GENERALIZED PASCAL TRIANGLES AND PYRAMIDS

THEIR FRACTALS, GRAPHS, AND APPLICATIONS

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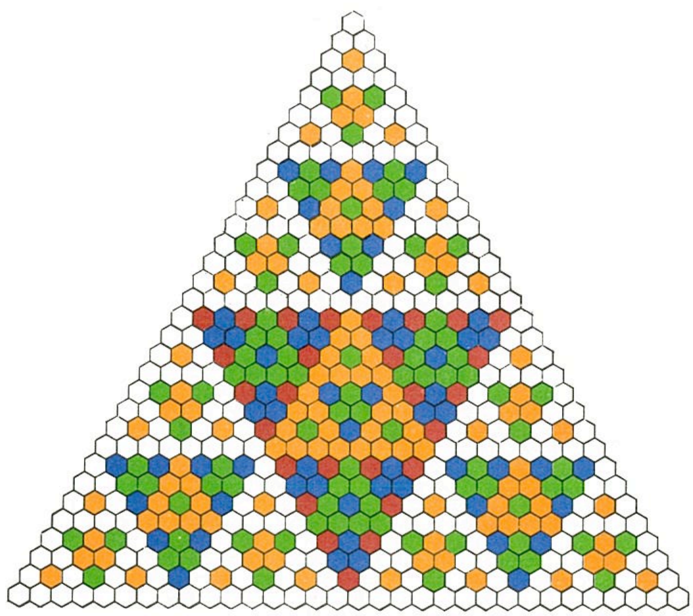

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TRANSLATOR'S PREFACE

Professor Bondarenko has done the mathematical community a valuable service in writing this useful and interesting compendium of results on the Pascal triangle and its ramifications, and in compiling the excellent and lengthy collection of references. My intention is to help make this work widely accessible.

This is meant to be a serviceable translation, rather than a re-exposition, and the English version follows the Russian closely. In the Russian language section (the first 54 items) of the References, I have replaced some translated entries first published in English by their original English citations.

For their indispensable help, I would like to thank Kathy Mauro, who typeset the manuscript, and Linda Bollinger, who proofread the final version.

Richard C. Bollinger

PREFACE

The discrete methods of combinatorial analysis, and their application to the construction of mathematical models and solutions of applied problems in technology and the natural sciences, have brought about a great deal of interest in the study of the arithmetic and geometric properties of the so-called "arithmetic triangles." The classical example of the arithmetic triangle is, of course, the Pascal triangle.

In recent decades there has been a widening circle of research on the Pascal triangle itself, as well as its planar and spatial analogs and generalizations. Although there are a large number of scientific and methodological papers devoted to the study of the Pascal triangle and other arithmetic triangles, there have been only a few isolated expository studies and books, chiefly methodological in character. Among these we might cite the small volume of V.A. Uspenskii "The Pascal Triangle" [50], which was translated into English and gives a popular account of the basic properties of the Pascal triangle, and also the excellent book of T.M. Green and C.L. Hamberg, "Pascal's Triangle " [162], which describes known and new properties of the Pascal triangle, and is intended for college students and amateur mathematicians.

The present monograph is devoted to rather more profound questions connected with the study of the Pascal triangle, and its planar and spatial analogs. There is an extensive discussion of the divisibility of the binomial, trinomial, and multinomial coefficients by a prime p, and of the distributions of these coefficients with respect to the modulus p, or p^s, in corresponding arithmetic triangles, pyramids, and hyperpyramids. Particular attention is given to those objects which today we speak of as fractals, and whose present extensive development arose from the works of Benoit Mandelbrot [270-272]. Fractals obtained from the Pascal triangle and other arithmetic triangles are described, as are also results from the study of the properties of the generalized arithmetic graph, a special case of which is the graph model of the generalized Pascal triangle. We also construct and investigate matrices and determinants whose elements may be binomial, generalized binomial, and trinomial coefficients, and other special values. Particular attention is given to the development of effective combinatorial methods and algorithms for the construction of basis systems of polynomial solutions of partial differential equations, including equations of high order and with mixed derivatives. The algorithms proposed are invariant with respect to the order, and the iteration, of operators arising in connection with differential equations. Finally, we discuss non-orthogonal polynomials of binomial type, and polynomials whose coefficients may be Fibonacci, Lucas, Catalan, and other special numbers.

This monograph consists of seven chapters, and there are many illustrations and specific examples. Fundamental results are formulated as theorems and algorithms, and as various equations and formulas. There is a detailed list of over four hundred references, covering almost all known works on arithmetic triangles and pyramids,

The author is deeply grateful to S.G. Mikhlin for his valuable advice and constant support of this work, and to A.A. Adylov for writing Chapter 5 on arithmetic graphs.

For their reviews of the manuscript and useful comments the author thanks F.B. Abutaliev, V.M. Maksimov, and V.K. Kabulov.

The solutions of the specific examples, and the constructions of the arithmetic triangles and their fractals, were carried out by Mariya Morozova, to whom the author gladly expresses his appreciation.

Boris A. Bondarenko

CHAPTER 1

THE PASCAL TRIANGLE AND ITS PLANAR AND SPATIAL GENERALIZATIONS

In this chapter we outline some of the history of the Pascal triangle and the binomial coefficients, and also describe some modern results obtained by mathematicians in recent decades. We consider, as well, generalized Pascal triangles of s^{th} order, Pascal pyramids and hyperpyramids, and triangles associated with the Fibonacci, Lucas, and Catalan numbers. Finally, we discuss generalized binomial coefficients of s^{th} order, multinomial coefficients, and Gauss-, Fibonacci-, and other analogs of the binomial coefficients.

1.1 THE PASCAL TRIANGLE AND ITS PROPERTIES

One of the most familiar objects in the history of mathematics is the so-called "arithmetical triangle", more commonly known today as the Pascal triangle in honor of the seventeenth century French mathematician and philosopher Blaise Pascal (1623-1662), who set forth his results in this area in his Traité du triangle arithmetique [303] (published after the author's death). Pascal generalized known results, and gave a number of new properties of the arithmetic triangle, which he formulated in nineteen theorems. [Figure 1 is an example from Pascal's work.] The various properties of the numbers generated in the arithmetic triangle were given by Pascal in descriptive form, rather than algebraically, but he made direct and significant use of the principles he had discovered, e.g., in the method of induction and the application of the arithmetic triangle to problems in the theory of probability.

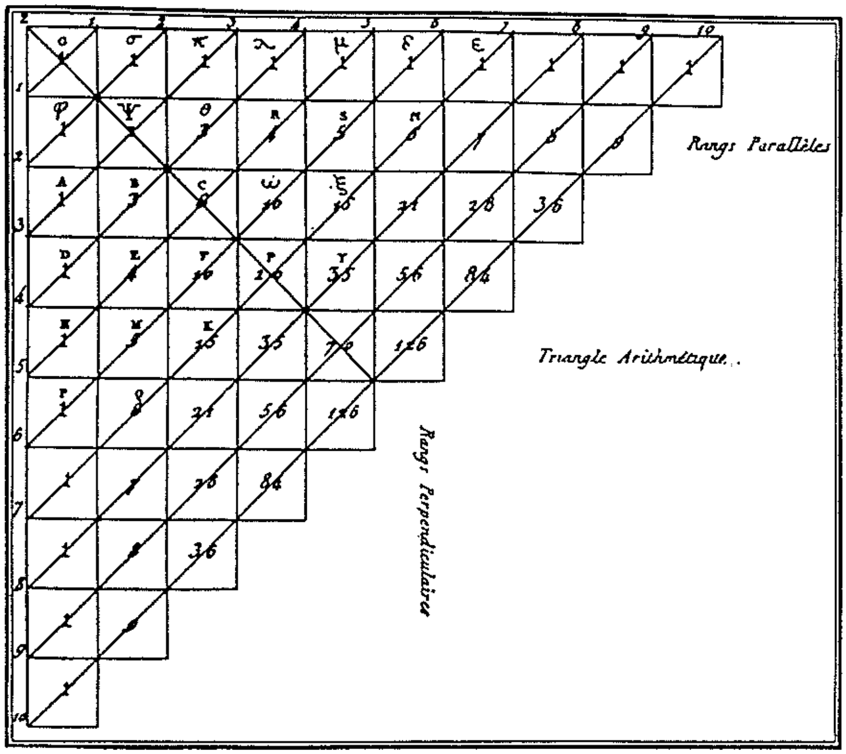


Figure 1

The arithmetic triangle and the additive rules for the formation of its entries were known in India virtually as we know them today. Its structure was also known to Omar Khayyám, the Persian mathematician, poet, and philosopher (c.1100). Later, the triangle appeared in China, and was depicted in a book of Chu Shin-Chien (1303).

In Europe, the arithmetic triangle had been known long before the publication of Pascal's work. It appeared, for example, on the title page of a book by A. Apian in 1529, and was used by many other mathematicians, among them M. Stifel (1544), G. Peletier (1549), K. Rudolph (1533), N. Tartaglia (1556), J. Cardan (1570), S. Stevin (1585), A. Girard (1629), W. Oughtred (1631), and G. Briggs (1633). More on the history of the Pascal triangle may be found in [17, 28, 31, 44, 50, 89, 90, 122, 141, 147, 241, 257, 265, 291, 292, 320, 379].

The familiar form of the table,

1	1	1	1	1	1	...
1	2	3	4	5	6	...
1	3	6	10	15	21	...
1	4	10	20	35	56	...
1	5	15	35	70	126	...
1	6	21	56	126	252	...

was published more than a century before Pascal's treatise in a work of the outstanding Italian mathematician Nicolo Tartaglia (1556). Subsequent investigations on the Pascal triangle and the binomial coefficients, and their connection with the origins and development of combinatorial analysis are connected with the names of Leibnitz, Bernoulli, Euler, Lucas, Legendre, and other prominent eighteenth- and nineteenth-century mathematicians.

Interest in the Pascal triangle has not diminished even up to the present, which accounts for the discovery of new and often unexpected properties related to divisibility and the distribution of the triangle's elements modulo a prime p, the construction and study of its fractals and graphs, and its application to important practical problems. We also depend on the triangle for a model in considering new types of arithmetic triangles, and rectangular, pyramidal, and other arithmetic tables.

The Pascal triangle is often presented in the form of an isosceles triangle whose sides are bordered by ones (Figure 2), and such that the remaining elements are the sums of the two entries just above to the left and right. The line numbered n consists of the coefficients in the binomial expansion of $(1+x)^n$. These coefficients are denoted in various ways in the literature, but here we will use the notation $\binom{n}{m}$, introduced as far back as Euler's time, and/or the notation C_m^n, which appeared in the nineteenth century.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
. . . .

Figure 2

The Pascal triangle may also be presented in right triangular form, as for instance,

a

1			
1	1		
1	2	1	
1	3	3	1

. . .

b

			1
		1	1
	1	2	1
1	3	3	1

. . .

c

1	1	1	1	...
1	2	3		.
1	3		.	
1		.		

...

d

...	1	1	1	1
.		1	2	3
	.		3	1
		.		1 .

...

Most common is the form

n \ m	0	1	2	3	4	.
0	1					
1	1	1				
2	1	2	1			
3	1	3	3	1		
4	1	4	6	4	1 .	
.	.	.	.	.	.	

Results on the properties of the Pascal triangle, including some questions of divisibility, may be found in Uspensky [50], in the literature on combinatorial analysis and number theory [4, 17, 21, 40, 41, 44, 45, 52, 54, 141, 255, 300, 316], and in mathematical reference books. The most complete description of the numerous elementary properties of the Pascal triangle is that of Green and Hamberg [162], with its many tables, figures, and diagrams, and interesting problems for independent study. Included, for example, is a table of prime factors of the binomial coefficients up through the 54^{th} row of the triangle.

We should also mention some results connected with direct applications of the Pascal triangle. T.M. Green [161] considered recurrent sequences connected with the triangle in the following way. Let the vertex of the triangle coincide with the origin of the usual coordinate system, and its elements with the lattice points of the first quadrant. This establishes a relation between the lattice points (x,y) and the elements of the Pascal triangle:

$$\binom{n}{r} = \frac{(x+y)!}{x!\ y!},$$

where n=x+y, r=y. Then, any set of parallel diagonals of the triangle having rational slopes gives rise to a recurrent sequence, the elements of which are sums of the triangle elements lying on the corresponding diagonals. That is, comparison of successive diagonals with the straight lines ax+by=n, where k=-a/b is the given slope, n=0,1,2,..., and a,b=1,2,..., leads to the sequence T_0, T_1, T_2, ... satisfying the relation

$$T_n = T_{n-a} + T_{n-b},$$

where T_n is the sum of the numbers on the n^{th} diagonal. The case a=2, b=1 gives the Fibonacci sequence.

In a series of works, the Pascal triangle has also been directly employed in problems involving the expansion of functions. Thus, M. Bicknell [71], using the column elements of the triangle, found an expansion for an exponential generating function; the result is used to construct the series expansion for some specific functions.

D.C. Duncan [126] showed that the n^{th} diagonal of the isosceles Pascal triangle gives the coefficients in the McLaurin series expansion of $(1-x)^{-n}$ for all positive n and $|x| < 1$. This expansion was also obtained in the work of A.R. Pargeter [302]. We also note that this interesting expansion allows us to find with any degree of precision the value of $(1+x)^{-n}$ for $x < 1$ and n a positive integer.

