_3L_6 \\ &= F_4(L_8 + 1) \\ &= F_6L_6. \end{aligned}$$

10. Verify that

$$\begin{aligned} F_{18} &= F_3(L_{15} - L_9 + L_3) \\ &= F_6(L_{12} + 1) \\ &= F_9L_9. \end{aligned}$$

11. Find the greatest common divisor of F_{16} and F_{24}.

12. Find the greatest common divisor of F_{24} and F_{36}.