Power series with coefficients situated in the vertical columns of the isosceles Pascal triangle are considered in the work of A.A. Fletcher [142]. The general expression of these expansions has the form

$$S_r = 1 + rx + \binom{r+2}{2} x^2 + \binom{r+4}{3} x^3 + \cdots + \binom{2n+r-2}{n} x^n + \cdots.$$

It can be shown that S_r satisfies the recurrence relation

$$S_{r+2} = \frac{1}{x} (S_{r+1} - S_r), \ r \geq 2,$$

where S_2 is the series corresponding to the central vertical column with elements $\binom{2n}{n}$, $n=1,2,\ldots$, which are the coefficients in the expansion of $(1-4x)^{-1/2}$ for $x < 1/4$.

L.K. Jones [230] estimated the magnitude of the sums of the reciprocals of the elements of the Pascal triangle. For the n^{th} row, if we write

$$a_n = \sum_{k=0}^{n} \binom{n}{k}^{-1},$$

he established an upper estimate of the form $2+O(n)$, and a lower estimate of 2; $2+O(n^{-1})$?

consequently, $\lim_{n\to\infty} a_n = 2$. He also proved that for the k^{th} diagonal,

$$\sum_{n=k}^{\infty} \binom{n}{k}^{-1} = \frac{k}{k-1}.$$

In the work of A.R. Turguette [380, 381] the Pascal triangle is used in the study of Post sets and the solution of problems of many-valued logic.

Others have employed the Pascal triangle in the solution of various problems. Thus, D.A. Holton [214] showed that the dimensions of stable orbits are the coefficients in the polynomial $[1+(r-1)x]^n$, where in the case of the n-dimensional cube $r=2$, and the orbit dimensions are found in the Pascal triangle. In the work of H. Gorenflo [153], it is used to obtain the lifting force of pulley blocks. R.L. Morton [288] suggested a simple method of obtaining certain powers of 11 with the aid of the rows of the triangle. J. Wlodarski [395] showed that certain multiples of the elements of the triangle are related to two well-known numerical sequences in nuclear physics. G. Hoyer [226] suggested ways of deriving various formulas and relations among the binomial coefficients directly from the Pascal triangle. C.W. Trigg [378] considered properties of the sequence of elements of the fifth column of the triangle, as for example the length of the period of the sequence of low order digits, the sums of the digits, and so on.

In references [63, 76, 97, 110, 170, 193, 229, 242, 263, 294, 329] are discussions of elementary properties of the Pascal triangle, alternate versions of its development, and geometric interpretations.

The numbers of Fibonacci, Lucas, Catalan, Fermat, Stirling, and others may be derived and investigated by making use of the Pascal triangle directly [96, 112, 184, 192, 295, 298, 321, 327-329, 337, 394, 396].

1.2 BINOMIAL COEFFICIENTS AND THEIR GENERALIZATIONS

As we know, the elements of the Pascal triangle are the binomial coefficients, which were already known before the appearance of the Pascal triangle. However, Pascal was the first to define and to apply them [303]. Some references on the history of the binomial coefficients and the binomial theorem are [17, 36, 40, 41, 44, 50, 111, 122, 141, 241, 242, 268, 292].

The binomial coefficients are the simplest combinatorial objects, being defined as the number of distinct combinations of m elements out of n. They may be obtained from the generating function as the coefficients in the expansion of the expression

$$(1 + x)^n = \sum_{m=0}^{n} \binom{n}{m} x^m, \tag{1.1}$$

where

$$\binom{n}{m} = \frac{n!}{m!\,(n-m)!}, \; n = 0,1,2,\ldots,m \le n.$$

The binomial coefficients satisfy the recurrence relation

$$\binom{n+1}{m} = \binom{n}{m-1} + \binom{n}{m}, \; \binom{0}{0} = 1, \tag{1.2}$$

as well as the simple equalities

$$\binom{n}{0} = \binom{n}{n} = 1, \quad \binom{n}{m} = \binom{n}{n-m},$$
$$\sum_{m=0}^{n} \binom{n}{m} = 2^n, \quad \sum_{m=0}^{n} (-1)^m \binom{n}{m} = 0, \tag{1.3}$$

$$\binom{n+m}{l} = \sum_{k=0}^{l} \binom{n}{k}\binom{m}{l-k}, \binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2. \qquad (1.4)$$

Hundreds of identities and relations among the binomial coefficients have been established; many of these may be found in [4, 21, 29, 42, 46, 52, 158, 233, 248, 292]. The greatest numbers of identities are collected in the books of J. Riordan [42], B.N. Sachkov [46], H.W. Gould [158], and E. Netto [292]. In recent decades, new relations among the binomial coefficient have also been obtained, some of which we mention below.

M. Boscarol [88] obtained for nonnegative integers m and n the relation

$$\sum_{i=0}^{m} \frac{\binom{n+i}{i}}{2^{n+i}} + \sum_{h=0}^{n} \frac{\binom{m+n-h}{m}}{2^{m+n-h}} = 2.$$

H. Scheid [338] proved that the number of distinct prime factors of the binomial coefficient $\binom{n}{m}$ is not less than (m log 2)/(log 2m) for $2 \le 2m \le n$.

S.M. Tanny and M. Zuker [371] studied the sequence of binomial coefficients of the form $\binom{n-r}{r}$ for $n \ge 0$, $0 \le r \le [n/2]$, and pointed out its importance for many combinatorial problems.

G. Zirkel [405] discussed a method for numerically approximating the binomial coefficients with the help of a table of areas under the normal curve approximating the corresponding binomial distribution.

In [376], C.A. Tovey discussed the problem of the existence of infinite sets of natural numbers N, each element of which is equal to t distinct binomial coefficients $\binom{n}{m}$, where n=0,1,2,..., and $1<m<[n/2]$. He showed that for t=2, the least such value is the number 120, which equals $\binom{10}{3}$ and $\binom{16}{2}$; the number 210, which equals $\binom{10}{4}$ and $\binom{21}{2}$, would also be in this set. The number 3003, for example, has three representations: $\binom{14}{6}$, $\binom{15}{5}$, $\binom{78}{2}$. For t=2, this problem is solved, i.e., it is known that there are infinitely many natural numbers N which have two representations as binomial coefficients.

G.H. Weiss and M. Dishon [391] proved that in the expansion

$$\frac{1}{2}\left[1 - u - v - \sqrt{1-2(u+v)+(u-v)^2}\right] = \sum_{r=1}^{\infty}\sum_{s=1}^{\infty} C_{r,s}\, u^r v^s$$

the values of the $C_{r,s}$ may be expressed in terms of binomial coefficients:

$$C_{r,s} = (r+s-1)^{-1}\binom{r+s-1}{r}\binom{r+s-1}{s}.$$

Various other new properties appear in references [49, 56, 91, 118, 128, 163, 183, 358, 393]. The binomial coefficients and their various identities and relations play a major role in the solution of many problems in mathematics, mechanics, and physics. They also serve as a model for various generalized binomial coefficients. Two of these, the generalized binomial coefficients of s^{th} order, $\binom{n}{m}_s$, and the multinomial coefficients, $(n; n_1, n_2, \ldots, n_s)$, will be discussed in detail in sections 1.3 and 1.4 of the present chapter; a few other generalizations we mention below.

S.W. Golomb [151] introduced the so-called "iterated binomial coefficients" by the scheme

$$(a_1) = a_1;\ (a_1;\ a_2) = \binom{a_1}{a_2};\ (a_1;\ a_2;\ a_3) = \left(\binom{a_1}{a_2} \atop a_3\right),\ \ldots,$$

$$(a_1;\ a_2;\ \cdots;\ a_{k-1};\ a_k) = ((a_1;\ a_2;\ \cdots;\ a_{k-1});\ a_k).$$

For these iterated binomial coefficients, for specified values k and a_i, $i = 1,2,\ldots,k$, the author establishes various identities, inequalities, transformation formulas, and asymptotic and other formulas and relations.

M. Sved [366] introduced a different kind of generalized binomial coefficient as follows. Let $S = [a_1, a_2, \ldots, a_n]$ be a set of n distinct elements. The "sequence" $A = a_1^{(m_1)} a_2^{(m_2)} \cdots a_n^{(m_n)}$ is formed from the elements of S taken with multiplicities $(m_1,m_2,\ldots,m_n)$, and the degree of A is the number $|m| = m_1 + m_2 + \cdots + m_n$. If we take the "subsequence" $B = a_1^{(k_1)} a_2^{(k_2)} \cdots a_n^{(k_n)}$, where $0 \le k_i \le m_i$, to be a subsequence of A, then the generalized binomial coefficient $G_r^n(m)$ is the number of such subsequences B of A ($G_r^n(m)=0$ for $r<0$ and $r>n$). In elementary number theory the introduction of these coefficients has the following meaning. From the factorization of a natural number into its prime factors we form a sequence, starting with the set of distinct prime divisors, and the degree of the sequence is the sum of the divisors occurring in the factorization. Then $G_r^n(m)$ enumerates the set of all divisors of fixed degree; this generalizes a known property of the binomial coefficients to the coefficients $G_r^n(m)$.

The Gaussian binomial coefficients, also known as the q-binomial coefficients are defined [48] by:

$$\begin{bmatrix} n \\ m \end{bmatrix}_q = \prod_{k=1}^{m} \frac{q^{n-k+1}-1}{q^k-1}, \; 0<m\leq n \tag{1.5}$$

$$\begin{bmatrix} n \\ 0 \end{bmatrix}_q = 1, \; \begin{bmatrix} n \\ m \end{bmatrix}_q = 0, \; m<0, \; m>n, \tag{1.6}$$

where m,n are nonnegative integers and q is a real number. We know that the q-binomial coefficients occur in the expansion

$$\prod_{m=1}^{n} (1+q^{m-1}x) = \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix}_q q^{\frac{1}{2}m(m-1)} x^m, \tag{1.7}$$

from which it follows that the q-binomial coefficient is itself a polynomial in q, which for $q \to 1$ reduces to the ordinary binomial coefficient. These coefficients satisfy the recurrence

$$\begin{bmatrix} n+1 \\ m \end{bmatrix}_q = \begin{bmatrix} n \\ m \end{bmatrix}_q + \begin{bmatrix} n \\ m-1 \end{bmatrix}_q q^{n-m+1}, \; \begin{bmatrix} n \\ 0 \end{bmatrix}_q = 1. \tag{1.8}$$

In [315] G. Polya and G.L. Alexanderson discuss various combinatorial interpretations and properties of the q-binomial coefficients, and construct their multinomial analogs.

M. Sved in [367] discusses known and new properties of the q-binomial coefficients, including their geometric significance, and gives for q=2,3,4,5 the triangular tables of these coefficients analogous to the Pascal triangle. Equations (1.5)-(1.8) summarize the basic relations for the q-binomial coefficients; these and others are to be compared with the corresponding formulas for ordinary binomial coefficients.

L. Carlitz [101] generalized various theorems for the q-binomial coefficients to the multinomial case. R.D. Fray [143] and F.T. Howard [224] studied the question of the divisibility of the q-binomials by prime divisors; we will take up divisibility questions at length in the next chapter.

Another generalization of the binomial coefficients is given by the so-called Fibonomial coefficients [57],

$$\left(\binom{n}{m}\right)_F = \frac{F_n F_{n-1} \cdots F_{n-m+1}}{F_m F_{m-1} \cdots F_1}, \tag{1.9}$$

where the F_n are the Fibonacci numbers [20], n and m are nonnegative integers, and

$$\left(\binom{n}{0}\right)_F = \left(\binom{n}{n}\right)_F = 1 \quad \text{for all } n=0,1,2,\ldots.$$

In [57] G.L. Alexanderson and L.F. Klosinski also introduce the Gaussian Fibonomial coefficients

$$\left[\begin{bmatrix} n \\ k \end{bmatrix}\right] = \frac{(x^{F_n}-1)(x^{F_{n-1}}-1)\ \cdots\ (x^{F_{n-k+1}}-1)}{(x^{F_k}-1)(x^{F_{k-1}}-1)\ \cdots\ (x^{F_1}-1)}, \tag{1.10}$$

where n,k are nonnegative integers, and

$$\left[\begin{bmatrix} n \\ 0 \end{bmatrix}\right]_F = \left[\begin{bmatrix} n \\ n \end{bmatrix}\right]_F = 1,\ n=0,1,2,\ldots.$$

These Gaussian Fibonomials satisfy a recurrence relation which for $x \to 1$ includes that of the Fibonomial coefficients, and similarly for other relations. They also examine the case of a more general Fibonacci sequence

$$g_{n+2} = p g_{n+1} + q g_n,\ n \geq 0, \tag{1.11}$$

where $g_0=0$, $g_1=1$, and p and q are arbitrary.

Other analogs and generalizations of the binomial coefficients will be discussed in 1.4 and 1.5, along with the corresponding analogs of the Pascal triangle.

1.3 GENERALIZED PASCAL TRIANGLES AND GENERALIZED BINOMIAL COEFFICIENTS

The generalized Pascal triangle of s^{th} order is the table of coefficients of powers of x in the expansion

$$(1+x+x^2+\dots+x^{s-1})^n = \sum_{m=0}^{(s-1)n} \binom{n}{m}_s x^m \quad , s \geq 2. \tag{1.12}$$

The coefficients $\binom{n}{m}_s$ are known as the generalized binomial coefficients of order s. For s=2, they become the ordinary binomial coefficients, $\binom{n}{m}_2 = \binom{n}{m}$, and the corresponding triangular table is <u>the</u> Pascal triangle. (We note that some authors speak of triangles of "kind" s rather than triangle of "order" s.) In the literature, the generalized Pascal triangle is sometimes referred to as the s-arithmetic triangle.

The generalized Pascal triangle of order s may be written, as is the Pascal triangle, in the form of a right triangle or an isosceles triangle. For example, we give the generalized Pascal triangles of order 3 and 4 in right triangle form:

n \ m	0	1	2	3	4	5	6	7	8	.
0	1									
1	1	1	1							
2	1	2	3	2	1					
3	1	3	6	7	6	3	1			
4	1	4	10	16	19	16	10	4	1	
.	.	.	.	.	.	.	.	.	.	.

n \ m	0	1	2	3	4	5	6	7	8	9	10	11	12	.
0	1													
1	1	1	1	1										
2	1	2	3	4	3	2	1							
3	1	3	6	10	12	12	10	6	3	1				
4	1	4	10	20	31	40	44	40	31	20	10	4	1	.
.	.	.	.	.	.	.	.	.	.	.	.	.	.	.

Figure 3a

In isosceles form, these are:

								1								
							1	1	1							
						1	2	3	2	1						
					1	3	6	7	6	3	1					
				1	4	10	16	19	16	10	4	1				
			.	.	.	.	.	.	.	.	.	.	.			

							1							
					1	1		1	1					
				1	2	3	4	3	2	1				
		1	3	6	10	12		12	10	6	3	1		
1	4	10	20	31	40	44		40	31	20	10	4	1	
.	.	.	.	.	.	.	.	.	.	.	.	.	.	.

Figure 3b

In the first triangle (s=3) of Fig. 3a every element is equal to the sum of three elements in the preceding row: the number just above and its two neighbors to the left. In the zero-th column, all elements are ones, and we assume any missing elements to the left are zeros. Similarly, in the second triangle (s=4) each element is the sum of four elements in the preceding row: the number just above and its three neighbors to the left. In like fashion we fill in the rows of the generalized Pascal triangle of any order.

Dozens of papers have been devoted to the properties and applications of the generalized Pascal triangle and generalized binomial coefficients of order s. We will give some of these references after we list some of the basic properties of the generalized binomial coefficients of order s.

The generalized binomial coefficient $\binom{n}{m}_s$ is the number of different ways of distributing m objects among n cells where each cell may contain at most s-1 objects.

We also note the recurrence relation for the generalized binomial coefficients:

$$\binom{n+1}{m}_s = \sum_{k=0}^{s-1} \binom{n}{m-k}_s, \binom{n}{0}_s = 1. \tag{1.13}$$

For s=2, this coincides with the recurrence relation (1.2) for the ordinary binomial coefficients. The generalized binomial coefficients satisfy many equalities, identities, and other relations analogous to those for the binomial coefficients. For example,

$$\left.\begin{aligned} &\binom{n}{0}_s = \binom{n}{n}_s = 1, \quad \binom{n}{m}_s = \binom{n}{(s-1)n-m}_s, \\ &\sum_{m=0}^{(s-1)n} \binom{n}{m}_s = s^n, \quad \sum_{m=0}^{(s-1)n} (-1)^m \binom{n}{m}_s = \begin{cases} 0, s=2t \\ 1, s=2t+1. \end{cases} \end{aligned}\right\} \tag{1.14}$$

The relation among the generalized binomial coefficients in successive triangles has the form:

$$\left.\begin{aligned} &\binom{n}{m}_{s+1} = \sum_{k=0}^{n} \binom{n}{k} \binom{k}{m-k}_s, \text{ where } s \geq 2, \\ &\text{and} \quad \binom{k}{m-k}_s = 0 \text{ for } k < \frac{m}{s}. \end{aligned}\right. \tag{1.15}$$

The generalized binomial coefficient of order s may be expressed in terms of the binomial coefficients as:

$$\binom{n}{m}_s = \sum_{k=0}^{[m/s]} (-1)^k \binom{n}{k} \binom{n+m-sk-1}{n-1}. \tag{1.16}$$

We introduce for the multinomial coefficient the notation

$$(n;m_1,m_2,\dots,m_{s-1}) = \frac{n!}{(n-m_1)!\,(m_1-m_2)!\,\dots\,(m_{s-1}-m_{s-2})!\,m_{s-1}!}$$

in place of the usual $(n;\ n_1,\ n_2,\ \dots,\ n_s)$; more detail will appear in section 1.6. Then it is true that

$$\binom{n}{m}_s = \sum (n;m_1,m_2,\dots,m_{s-1}), \tag{1.17}$$

where $n \geq 0$, $0 \leq m \leq (s-1)n$, $s \geq 3$, and the summation is over all m_k such that $m_1+m_2+\dots+m_{s-1}=m$, $m_k \leq m_{k-1}$.

Let $C_{n,s} = \sup_m \binom{n}{m}_s$. Then for any n and $s \geq 2$, the correct asymptotic formula is

$$\lim_{n\to\infty} C_{n,s} \frac{\sqrt{n}}{s^n} = \sqrt{\frac{6}{\pi(s^2-1)}}. \tag{1.18}$$

The derivation of (1.13)-(1.18) is fairly straightforward and is omitted here.

In the Pascal triangle of order s, denote by $N_{n,s}$ the number of generalized binomial coefficients in the row numbered n, and by $Q_{n,s}$ the total number of coefficients in the triangle up to and including row n; then

$$N_{n,s} = n(s-1)+1, \quad Q_{n,s} = \frac{1}{2}(n+1)[(s-1)n+2]. \tag{1.19}$$

For s=2, $N_{n,2}=n+1$, and $Q_{n,2} = \frac{1}{2}(n+1)(n+2)$.

The generalized binomial coefficients have other interesting properties, as well; in succeeding chapters we will consider their divisibility properties, and the construction of their fractals and graphs. The applications of these triangles and coefficients in various mathematical contexts originated in the 1950's, and below we list in chronological order some works which were fundamental in this period and up to the present.

In considering these works, we must emphasize the original articles of J.E. Freund [144], and J.E. Freund and A.N. Pozner [145], in which they construct the generalized Pascal triangle, set forth the recurrence (and other) relations for the generalized binomial coefficients (which they denote by $N_m(r,k)$), and apply the results to some occupancy problems. J.D. Bankier [64] also used the results of [144] to find the coefficients in the expansion of $(x^2-x)(1+x+x^2)^k$.

V.E. Hoggatt and M. Bicknell [200,203] obtained difference relations and derived formulas for the sums of the elements in the generalized Pascal triangle which lie on the diagonals. A.K. Gupta in [164] explicitly expressed generalized binomial coefficients of arbitrary order by means of binomial coefficients. J.M. Deshouillers [117] derived asymptotic formulas for the generalized binomial coefficients, with integral estimates of their increase with increasing n.

V.E. Hoggatt and G.L. Alexanderson [197] worked out a method for determining partial sums of generalized binomial coefficients:

$$S(n,s,q,r) = \sum_{i=0}^{N} \binom{n}{r+iq}_s, \quad N=\left[\frac{(s-1)n-r}{q}\right].$$

In the special cases s=2, q=3,4,5,8; s=3, q=5, the expressions for the sums take the form of simple formulas involving the Lucas numbers, or the Pell-Lucas numbers, or their powers. These partial sums are also considered, for s=2,3,4,6, in C. Smith and V.E. Hoggatt [354-356].

T.B. Kirkpatrick [239] took the ascending diagonals of the generalized Pascal triangle to be the lines of a new triangle; iterating this operation R times, he obtains the additive triangle of order s and "degree" R. He then shows that if the diagonal sums of the elements of this triangle form the sequence $\{T_i\}_1^\infty$, this sequence has the recurrence relation

$$T_{N+(k-1)R+1} = T_N + T_{N+R} + T_{N+2R} + \cdots + T_{N+(k-1)R},$$

where $k \geq 2$, $R \geq 1$, and $T_1 = T_2 = \cdots = T_{R+1} = 1$.

In [78-80], R.C. Bollinger considers a number of properties of generalized Pascal triangles (there called Pascal-T triangles) and their coefficients. In [78] he constructs (modified) Fibonacci sequences of order k and uses them to solve various enumeration problems, which he calls "k-in-a-row" problems. In [79] the connection between the generalized binomial coefficients and the multinomials is found to have the form

$$C_m(n,k) = \sum \binom{n}{n_1, n_2, \cdots, n_m},$$

where the sum is taken over all $n_1, n_2, \ldots, n_m$ satisfying $n_1 + n_2 + \cdots + n_m = m$ and $0n_1 + 1n_2 + \cdots + (m-1)n_m = k$. Also given is the recurrence relation

$$C_m(n,k) = \sum_{j=0}^{n} \binom{n}{j} C_{m-1}(j,k-j). \tag{1.20}$$

In [80] Bollinger generalized the Mann-Shanks primality condition [274] for a natural number m to be a prime. Mann and Shanks gave this novel criterion in terms of the displaced entries of the Pascal triangle as follows. Each row of the Pascal triangle is displaced successively two places to the right, so that the n+1 entries in row n occupy columns m=2n to m=3n. Also, we underline the entries in row n which are divisible by n. Then the criterion is: column number m is a prime if and only if all the entries in column m are underlined. The table below shows how this works for the Pascal triangle (the triangle of order 2).

n \ m	0	1	2*	3*	4	5*	6	7*	8	9	10	11*	12	13*	14	15	16	17*	.
0	1																		
1			$\underline{1}$	$\underline{1}$															
2					1	$\underline{2}$	1												
3							1	$\underline{3}$	$\underline{3}$	1									
4									1	$\underline{4}$	6	$\underline{4}$	1						
5											1	$\underline{5}$	$\underline{10}$	$\underline{10}$	$\underline{5}$	1			
6													1	$\underline{6}$	15	20	15	$\underline{6}$	.
7															1	$\underline{7}$	$\underline{21}$	$\underline{35}$	.
8																	1	$\underline{8}$	.
.																			.

Using (1.20) and a theorem of G. Ricci [322], Bollinger in [80] shows that if the same displacements are applied to the generalized Pascal triangle of order three (so that the $2n+1$ elements in row n occupy columns $m=2n$ to $m=4n$), and the entries are underlined in the same way, then it is again true that the column number m is a prime if and only if all the entries in column m are underlined. The table below shows how this works for the triangle of order three. He also conjectured that the criterion is true for the generalized Pascal triangle of any order.

n \ m	0	1	2*	3*	4	5*	6	7*	8	9	10	11*	12	13*	14	15	16	17*	·
0	1																		
1			$\underline{1}$	$\underline{1}$	$\underline{1}$														
2					1	$\underline{2}$	3	$\underline{2}$	1										
3							1	$\underline{3}$	$\underline{6}$	7	$\underline{6}$	$\underline{3}$	1						
4									1	$\underline{4}$	10	$\underline{16}$	19	$\underline{16}$	10	$\underline{4}$	1		
5											1	$\underline{5}$	$\underline{15}$	$\underline{30}$	$\underline{45}$	51	$\underline{45}$	$\underline{30}$	·
6													1	$\underline{6}$	21	50	$\underline{90}$	$\underline{126}$	·
7															1	$\underline{7}$	28	$\underline{77}$	·
8																	1	$\underline{8}$	·
·																			·

R.C. Bollinger and C.L. Burchard in [81] showed there is, for the generalized binomial coefficients, an analog of Lucas's Theorem for the binomial coefficients, namely,

$$C_m(n,k) \equiv \sum_{(s_0,\dots,s_r)} \prod_{i=0}^{r} C_m(n_i,s_i) \ (mod\ p),$$

where p is a prime, $n = (n_r n_{r-1} \cdots n_1 n_0)_p$, $k = (k_t k_{t-1} \cdots k_1 k_0)_p$, $0 \le n_i < p$, $0 \le k_i < p$, $0 \le k \le (m-1)n$, and the summation is over all s_i for which $s_0 + s_1 p + \cdots + s_r p^r = k$, $0 \le s_i \le (m-1)n$. If we denote by $N_m(n,p)$ the number of generalized binomial coefficients for which $C_m(n,k) \not\equiv 0$ (mod p) and apply the extended Lucas's Theorem, the authors found exact formulas for $N_m(n,p)$ in the cases m=p and $m=p^\ell$. Let $(p-1)n = (a_r a_{r-1} \cdots a_1 a_0)_p$; then

$$N_m(n,p) = (1+a_0)(1+a_1)\cdots(1+a_r),$$

$$N_m(n,p^\ell) = N_p(n(p^\ell-1)/(p-1),p).$$

They also established, for the generalized Pascal triangle of order p, that for large n "almost all" coefficients $C_p(n,k)$ are divisible by p.

Other questions connected with the application of the generalized binomial coefficients and generalized Pascal triangle of order s are discussed in [119, 154, 164, 212, 231, 232, 243, 287, 308, 314, 357].

1.4 LUCAS, FIBONACCI, CATALAN, AND OTHER ARITHMETIC TRIANGLES

In sections 1.1 and 1.3 we discussed Pascal triangles and generalized Pascal triangles of order s. We now turn our attention to the construction and application of other forms of arithmetic triangles: the triangles associated with the names of Lucas, Fibonacci, Catalan, Stirling, and others.

M. Feinberg [138] constructed the arithmetic triangle whose elements are the coefficients in the expansion of $(a+2b)(a+b)^{n-1}$; the result is what might be called the Lucas triangle, in which the sums of the elements on the ascending diagonals give the sequence of Lucas numbers 1,3,4,7,11,18,29,....

The Lucas triangle and its properties were studied in detail by H.W. Gould and W.E. Greig [160]. In this triangle (nine rows of which are shown below), the elements satisfy

n \ k	0	1	2	3	4	5	6	7	8	9	.
1	1	2									
2	1	3	2								
3	1	4	5	2							
4	1	5	9	7	2						
5	1	6	14	16	9	2					
6	1	7	20	30	25	11	2				
7	1	8	27	50	55	36	13	2			
8	1	9	35	77	105	91	49	15	2		
9	1	10	44	112	182	196	140	64	17	2	
.	.	.	.	.	.	.	.	.	.	.	.

see pg 39

the recurrence relation

$$A(n+1,k) = A(n,k) + A(n,k-1), \tag{1.21}$$

with initial conditions A(1,0) = 1, A(1,1) = 2, and A(n,k) = 0 for k<0 or k>n. The relation between the numbers A(n,k) and the binomial coefficients is

$$A(n,k) = \binom{n}{k} + \binom{n-1}{k-1}. \tag{1.22}$$

There are also four criteria given, the proofs being based on the properties of the Lucas triangle and its elements, for deciding whether a given natural number $d \geq 2$ is a prime.

V.E. Hoggatt [194] constructed a new triangle from the Lucas triangle by shifting the i^{th} column down k places (k=1,2,3,...), and derived various results, including the Lucas numbers, for the elements of this triangle.

H. Hosoya [216] constructed the arithmetic triangle (Figure 4) for the numbers $\{f_{m,n}\}$ satisfying the equations

$$\left.\begin{aligned} f_{m,n} &= f_{m-1,n} + f_{m-2,n} \\ f_{m,n} &= f_{m-1,n-1} + f_{m-2,n-2}\ , \quad m \geq 2,\ m \geq n \geq 0 \end{aligned}\right\}, \tag{1.23}$$

with initial conditions $f_{0,0} = f_{1,0} = f_{1,1} = f_{2,1} = 1$. He showed that $f_{m,n} = f_n f_{m-n}$ $(m \geq n \geq 0)$, where f_n is the n^{th} Fibonacci number, and called the resulting triangle a Fibonacci triangle. He studied the topological properties of its graph, obtained using the triangle, and applied the results to the classification of chemical formulas.

1

1 1

2 1 2

3 2 2 3

5 3 4 3 5

8 5 6 6 5 8

13 8 10 9 10 8 13

21 13 16 15 15 16 13 21

. . . .

Figure 4

J. Turner [382] suggested and studied what he called the Fibonacci-T triangle.

J. Šána in [335] considered a sequence $\{g_{m,n}\}$ like that of Hosoya [216],

$$\left.\begin{aligned} g_{m,n} &= g_{m-1,n} + g_{m-2,n} \\ g_{m,n} &= g_{m-1,n-1} + g_{m-2,n-2}, \quad m \geq 2,\ m \geq n \geq 0 \end{aligned}\right\}, \tag{1.24}$$

with initial conditions $g_{0,0}=2$, $g_{1,0}=1$, $g_{1,1}=1$, $g_{2,1}=2$, and constructed the arithmetic triangle in Figure 5, which he called a Lucas triangle. It has properties analogous to those obtained in [216]; some of these are investigated, and also the graph equivalent to the Lucas triangle is constructed.

							2							
						1		1						
					3		2		3					
				4		3		3		4				
			7		5		6		5		7			
		11		8		9		9		8		11		
	18		13		15		14		15		13		18	
29		21		24		23		23		24		21		29
.	.	.	.	.	.	.	.	.						

Figure 5

The elements of these Fibonacci and Lucas triangles have a recurrence relation of the form (1.23) or (1.24), in which each element is the sum of two preceding elements on an ascending or descending diagonal. Other relevant references here are [11, 138, 207].

M. Sved [367] also discussed the arithmetic triangle whose elements are the Gaussian binomial coefficients $\begin{bmatrix} n \\ r \end{bmatrix}_q$, and obtained the Gaussian triangles for $q = 2,3,4,5$.

L.W. Shapiro [342] constructed the arithmetic triangle whose elements are the numbers $B_{n,k}$ satisfying the recurrence relation

$$B_{n,k} = B_{n-1,k-1} + 2B_{n-1,k} + B_{n-1,k+1}$$

with the conditions $B_{1,1}=1$, $B_{n,0}=0$, $B_{n,m}=0$, $m>n+1$. The first several rows are shown below.

n \ k	1	2	3	4	5	6	.
1	1						
2	2	1					
3	5	4	1				
4	14	14	6	1			
5	42	48	27	8	1		
6	132	165	110	44	10	1	
.	.	.	.	.	.	.	.

The sequence of numbers $\{C_n\} = \{1,2,5,14,42,132,...\}$ in the first column are the Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$. It is not difficult to show that the solution of the recurrence relation is $B_{n,k} = \frac{k}{n}\binom{2n}{n-k}$, and for k=1,

$$B_{n,1} = C_n = \frac{1}{n}\binom{2n}{n-1} = \frac{1}{n+1}\binom{2n}{n}. \tag{1.25}$$

The article also shows that the $B_{n,k}$ may be expressed as a sum of products of Catalan numbers by means of the formula $B_{n,k} = \sum C_{i_1} C_{i_2} \cdots C_{i_k}$, where the summation is over values for which $i_1+i_2+\cdots+i_k=n$. As a result, each element of the Catalan triangle may be expressed in terms of the Catalan numbers; the name arises because of this connection. Various properties of the Catalan triangle, analogous to those of the Pascal triangle, are also discussed.

D.G. Rogers [328] studied questions connected with renewal sequences which led to various generalized Pascal and Catalan triangles. These are connected with the introduction of the generalized Catalan sequence $\{C_t(n)\}$, where

$$C_t(n) = \frac{1}{tn+1}\binom{(t+1)n}{n}, \; n \geq 0, \; t \geq 0. \tag{1.26}$$

For $t=1$, we have $C_1(n) = C_n$, the Catalan numbers. The introduced sequence and the related generalized Catalan triangle are applied in the solution of some combinatorial problems.

A number of authors have constructed arithmetic triangles by choosing as their elements the numbers which satisfy a recurrence relation of the form

$$f(n+1,m) = p(n,m)f(n,m-1) + q(n,m)f(n,m) \tag{1.27}$$

with appropriate coefficients p, q and initial conditions.

C. Cadogan [98] considered the case of this equation where $p, q \in R$ and with initial conditions $f(0,k) = d_k \in R$; he found then,

$$f(n,k) = \sum_{m=0}^{n} \binom{n}{m} p^{n-m} q^{m} f(0,n-m). \tag{1.28}$$

By choosing as the values of the d_k the cases: $d_0=1$, $d_k=0$ $(k\neq 0)$; $d_0=a$, $d_{-1}=d$, $d_k=0$ $(k\neq 0,-1)$; $d_k = a(m-1)^{-k}$, $k\leq 0$ $(d_k=0,\ k>0)$, the author constructs the corresponding Pascal triangle, a triangle with elements which form an arithmetic progression, and a triangle with elements which form a geometric progression. The results are also generalized to the three-dimensional case.

In [240] M. Klika considered (1.27) for integer-valued functions p(m), q(n) and initial conditions $f(0,0)=1$, $f(i,j)=0$ for $i<j$, $j<0$, where i,j are nonnegative whole numbers, and constructed the corresponding generalized Pascal triangle P(p,q). For $p=q=1$ we get the Pascal triangle itself, and for $p=m+1$, $q=1$ the triangle whose elements are the Stirling numbers of the second kind; the author also discusses the triangle P(p,q) for various other conditions. In [227], S.K. Janardan and K.G. Janardan also investigate this kind of Stirling triangle.

H. Ouellette and G. Bennett [301] considered the triangle whose elements are the absolute values of the Stirling numbers of the first kind.

In a dissertation [24] V.N. Dokina studied the special cases of (1.27) consisting of: $p=1$, $q=\mu_n$; $p=1$, $q=\mu_m$; $p=1$, $q=\mu_n+\mu_m$ and initial conditions equal to unity. He formed the corresponding triangles consisting of generalized Stirling numbers of the first and second kind, and Lah numbers. He also extended the discussion to the case when p(n,m) and q(n,m) are not merely numerical, but are operators operating on a linear space of polynomials in t with real coefficients. In these cases the elements of the generalized Pascal triangle are functions of t. The results are applied to various probability problems, problems connected with population growth, and others.

V.L. Jannelli [228] constructed and studied the triangle formed from the coefficients in the expansion of $(x+a_1)(x+a_2)\cdots(x+a_n)$. For $a_1=0$, $a_2=1$, ..., $a_n=-(n-1)$ the author arrives at the triangle of Stirling numbers of the first kind; other cases, when $a_k=k$, are discussed in [120, 133].

In [333], M. Rumney and E.J. Primrose studied the triangle whose rows are the coefficients in the expansions of 1, 1+x, (1+x)(2+x), (1+x)(2+x)(3+x),...; a portion of this triangle is:

n \ m	0	1	2	3	4	5	.
0	1						.
1	1	1					
2	2	3	1				
3	6	11	6	1			
4	25	50	35	10	1		
5	120	274	225	85	15	1	
.	.	.	.	.	.	.	.

The elements, denoted by $e_{n,m}$, satisfy the recurrence

$$e_{n+1,m} = e_{n,m-1} + (n+1)e_{n,m}, \tag{1.29}$$

which gives a simple rule for forming the triangle. It is also not difficult to show that

$$\sum_{m=0}^{n} e_{n,m} = (n+1)!,$$

and other relations are given. The authors also study in great generality the triangle composed of the numbers in the harmonic series.

C.W. Puritz [318] generalized the binomial coefficient $\binom{n}{m}$ to the case of n negative, using the notation C(n,m). He used the arithmetic and symmetry properties of the recurrence

$$C(n,m) = C(n+1,m) - C(n,m-1)$$

and found that

$$C(-n,m) = (-1)^m \binom{n+m-1}{m},$$

writing out a portion of the complementary Pascal triangle as below.

n \ m	0	1	2	3	4	.
.	.	.	.	.	.	.
-4	1	-4	10	-20	35	.
-3	1	-3	6	-10	15	.
-2	1	-2	3	-4	5	.
-1	1	-1	1	-1	1	.
0	1	0	0	0	0	.
1	1	1	0	0	0	.
2	1	2	1	0	0	.
3	1	3	3	1	0	.
4	1	4	6	4	1	.
.	.	.	.	.	.	.

Other variants of the Pascal triangle, in which the elements come from the coefficients in the expansion of

$$(a\mp b)(a\pm b)(a\mp b)\ \cdots\ (a\pm(-1)^n b),$$

were considered by P. Sahmel [334]. For n=2m and n=2m+1, we obtain the corresponding expansions of $(a^2-b^2)^n$ and $(a\mp b)(a^2-b^2)^m$.

a						
a	b					
a	a	b				
a	a	a+b	b			
a	a	2a+b	a+b	b		
a	a	3a+b	2a+b	a+2b	b	
a	a	4a+b	3a+b	3a+3b	a+2b	b
.	.	.	.	.	.	.

Figure 6

H.W. Gould [156] constructed and studied the Pascal triangle (Fig. 6) in which the elements are defined by the recurrence relation

$$C_m^{n+1} = C_{m-1}^n + \frac{1+(-1)^m}{2}\,C_m^n,\ n\geq 1,\ m\geq 0, \tag{1.30}$$

and the conditions

$$C_0^0 = C_0^1 = a,\ C_1^1 = b,\ C_m^n = 0 \text{ for } m>n,\ m<0.$$

The cases a=b=1 and a=1, b=2 are studied in detail. The coefficients are denoted by A_m^n in the first case and by B_m^n in the second, m=0,1,2,...,n. Using (1.30), the following values are calculated for n=0,1,2,....

$$A_{2k}^n = \binom{n-k}{k},\ A_{2k+1}^n = \binom{n-k-1}{k},\ A_0^n = 1,\ A_1^1 = 1,$$

$$B_{2k}^n = \frac{n}{n-k}\binom{n-k}{k},\ B_{2k+1}^n = \frac{n-1}{n-k-1}\binom{n-k-1}{k},\ B_0^n = 1,\ B_1^1 = 2.$$

The A_m^n and B_m^n can be used to express the Fibonacci and Lucas numbers as

$F_{n+2} = \sum_{m=0}^{n} A_m^n$ and $L_{n+1} = \sum_{m=0}^{n} B_m^n$, for $n \geq 0$. It should be noted that

$$\sum_{m=0}^{n} C_m^n = aF_{n+1} + bF_n,\ n\geq 0,$$

$$\sum_{m=0}^{n} (-1)^m C_m^n = aF_{n-2} + bF_{n-3},\ n\geq 1.$$

In [77] M.B. Boisen considers two tables A and B,

$$\begin{array}{rrrrl} & & & a_{44} & , \\ & & a_{33} & a_{34} & , \\ & a_{22} & a_{23} & a_{24} & , \\ a_{11} & a_{12} & a_{13} & a_{14} & , \end{array}$$

$$\begin{array}{llll} b_{11} & & & \\ b_{12} & b_{22} & & \\ b_{13} & b_{23} & b_{33} & \\ b_{14} & b_{24} & b_{34} & b_{44} \\ \cdot & \cdot & \cdot & \cdot\ \ \cdot \end{array}$$

where the a's and b's are integers, and defines the superposition of A on B, which then generates the sequence $C = \{c_1, c_2, \ldots\}$ with elements of general form

$$c_i = \sum_{k=0}^{\left[\frac{i}{2}\right]} \sum_{l=k+1}^{i-k} a_{k+1,l}\, b_{l,i-k}. \tag{1.31}$$

With the sequence $\{c_i\}$ defined he takes the following approach. Let

$$P_k(x) = a_{1k} + a_{2k}x + \cdots + a_{kk}x^{k-1}$$

and let $G_k(x)$ be the generating function of the k^{th} column of table B, $k=0,1,2,\ldots$. Then $\sum_{i=0}^{\infty} P_i(x)G_i(x)$ is the generating function of $\{c_i\}$. Several examples are considered in which A and B are chosen to be the Pascal triangle or its generalizations; in one of these, for example, the sequence $\{c_i\}$ turns out to be the Fibonacci sequence.

C.K. Wong and T.W. Maddocks [399] studied the numbers $M_{k,r}$ satisfying the recurrence relation

$$M_{k+1,r+1} = M_{k+1,r} + M_{k,r+1} + M_{k,r} \tag{1.32}$$

with initial conditions $M_{0,0} = M_{1,0} = M_{1,1} = 1$. The numbers $M_{k,r}$, for which the condition $M_{k,r} = M_{r,k}$ clearly holds, constitute an analog of the Pascal triangle (Fig. 7).

1

1 1

1 3 1

1 5 5 1

1 7 13 7 1

1 9 25 25 9 1

1 11 41 63 41 11 1

. . . .

Figure 7

In this triangle, k is the number of the line parallel to the right side of the triangle, k=0,1,2,..., and r is the number of the line parallel to the left side of the triangle, r=0,1,2,.... If we introduce the number of the line parallel to the base of the triangle and denote it by n=0,1,2,..., then the law of formation of the elements is simple: any value in the n^{th} row is the sum of the two elements above in the $(n-1)^{st}$ row and the element directly above in the $(n-2)^{nd}$ row. Thus, 41 is the sum 9+25+7. The author also shows that the sums of the elements on the ascending diagonals form the "Tribonacci" numbers, 1, 1, 2, 4, 7, 13, 24, 44, ...

M. Bicknell-Johnson in [73] writes on the Leibnitz harmonic triangle (Fig. 8), whose diagonals are the products of the reciprocals of the n^{th} row elements by the reciprocals of the row numbers (assumed to begin with one) in the Pascal triangle. The sums of the row elements, and of the ascending diagonal elements are found.

$$\begin{array}{ccccccc}
\frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \cdot \\
\frac{1}{2} & \frac{1}{6} & \frac{1}{12} & \frac{1}{20} & \frac{1}{30} & \cdot & \\
\frac{1}{3} & \frac{1}{12} & \frac{1}{30} & \frac{1}{60} & \cdot & & \\
\frac{1}{4} & \frac{1}{20} & \frac{1}{60} & \cdot & & & \\
\frac{1}{5} & \frac{1}{30} & \cdot & & & & \\
\frac{1}{6} & \cdot & & & & & \\
\cdot & & & & & &
\end{array}$$

Figure 8

D. Logothetti [259] formed a new, truncated triangle (Fig. 9) (without a vertex) by taking groups of four elements at the vertices of a rhombus in the Pascal triangle and forming the numbers

$$I(n,k) = \binom{n-2}{k-1} + \binom{n-2}{k} + \binom{n-1}{k-1} + \binom{n-1}{k},$$

where $n=1,2,3,\ldots$, $k=0,1,2,\ldots,n$.

2 1

2 3 1

2 5 4 1

2 7 9 5 1   see pg 25

2 9 16 14 6 1

2 11 25 30 20 7 1

2 13 36 55 50 27 8 1

. . . .

Figure 9

Although there is no symmetry, the triangle and its elements have interesting properties, as, e.g.,

$$I(n,k) = I(n-1,k-1) + I(n-1,k),$$

$$\sum_{k=0}^{n} I(n,k) = 3 \cdot 2^{n-1}, \sum_{k=0}^{n} (-1)^k I(n,k) = 0,$$

$$(2x+1)(x+1)^{n-1} = \sum_{k=0}^{n} I(n,k) x^{n-k}.$$

The truncated triangle of Fig. 9 may be considered as a special case of a more general triangle (Fig. 10).

$$
\begin{array}{ccccccccccc}
 & & & & a & & 1 & & & & \\
 & & & a & & a+1 & & 1 & & & \\
 & & a & & 2a+1 & & a+2 & & 1 & & \\
 & a & & 3a+1 & & 3a+3 & & a+3 & & 1 & \\
a & & 4a+1 & & 6a+4 & & 4a+6 & & a+4 & & 1 \\
\end{array}
$$

. . . .

Figure 10

Here, the elements G(n,k) are the coefficients of x^{n-k} in the expansion of $(ax+1)(x+1)^{n-1}$, and satisfy the recurrence G(n,k) = G(n-1,k-1) + G(n-1,k), G(n,0)=a, G(n,n)=1.

H. Harborth [173] considered triangles composed of plus and minus signs, to every pair of which is assigned a (+) or a (-) sign according to Pascal's rule. Such a triangle for a given n contains N = ½n(n+1) signs and we assign for that n the signs in the first row. His results solve the Steinhaus problem [53] on the existence of numbers n, where n ≡ 0,3 (mod 4), for which the generated triangle has plus signs as half of its elements. For example, for n=11 Figure 11 shows such a triangle, with 33 of its 66 elements being plus signs. Variants of this problem were also solved and studied by M. Bartsch [65].

$$\begin{array}{ccccccccccccccccccccc}
+ & & - & & + & & + & & - & & + & & - & & - & & + & & + & & + \\
& - & & - & & + & & - & & - & & - & & + & & - & & + & & + & \\
& & + & & - & & - & & + & & + & & - & & - & & - & & + & & \\
& & & - & & + & & - & & + & & - & & + & & + & & - & & & \\
& & & & - & & - & & - & & - & & - & & + & & - & & & & \\
& & & & & + & & + & & + & & + & & - & & - & & & & & \\
& & & & & & + & & + & & + & & - & & + & & & & & & \\
& & & & & & & + & & + & & - & & - & & & & & & & \\
& & & & & & & & + & & - & & + & & & & & & & & \\
& & & & & & & & & - & & - & & & & & & & & & \\
& & & & & & & & & & + & & & & & & & & & &
\end{array}$$

Figure 11

Arithmetic triangles of Stirling numbers of the first kind $S_n^{(m)}$, satisfying the recurrence $S_{n+1}^{(m)} = S_n^{(m-1)} - nS_n^{(m)}$, where $S_1^{(1)} = 1$, and $S_n^{(m)} = 0$ for $n<1$, $m<1$, $m<n$, have the form

$$\begin{array}{ccccccc}
1 & & & & & & \\
\underline{1} & 1 & & & & & \\
2 & \underline{3} & 1 & & & & \\
\underline{6} & 11 & \underline{6} & 1 & & & \\
24 & \underline{50} & 35 & \underline{10} & 1 & & \\
\underline{120} & 274 & \underline{225} & 85 & \underline{15} & 1 & \\
. & . & . & . & . & . & .
\end{array}$$

where the negative elements are underlined.

Triangles of Stirling numbers of the second kind $\sigma_n^{(m)}$, satisfying $\sigma_{n+1}^{(m)} = \sigma_n^{(m-1)} + m\sigma_n^{(m)}$, where $\sigma_1^{(1)} = 1$ and $\sigma_n^{(m)} = 0$ for $n<1$, $m<1$, $m<n$, have the form

1						
1	1					
1	3	1				
1	7	6	1			
1	15	25	10	1		
1	31	90	65	15	1	
1	63	301	350	140	21	1
.	.	.	.	.	.	.

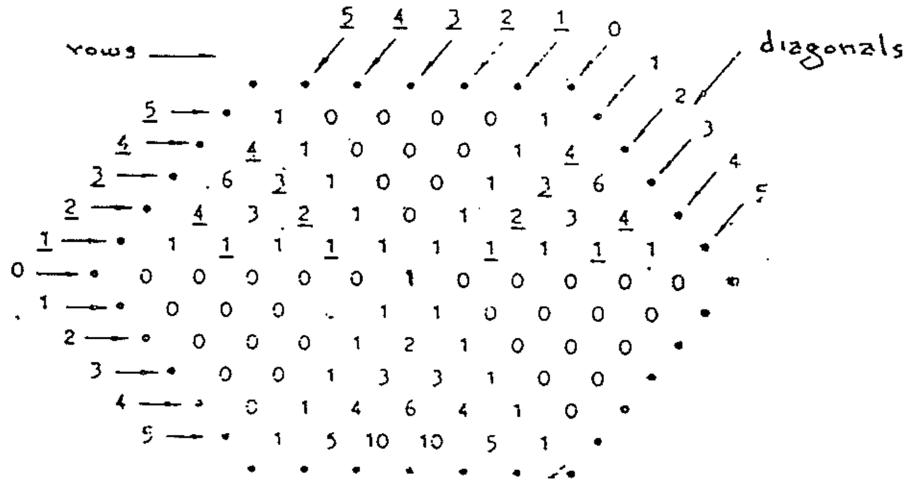
P. Hilton and J. Pederson [188-190] obtained new arithmetic and geometric properties of the binomial coefficients $\binom{n}{m}$, including the case of negative values of m and n, by extending the definition as follows:

$$\binom{n}{r} = 0 \text{ for } n \geq 0,\ r>n,\ r<0,$$

$$\binom{-n}{r} = (-1)^r \binom{n+r-1}{r} \text{ for } n>0,\ r\geq 0,$$

$$\binom{-n}{-r} = (-1)^{r-n} \binom{r-1}{n-1} \text{ for } n>0,\ r>0.$$

As a result of this generalization the authors construct the hexagon (Fig. 12) consisting of the binomial coefficients for both positive and negative values of m and n, and call it the Pascal hexagon.



They considered the geometric properties of the Pascal hexagon and other figures such as the arrangement in Figure 13.

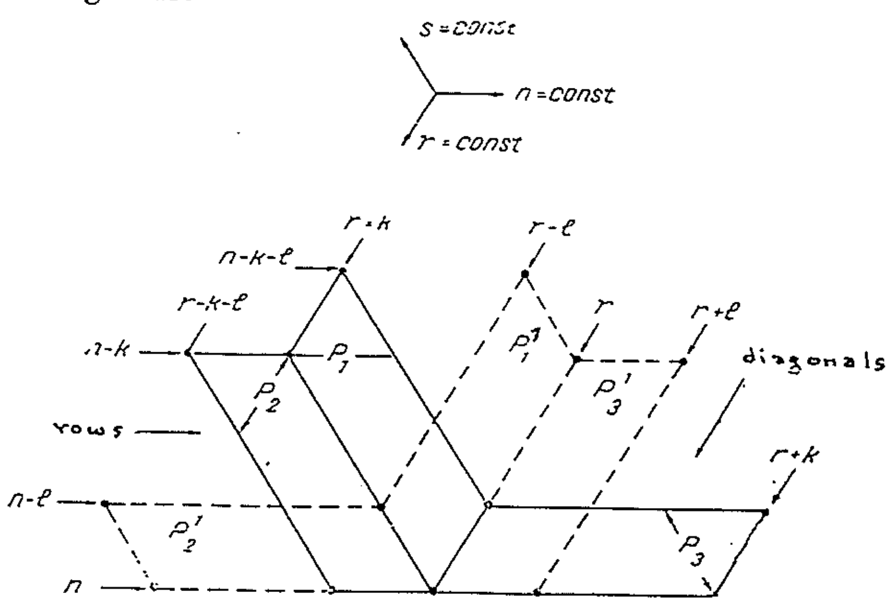


Figure 13

In [191] they also discuss the Leibnitz harmonic coefficients and the q-binomials for positive and negative values of m and n, as well as describing the properties of the Pascal hexagon and constructing the generalized star of David, the harmonic triangle, and the Pascal hexagon.

K. Dilcher [123] replaced the partial differential equation $u_{xx} = u_{tt} + u_t$ by a difference equation and, after appropriate normalization, constructed the triangle in Figure 14, which is a kind of generalized Pascal triangle of order three (discussed in 1.3). The elements $C_{n,m}$ of this triangle (cf. Fig. 14) in the n^{th} row are combinations of three elements in the $(n-1)^{st}$ row and one in the $(n-2)^{nd}$ row, according to the recurrence

$$C_{n,m} = C_{n-1,m-1} + C_{n-1,m} + C_{n-1,m+1} - 2C_{n-2,m},\ C_{0,0} = 1.$$

1

1 1 1

1 2 1 2 1

1 3 2 3 2 3 1

1 4 4 4 5 4 4 4 1

1 5 7 6 9 7 9 6 7 5 1

. . . .

Figure 14

These coefficients may also be generalized by introducing the parameters λ, ν, in which case they satisfy

$$C_{n,m}^{\lambda,\nu} = \left(1 + \frac{\nu-1}{n}\right)\left(C_{n-1,m-1}^{\lambda,\nu} + C_{n-1,m}^{\lambda,\nu} + C_{n-1,m+1}^{\lambda,\nu}\right)$$
$$- \left(1 + 2\,\frac{\nu-1}{n}\right)\lambda\, C_{n-2,m}^{\lambda,\nu} \,,$$

where $C_{n,m}^{\lambda,\nu} = C_{n,n-m}^{\lambda,\nu}$; for $\lambda=2$, $\nu=1$, $C_{n,m}^{2,1} = C_{n,m}$. The properties of the $C_{n,m}^{\lambda,\nu}$ and their arithmetic triangles are considered in detail.

Arithmetic triangles also appear in the references [69, 109, 120, 125, 281, 357, 385, 396].

1.5 PASCAL PYRAMIDS AND TRINOMIAL COEFFICIENTS

As we have seen, the binomial coefficients $\binom{n}{m}$ arise as a result of the expansion of $(1+x)^n$, and can be written in the form of a Pascal triangle of one sort or another. If we write the binomial in terms of x_0, x_1, the expansion takes the form

$$(x_0 + x_1)^n = \sum_{m=0}^{n} \binom{n}{m} x_0^{n-m}\, x_1^{m}.$$

If we denote the trinomial coefficients by $(n;\, m_1,\, m_2)$, where n, m_1, m_2 are nonnegative integers, and set

$$(n;\ m_1,m_2) = \frac{n!}{(n-m_1)!\ (m_1-m_2)!\ m_2!}\,, \tag{1.33}$$

we can write the expansion of $(x_0+x_1+x_2)^n$ in the form

$$(x_0+x_1+x_2)^n = \sum_{m_1=0}^{n} \sum_{m_2=0}^{n} (n;\ m_1,m_2)\ x_o^{n-m_1}\ x_1^{m_1-m_2}\ x_2^{m_2}\ . \tag{1.34}$$

The trinomial coefficients are often written as

$$(n;\ n_1,\ n_2,\ n_3) = \frac{n!}{n_1!\,n_2!\,n_3!},\ n_1+n_2+n_3 = n\ ; \tag{1.35}$$

however, in many contexts in which one constructs and uses multi-harmonic, multi-wave, and other polynomials, the representation (1.33) is more convenient than (1.35), since (1.33) orders the polynomial terms and trinomial coefficients of the Pascal pyramid and its cross sections (Figure 15).

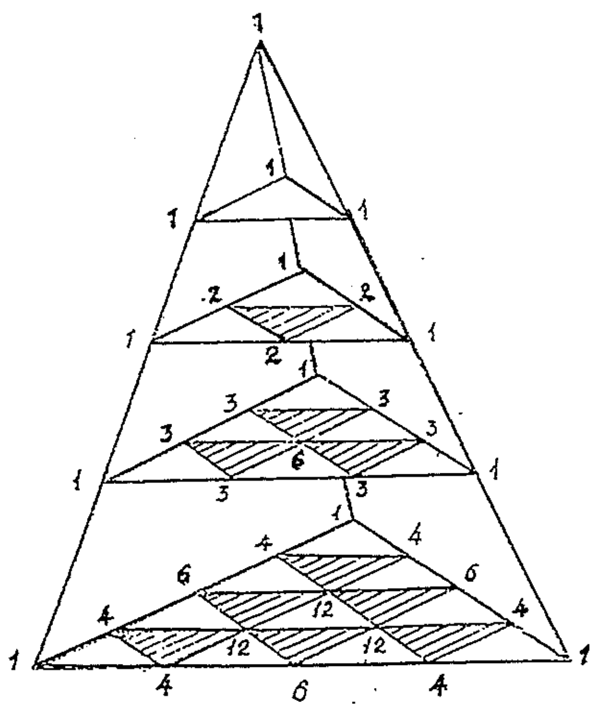


Figure 15

It is not difficult to show that the trinomial coefficients of (1.33) satisfy the recurrence relation

$$(n+1;\ m_1,m_2) = (n;\ m_1,m_2) + (n;\ m_1-1,\ m_2) + (n;\ m_1-1,\ m_2-1) \tag{1.36}$$

with initial conditions (0; 0,0) = 1, and where $(n;\ m_1,m_2) = 0$ for $n<0$, m_1 or $m_2<0$, $m_1>n$, $m_2>m_1$. We can also verify from Figure 15 the presence of three axes of symmetry.

Much like the binomial coefficients, the trinomial coefficients satisfy the conditions (n; 0,0) = (n; n,0) = (n; n,n) = 1, and the equations

$$\left.\begin{aligned}(n;\ m_1,m_2) &= (n;\ m_1,m_1-m_2),\\ (n;\ m_1,m_2) &= (n;\ n-m_1+m_2,m_2),\\ (n;\ m_1,m_2) &= (n;\ n-m_2,\ n-m_1).\end{aligned}\right\} \tag{1.37}$$

Some special sums are

$$\sum_{m_1=0}^{n}\sum_{m_2=0}^{m_1}(n;\ m_1,m_2) = 3^n,\ \sum_{m_1=0}^{n}\sum_{m_2=0}^{m_1}(-1)^{m_2}(n;\ m_1,m_2) = 1 \tag{1.38}$$

and the analog of the Cauchy summation formula is

$$\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{k_1}(n_1;\ k_1,k_2)(n_2;\ m_1-k_1,m_2-k_2) = (n_1+n_2;\ m_1,m_2). \tag{1.39}$$

The Pascal pyramid can be considered as a regular tetrahedron, or as a pyramid with unequal dihedral angles as shown. In the n^{th} cross section (n=0,1,2,...) parallel to the base, which is itself a triangle, we arrange the ½(n+1)(n+2) coefficients $(n;\ m_1,m_2)$. At the outer edges the entries are ones, and each of the sides (faces) is itself a Pascal triangle. The relation (1.36) allows us to conclude that each interior element of a cross section is the sum of three elements in the triangular element which forms the (n-1)st cross section.

The rule for constructing the elements in the nth cross section can also be thought of in terms of the equation

$$(n;\ m_1,m_2) = \binom{n}{m_1}\binom{m_1}{m_2}, \tag{1.40}$$

where $n=0,1,2,\ldots$; $m_1=0,1,2,\ldots,n$; $m_2=0,1,2,\ldots,m_1$. This says, in effect, that we get the entries in the nth cross section by taking the ordinary Pascal triangle for that n, rotating its last row counterclockwise through the angle $\pi/2$, and then multiplying the resulting row entries on the rows of the triangle, as shown for n=4 by the example in Figure 16(a); the result is Figure 16(b). If the cross section is considered an equilateral triangle its axes of symmetry are as shown in Figure 17.

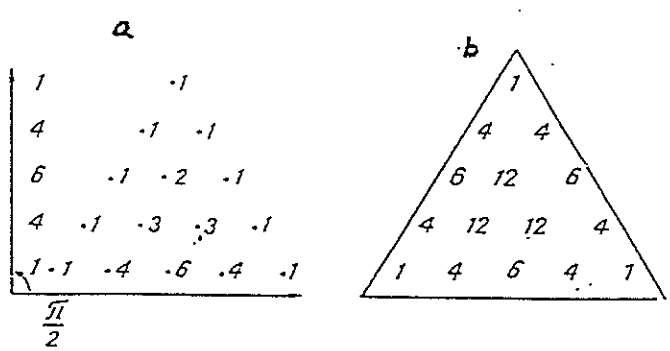


Figure 16

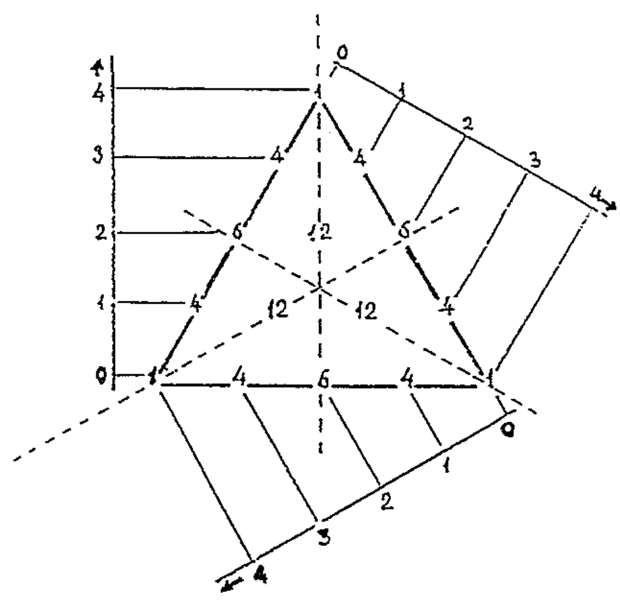


Figure 17

When the cross sections are taken to be right triangles, an algorithm for constructing the entries is given in [6].

These ideas can be extended to the multi-dimensional case. In particular, the coefficients in the expansion of $(x_0+x_1+x_2+x_3)^n$ form a four-dimensional Pascal pyramid, bounded by five tetrahedrons. Analogously, we can think of multi-dimensional Pascal pyramids bounded by Pascal pyramids of dimension one less.

Pascal pyramids and hyperpyramids have been used in the solution of problems on probability theory, polyharmonic polynomials, generalized Fibonacci sequences, and so on. The ideas of the construction and use of these objects appear in the works of many authors, and below we give a brief chronological survey of some of these papers and the results obtained.

One of the first occurrences of the Pascal pyramid, apparently, is in the work of E.B. Rosenthal [330], who suggested and wrote out the trinomial coefficients in an array which he called a Pascal pyramid.

The author of the present volume worked out an algorithm for constructing the cross sections of the Pascal pyramid, discussed the multi-dimensional case, and applied the results to the construction of harmonic and polyharmonic polynomials, and polynomial solutions to some problems in elasticity theory [5,6].

G. Garcia [146] geometrically formed the Pascal pyramid in the course of considering the coefficients in the expansion of $(a+b+c)^n$, and discussed the possibility of extending the example to the four-dimensional case.

A note of M. Basil [66] considers some properties of the trinomial coefficients written in the form of a Pascal pyramid.

R.L. Keeney [236] derived an algorithm for the construction of the elements in the cross sections, noted their symmetry, and the possibility of extension to the multi-dimensional case.

A note of S. Mueller [289] discusses the relations among the trinomial coefficients by means of the Pascal pyramid.

J. Staib and L. Staib [359] gave an algorithm for constructing the cross section elements in the trinomial case, and discussed the question of extension to the multi-dimensional case.

V.E. Hoggatt [195] discussed Pascal pyramids having as the elements of their cross sections the numbers in the expansion of $(a+b+c)^n$, and gave as the generating function of the columns

$$G^*_{m,n} = \frac{x^{pm+qn} b^m c^n \binom{m+n}{n}}{(1-ax)^{m+n+1}} .$$

He also showed that

$$G = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} G^*_{m,n} = \frac{1}{1-ax-bx^p-cx^q} ,$$

and particular choices of the parameters give the generating function for the Fibonacci numbers (a=1, b+c=1, p=q=2), the Tribonacci numbers (a=b=c=1, p=2, q=3), generalized Fibonacci numbers (a=b=c=1, p=t+1, q=2t+1), and other sequences.

In [341] A.G. Shannon used the Pascal pyramid to construct the Tribonacci numbers by summing the diagonal elements.

M. Alfonso and P. Hartung [58] emphasized the analogies between the Pascal pyramid and Pascal triangle, and used this approach to obtain some results in probability theory.

J.F. Putz [319, 320] discussed in detail the extension of the Pascal triangle, and the construction and properties of the pyramid, there called a Pascal polytope. He obtained a graphical representation, and also established the possibility of applying the polytope to the study of k-Fibonacci sequences. In [319], he generalized all 19 theorems of Pascal to the multi-dimensional case.

J. Shorter and F.M. Stein [343] constructed the Pascal tetrahedron, examined its properties, and discussed the possibility of extension to the multi-dimensional case.

The question of studying some special function values with the help of the pyramid is discussed in [267] by H.F. Lucas.

R.C. Bollinger [79] obtained some results on generalized binomial coefficients of order m, discussed the construction of the pyramid and its cross sections, and gave a method for computing the trinomial and multinomial coefficients.

1.6 MULTINOMIAL COEFFICIENTS AND PASCAL HYPERPYRAMIDS

As we know, the multinomial (also called polynomial) coefficients occur in the expansion of the polynomial $(x_0+x_1+\cdots+x_{s-1})^n$; the usual notation is $(n;\ n_1,n_2,\ldots,\ n_s)$, which stands for the form

$$(n;\ n_1,n_2,\cdots,n_s) = \frac{n!}{n_1!\,n_2!\cdots n_s!}\ , \tag{1.41}$$

$$\text{where} \quad n_1+n_2+\cdots+n_s = n\ . \tag{1.42}$$

The combinatorial sense of the multinomial coefficient may be expressed as: $(n;\ n_1,n_2,\ldots,n_s)$ gives the number of ways that n different objects may be distributed among s cells, where the number of objects in the k^{th} cell is n_k, $k=1,2,\ldots,s$.

Here we will denote the multinomial coefficients by $(n;\ m_1,m_2,\ldots,m_{s-1})$, defined as

$$(n;\ m_1,m_2,\cdots,m_{s-1}) = \frac{n!}{(n-m_1)!(m_1-m_2)!\cdots(m_{s-2}-m_{s-1})!m_{s-1}!}\ , \tag{1.43}$$

and introduced in [6]. Using this definition, condition (1.42) will be satisfied, and we have the ordered expansion

$$\begin{aligned} H_s(x,n) &\equiv (x_0 + x_1 + \cdots + x_{s-1})^n \\ &= \sum_{m_1=0}^{n} \sum_{m_2=0}^{m_1} \cdots \sum_{m_{s-1}=0}^{m_{s-2}} (n;\ m_1,\ m_2,\ \cdots,\ m_{s-1}) \cdot \\ &\qquad x_0^{n-m_1}\, x_1^{m_1-m_2} \cdots x_{s-2}^{m_{s-2}-m_{s-1}}\, x_{s-1}^{m_{s-1}}\ . \end{aligned} \tag{1.44}$$

We use (1.43) and (1.44) in the ordered construction of the Pascal hyperpyramid of multinomial coefficients, polyharmonic and other polynomial systems, and in discussing relations among the coefficients themselves. The multinomial expansion (1.44) appears in the literature of combinatorial analysis, algebra, statistics, and number theory [22, 23, 25, 38, 47].

We mention some basic formulas (omitting the proofs) for the multinomial coefficients (1.43), and then turn to a review of some references devoted to multinomial coefficients, the multinomial theorem, and connections with related matters.

The recurrence relation is

$$(n+1;\ m_1,m_2,\cdots,m_{s-1}) = (n;\ m_1,m_2,\cdots,m_{s-1}) + (n;\ m_1-1,m_2,\cdots,m_{s-1})$$
$$+ (n;\ m_1-1,m_2-1,\cdots,m_{s-1}) + \cdots$$
$$+ (n;\ m_1-1,m_2-1,\cdots,m_{s-2}-1,m_{s-1})$$
$$+ (n;\ m_1-1,m_2-1,\cdots,m_{s-1}-1), \tag{1.45}$$

with initial condition (0; 0,0,...,0) = 1, and $(n;\ m_1,\ldots,m_{s-1}) = 0$ for $n<0$, or $m_k<0$, for at least one value of k, and for $m_1>n$, $m_k>m_{k-1}$. The coefficients also satisfy the conditions

$$(n;\ 0,0,\cdots,0) = (n;\ n,0,\cdots,0) = (n;\ n,n,0,\cdots,0) = \cdots$$
$$= (n;\ n,n,\cdots,n) = 1,$$

and s equalities, the first and last of which are

$$\left.\begin{aligned} (n;\ m_1,\cdots,m_{s-1}) &= (n;\ m_1,m_2,\cdots,m_{s-2},m_{s-2}-m_{s-1}) \\ &\vdots \\ (n;\ m_1,\cdots,m_{s-1}) &= (n;\ n-m_{s-1},n-m_{s-2},\cdots,n-m_2,n-m_1). \end{aligned}\right\} \tag{1.46}$$

We have also the summation formulas

$$\sum_{m_1=0}^{n}\sum_{m_2=0}^{m_1}\cdots\sum_{m_{s-1}=0}^{m_{s-2}} (n;\ m_1,\cdots,m_{s-1}) = s^n, \tag{1.47}$$

$$\sum_{m_1=0}^{n}\sum_{m_2=0}^{m_1}\cdots\sum_{m_{s-1}=0}^{m_{s-2}} \delta(m_1,\cdots,m_{s-1})(n;\ m_1,\cdots,m_{s-1})$$
$$= \begin{cases} 0, & s=2l \\ 1, & s=2l+1, \end{cases} \tag{1.48}$$

where $\delta(m_1,\cdots,m_{s-1}) = (-1)^{m_1-m_2+m_3-m_4+\cdots+(-1)^s m_{s-1}}$.

We obtain (1.47) from (1.44) by taking $x_0=x_1=\cdots=x_{s-1}=1$, and (1.48) by taking $x_0=x_2=\cdots=x_{r(s)}=1$ and $x_1=x_3=\cdots=x_{r(s)+1}=-1$, where $r(s) = 2\left[\frac{s-1}{2}\right]$.

The multi-dimensional analog of the Cauchy summation formula is

$$\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{k_1}\cdots\sum_{k_{s-1}=0}^{k_{s-2}}(n_1;\, k_1,\cdots,k_{s-1})\,(n_2;\, m_1-k_1, m_2-k_2,\cdots,m_{s-1}-k_{s-1})$$

$$= (n_1+n_2;\, m_1,\cdots,m_{s-1}), \tag{1.49}$$

where $(n;m_1,\cdots,m_{s-1})=0$ if at least one of the $m_k<0$.

As mentioned in the book of E. Netto [292], the multinomial theorem was first mentioned in a letter from Leibnitz to Johann Bernoulli in 1695. Its proof has been given by a number of authors using various methods, one of which is the combinatorial argument. There are many works devoted to the study of the multinomial coefficients, and reviews of earlier results may be found in [41, 122, 292, 322, 372]. Below, we give in chronological order a survey of some results from recent decades.

With the coefficients written in the form

$$\frac{n!}{i_1!\,i_2!\cdots i_r!\,(n-k)!}\,,\quad i_1+i_2+\cdots+i_r=k$$

for $i_1\le i_2\le\cdots\le i_r\le n-k\le n-2$, P. Erdős and I. Niven [136] obtained a formula for $f(x)$, the number of coefficients less than the positive number x, of the form

$$f(x) = (1+\sqrt{2})x^{1/2} + O(x^{3/2}).$$

Two works of S. Tauber [372, 373] are devoted to the study of the multinomial coefficients in the form (1.41). The first gives material of a historical nature and establishes basic summation formulas; the second contains proof of some summation formulas similar to those for binomial coefficients.

M. Abramson [55] discussed the multinomial coefficients in the form (1.41), and established the basic formulas and relations by using their combinatorial interpretations.

In the author's book [6], he studies the multinomial coefficients in the form (1.43), establishes their basic relations, and gives applications to the construction and study of multi-dimensional harmonic and polyharmonic polynomials.

V.E. Hoggatt and G.L. Alexanderson [196] defined for each multinomial coefficient (1.41) the $s(s+1)$ neighboring coefficients for which their product is N^m, where N is an integer such that there exists a partition of these coefficients into s sets of $(s+1)$ coefficients whose product equals N, and where any such set may be obtained from another such set by a cyclic permutation of indices.

In [296] A. Nishiyama discussed values $f(n)$ which occur as sums of multinomial coefficients $(n; j_1,\ldots,j_p)$, when the j_n satisfy some condition. For example, for $p=2$ we have the binomial coefficients, and if the condition is that they should lie on the Pascal triangle diagonals, then $f(n)$ is the Fibonacci sequence.

D.L. Hilliker [185] extended the binomial theorem for complex values, established by Abel in the binomial case, to the multinomial theorem, and gave [186] various representations of the expansion of $(a_1+a_2+\cdots+a_r)^n$ by means of binomial coefficients.

A.N. Philippou [309] proved a theorem on the representation of the terms of the Fibonacci sequence of order k, $\{f_n^{(k)}\}$ by means of multinomial coefficients. The result is

$$f_{n+1}^{(k)} = \sum (n_1+n_2+\cdots+n_k;\ n_1,n_2,\cdots,n_k),\ n\geq 0,$$

where the summation is over all nonnegative numbers $n_1,\ldots,n_k$ for which $n_1+2n_2+\cdots+kn_k=0$.

Further results on the multinomial coefficients connected with questions of divisibility and other properties may be found in [14-16, 18, 67, 121, 225, 261, 275, 348, 352], a review of which we turn to in the following chapter.

CHAPTER 2

DIVISIBILITY AND THE DISTRIBUTION WITH RESPECT TO THE MODULUS p, AND ITS POWERS, OF BINOMIAL, TRINOMIAL, AND MULTINOMIAL COEFFICIENTS

In this chapter we discuss questions of the divisibility of binomial, trinomial, and multinomial coefficients by a prime p and its powers for the Pascal triangle, pyramid, and hyperpyramid. We also consider the number and distribution of these coefficients with respect to the modulus p and its powers in a row, triangle, or cross section of a pyramid.

A great number of works have been devoted to the study of the divisibility of these coefficients. Fundamental in these investigations are the theorems of Legendre, Lucas, and Kummer, and other important results are those of L. Carlitz [99-105], P. Erdös [131-133], N.J. Fine [139], H. Harborth [172-181], F.T. Howard [221-225], D. Singmaster [346-352], M. Sved [366-369], and the present author [11, 12, 14-16]. A survey of early divisibility results may be found in L.E. Dickson [122], and work from more recent decades is reviewed in the detailed article of D. Singmaster [352].

2.1 DIVISIBILITY OF BINOMIAL COEFFICIENTS

References containing material on the divisibility of binomial coefficients by a prime p and its powers are [11, 103, 109, 139, 148, 149, 176, 221-223, 233, 238, 256, 265, 297, 325, 365, 369, 401]. In dealing with the arithmetic properties of binomial coefficients and other coefficients containing factorials, it is convenient to have Legendre's Theorem:

Theorem 2.1. Let p be a prime, and s the highest power of p such that p^s divides n!. Then

$$s = \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \cdots = \frac{n-a}{p-1}, \qquad (2.1)$$

where the p-ary representation of n is $n=(a_r a_{r-1} \cdots a_1 a_0)_p$, and $a=a_0+a_1+\cdots+a_r$.

To obtain the residue mod p of the binomial coefficients we have Lucas's Theorem:

Theorem 2.2. Let p be a prime, n and m nonnegative integers ($m=0,1,2,\ldots,n$), and let the p-ary representations of these be $n=(a_r a_{r-1} \cdots a_0)$, $m=(b_r b_{r-1} \cdots b_0)$, where $a_r \neq 0$, and $0 \le a_k < p$, $0 \le b_k < p$. Then

$$\binom{n}{m} \equiv \binom{a_0}{b_0}\binom{a_1}{b_1} \cdots \binom{a_r}{b_r} \pmod{p}, \qquad (2.2)$$

where $\binom{a_k}{b_k} = 0$ if $b_k > a_k$.

Using Theorem 2.1, E. Kummer [246] obtained a formula for determining the highest power s of the prime p for which $\binom{n}{m}$ is exactly divisible by p^s (and not by p^{s+1}):

Theorem 2.3. Let p, m, n and the p-ary representations be as in Theorem 2.2, and let $n-m = (c_r c_{r-1} \cdots c_0)_p$. Then $\binom{n}{m}$ is exactly divisible by p^s if and only if

$$s = \frac{1}{p-1} \sum_{k=0}^{r} (b_k + c_k - a_k). \qquad (2.3)$$

Let us denote by h(n,p) the number of binomial coefficients in the n^{th} row of the Pascal triangle which are divisible by p, and g(n,p) the number of these coefficients not divisible by p. Also, denote by $g_j(n,p)$ the number of these coefficients which when divided by p have the remainder $j \le p-1$, and by $h_s(n,p)$ the number of these coefficients exactly divisible by p^s. Then we have

$$g(n,p) = g_1(n,p) + g_2(n,p) + \cdots + g_{p-1}(n,p)$$
$$h(n,p) = h_1(n,p) + h_2(n,p) + \cdots + h_{q_n}(n,p),$$

where $q_n = \max\{s\}$ in the n^{th} row. Since row n has n+1 entries, we have

$$h(n,p) = (n+1) - g(n,p).$$

Theorem 2.4. Let p be a prime, and n a row number of the Pascal triangle, with $n=(a_r a_{r-1} \cdots a_0)_p$. Then

$$g(n,p) = (a_r+1)(a_{r-1}+1) \cdots (a_1+1)(a_0+1).$$

The proof of this theorem based on Lucas's Theorem 2.2 was first given in [139].

For the calculation of $h_s(n,p)$, L. Carlitz [103, 104] introduced the functions $\theta_s(n,p)$ and $\phi_s(n,p)$, where the first is the number of binomial coefficients $\binom{n}{m}$ exactly divisible by p^s, and the second is the number of products $(n+1)\binom{n}{m}$ divisible by p^s, with p a prime and $m=0,1,\ldots,n$. For these functions he derived a system of recurrence relations and found the generating functions. He proved that for s=1,

$$h_1(n,p) = \sum_{k=0}^{r-1} (a_0+1)(a_1+1) \cdots (a_{k-1}+1)(p-a_k-1)a_{k+1}(a_{k+2}+1) \cdots (a_r+1) \qquad (2.5)$$

and for $s>2$ established a formula for $h_s(n,p)$ when n has the form(s)

$$n = ap^r + bp^{r+1}, \quad 0 \le a < p,\ 0 \le b < p;$$

$$n = a(1+p+\cdots+p^{r+s}), \qquad 0<a<p$$

$$n = a(1+p+\cdots+p^{r+s})-1.$$

F.T. Howard [221, 222] found a formula for $h_s(n,p)$ when $s=0,1,\ldots,4$; in the case of $s>4$, the formula requires further conditions. In [223] he found an exact formula for $h_2(n,p)$ and for $s>2$ values of $h_s(n,p)$ valid when n is of the form(s):

$$n = ap^k + bp^r,\ 0<a<p,\ 0<b<p,\ k<r,$$

$$n = c_1p^{k_1}+\cdots+c_mp^{k_m},\ 0<c_i<p,\ k_1 \ge s,\ k_{i+1}-k_i>s.$$

The extension of the divisibility results noted here to the trinomial case will be discussed in 2.3 and 2.4. We mention two examples of the application of Theorem 2.4 to the enumeration of the number of binomial coefficients not divisible by p. Let $p=2$ and $n=13$; we write the binary representation $13=(1101)_2$ and find $g(13,2) = (1+1)(1+1)(0+1)(1+1) = 8$. And if $p=3$, $n=14$ we write the ternary representation $14=(112)_3$ and find $g(14,3) = (1+1)(1+1)(2+1) = 12$.

We now turn to the Pascal triangle whose base is the row numbered n. Denote by $H(n,p)$ the (total) number of coefficients divisible by p in this triangle, and by $G(n,p)$ the number not divisible by p. Also, let $G_j(n,p)$ denote the number of coefficients in this triangle whose remainder after division by p is $j \le p-1$, and let $H_s(n,p)$ be the number exactly divisible by p^s. Then

$$G(n,p) = G_1(n,p) + G_2(n,p) + \cdots + G_{p-1}(n,p),$$

$$H(n,p) = H_1(n,p) + H_2(n,p) + \cdots + H_{q_n}(n,p),$$

where $q_n = \max\{s\}$ over the triangle. We note also that the triangle contains $N(n)=\frac{1}{2}(n+1)(n+2)$ entries; thus $H(n,p)=N(n)-G(n,p)$.

Theorem 2.5. Let n be the row number of the base of the Pascal triangle, and let p be a prime. Then

$$G(n,p) = \frac{1}{2} \sum_{i=0}^{r} b_{r-i} \binom{p+1}{2}^{r-i} \prod_{j=0}^{i} (b_{r-j}+1), \tag{2.6}$$

where $n+1 = (b_r b_{r-1} \cdots b_0)_p$.

Proof: For any n, we have

$$G(n+1,\ p) = G(n,p) + g(n+1,\ p), \tag{2.7}$$

and so

$$G(n,p) = G(n+1,\ p) - g(n+1,\ p). \tag{2.8}$$

From (2.4), we can write

$$G(n+1,\ p) = \sum_{k=0}^{n+1} \prod_{j=0}^{r} (\beta_{r-j}+1), \tag{2.9}$$

where $k = (\beta_r \beta_{r-1} \cdots \beta_0)_p$. If we now pass from the single sum with index k to a multiple sum with indices $\beta_r, \cdots, \beta_0$ and take into account the implied limits of summation we have

$$G(n+1,\ p) = \sum_{\beta_r=0}^{b_r-1} \sum_{\beta_{r-1}=0}^{p-1} \cdots \sum_{\beta_1=0}^{p-1} \sum_{\beta_0=0}^{p-1} \prod_{j=0}^{r} (\beta_{r-j}+1)$$

$$+ (b_r+1) \sum_{\beta_{r-1}=0}^{b_{r-1}-1} \sum_{\beta_{r-2}=0}^{p-1} \cdots \sum_{\beta_1=0}^{p-1} \sum_{\beta_0=0}^{p-1} \prod_{j=1}^{r} (\beta_{r-j}+1)$$

$$+ \left[(b_r+1)(b_{r-1}+1)\right] \sum_{\beta_{r-2}=0}^{b_{r-2}-1} \sum_{\beta_{r-3}=0}^{p-1} \cdots \sum_{\beta_1=0}^{p-1} \sum_{\beta_0=0}^{p-1} \prod_{j=2}^{r} (\beta_{r-j}+1)$$

$$+ \cdots + \left[(b_r+1)(b_{r-1}+1) \cdots (b_2+1)\right] \sum_{\beta_1=0}^{b_1-1} \sum_{\beta_0=0}^{p-1} \prod_{j=r-1}^{r} (\beta_{r-j}+1)$$

$$+ \left[(b_r+1)(b_{r-1}+1) \cdots (b_1+1)\right] \sum_{\beta_0=0}^{b_0-1} (\beta_0+1)$$

$$+ (b_r+1)(b_{r-1}+1) \cdots (b_1+1)(b_0+1) \ . \tag{2.10}$$

Each of the sums in (2.10) is an elementary calculation. It follows that

$$G(n+1,\ p) = \frac{b_r(b_r+1)}{2} \binom{p+1}{2}^{r} + (b_r+1) \frac{b_{r-1}(b_{r-1}+1)}{2} \binom{p+1}{2}^{r-1}$$

$$+ (b_r+1)(b_{r-1}+1) \frac{b_{r-2}(b_{r-2}+1)}{2} \binom{p+1}{2}^{r-2} + \cdots$$

$$+ \left[(b_r+1)(b_{r-1}+1) \cdots (b_2+1)\right] \frac{b_1(b_1+1)}{2} \binom{p+1}{2}^{1}$$

$$+ \left[(b_r+1)(b_{r-1}+1) \cdots (b_1+1)\right] \frac{b_0(b_0+1)}{2} \binom{p+1}{2}^{0}$$

$$+ (b_r+1)(b_{r-1}+1) \cdots (b_1+1)(b_0+1)$$

$$= \frac{1}{2} \sum_{i=0}^{r} b_{r-i} \prod_{j=0}^{i} (b_{r-j}+1) \binom{p+1}{2}^{r-i}$$

$$+ (b_r+1)(b_{r-1}+1) \cdots (b_1+1)(b_0+1) = G(n,p) + g(n+1,\ p) \ ,$$

which proves the theorem. From Theorem 2.5 it follows that if $n=p^r-1$, then

$$\left.\begin{aligned} G(p^r-1,\ p) &= \binom{p+1}{2}^r,\ G(ap^r-1,\ p) = \binom{a+1}{2}\binom{p+1}{2}^r, \\ G(ap^r+b-1,\ p) &= G(ap^r-1,\ p) + (a+1)G(b-1,\ p)\ , \end{aligned}\right\} \tag{2.11}$$

where $0 \le a < p-1$, $1 \le b \le p^r$. If $c \ge p$, then

$$G(cp^r-1,\ p) = \binom{p+1}{2}^r G(c-1,\ p)\ .$$

Let $p=2$. Then $n+1$ may be written in the form

$$n+1 = b_{r_1}2^{r_1} + b_{r_2}2^{r_2} + \dots + b_{r_q}2^{r_q}\ ,$$

and it follows from Theorem 2.5 that

$$G(n,2) = \sum_{i=1}^{q} 2^{i-1}3^{r_i}\ . \tag{2.12}$$

If $n=2^r-1$, then $G(2^r-1,2) = 3^r$, $G(2^r+b-1,2) = G(2^r-1,2) + 2G(b-1,2)$, and $G(c2^r-1,2) = 3^rG(c-1,2)$.

It also follows that if we subtract $G(n,p)$ from the total number of elements, we have

$$H(n,p) = N(n) - G(n,p) = \binom{n+2}{2} - G(n,p)\ . \tag{2.13}$$

For each p, from some n onward $H(n,p) >> G(n,p)$. Thus, for $p=3$ we have $G(26,3) = 216$, $H(26,3) = 162$; $G(80,3) = 1296$, $H(80,3) = 2025$; $G(242,3) = 7776$, $H(242,3) = 21868$; $G(728,3) = 46656$, $H(728,3) = 485514$. We need to clarify this order of increase of H(n,p) and G(n,p). For this, it is sufficient to consider, rather than $\{n\}$, the subsequence $\{p^r-1\}$ for $r\to\infty$.

Theorem 2.6. For $p\geq 2$, $\lim_{n\to\infty} G(n,p)/H(n,p)=0$.

Proof: Since G and H are nondecreasing functions of n, then for $p^r-1\leq n<p^{r+1}-1$, using the first equation in (2.11) and equation (2.13), we have

$$G(n,p)/H(n,p) \leq G(p^{r+1}-1,\ p)/H(p^r-1,\ p)$$

$$= \binom{p+1}{2}^{r+1} \Big/ \left[\binom{p^r+1}{2} - \binom{p+1}{2}^r\right]$$

$$= p(p+1)\Big/\left[\left(\frac{2p}{p+1}\right)^r + \left(\frac{2}{p+1}\right)^r - 2\right].$$

Since for $p\geq 2$, $\frac{2p}{p+1}>1$, $\frac{2}{p+1}<1$, then as $r\to\infty$ $\left(\frac{2p}{p+1}\right)^r\to\infty$, $\left(\frac{2}{p+1}\right)^r\to 0$, and it follows that

$$\lim_{n\to\infty} G(n,p)/H(n,p) = 0, \tag{2.14}$$

which proves the theorem.

We give below a short survey of some basic works on the divisibility of the binomial coefficients.

J.W. Glaisher [148, 149] discussed questions of exact divisibility of the binomial coefficients by powers of a prime and established a formula for the numbers of entries not divisible by p in the rows of the Pascal triangle.

N.J. Fine [139] obtained a formula for the number of binomial coefficients $\binom{n}{m}$, $0\leq m\leq n$, not divisible by p, and gave necessary and sufficient conditions for divisibility by p and for non-divisibility by p. He also proved that as $n\to\infty$ almost all binomial coefficients are divisible by p.

J.B. Roberts [325] discussed the problem of obtaining the number $\theta_j(n)$ of binomial coefficients in the Pascal triangle congruent to j, $0\le j\le p-1$, modulo the prime p. He reduced the problem to the solution of a linear difference equation with constant coefficients, and gave a formula for $\theta_j(n)$ for $p=2$ and any n, and also for $p=3,5$ and $n=p^k-1$, $k\ge0$.

In [177], H. Harborth studied the problem of the number A(n) of binomial coefficients $\binom{n}{m}$ in the Pascal triangle which are divisible by their row number n, as $n\to\infty$. He proved that almost all binomial coefficients are divisible by their row number; the distribution of divisibility by the row number is also considered in section 3.2.

N. Robbins [323, 324] looked at the connection between the function A(n) mentioned above and Euler's function $\varphi(n)$. In [323] he proved that $A(n)\ge\varphi(n)$ for all n, and $A(n)=\varphi(n)$ if $n=p^s (s\ge1)$ or if n is twice a prime Mersenne number. In [324] he found necessary and sufficient conditions for the equality $A(n)=\varphi(n)$, when n is square-free, and also discussed the case when n is a product of three distinct primes.

Consider the number of binomial coefficients $\binom{n}{m}$, for $0\le m\le n\le N$, not divisible by the product $(n)(n-1)\cdots(n-s+1)$, $s\ge1$. H. Harborth [180] proved that for fixed $s\ge1$ and $N\to\infty$, almost all binomial coefficients are divisible by this product. From this, he concludes that almost all binomial coefficients are divisible by $\binom{n}{s}$, and for $s=1$ this is the row number.

L. Carlitz [99] proved that if $n=(a_r a_{r-1}\cdots a_0)_p$, $\sigma(n)=a_r+a_{r-1}+\cdots+a_0$, and $(p-1)$ is divisible by $k\ge\sigma(n)$, then all binomial coefficients $\binom{n}{km}$, $0<km<n$, are divisible by p. He also considered in [100] the number of binomial coefficients $\binom{n}{m}$ satisfying the conditions:

$$\binom{n}{m} \equiv \binom{n}{m-1} \not\equiv 0 \pmod{p},\ m \not\equiv 0 \pmod{p},$$

$$\binom{n}{m} \equiv \binom{n}{m-1} \equiv 0 \pmod{p},\ m = 1,2,\ldots,n.$$

In [360, 362, 363] K.B. Stolarsky studied various problems connected with the function B(n) defined as the number of ones in the binary representation of n. In [360] he discussed the recurrence relation $y_{n+1}=y_n+B(n)$, n=1,2,..., and established the asymptotic behavior $y_m \sim (m \log m)/2 \log 2$. In [362], he studied the function $r_h=B(m^h)/B(m)$, where h is positive. He showed that the maximal order of magnitude of $r_h(m)$ is $c(h)\ (\log\ m)^{\frac{h-1}{h}}$, where $c(h)>0$ depends only on h; the minimal order of magnitude of $r_2(m)$ is not greater than $c(\log \log m)^2/\log m$, where $c>0$ is an absolute constant. In [363], he compared the behavior of the functions B(kn) and B(n), and called n "strong" if B(kn) > B(n); he also studied the question of the number of solutions of $B(3n)-B(n)=a$ for $2^r \le n < 2^{r+1}$. If we denote by F(n) the number of odd binomial coefficients in the first n rows of the Pascal triangle, Stolarsky also studied [361] the asymptotic behavior of F(n), using the expressions

$$\alpha = \limsup_{n\to\infty} F(n)/n^{\theta}, \quad \beta = \liminf_{n\to\infty} F(n)n^{\theta},$$

where $\theta = \log 3/\log 2 = 1.58496\ldots$. He established that α and β satisfy the conditions $1 \le \alpha \le 1.052$, $0.72 \le \beta \le 0.815$, and that $n^{\theta}/3 < F(n) < 3n^{\theta}$.

These results were sharpened by H. Harborth [176], who showed that $\alpha=1$, $\beta=0.812556\ldots$.

In a series of papers [346-352], D. Singmaster studied various properties of the binomial and multinomial coefficients. In [346] he discussed the problem of the number of ways a whole number a may be represented as a binomial coefficient, and showed that $N(a)=O(\log a)$. In [347] he introduced the functions E(n) and F(n), where $E(n)=s$ if p^s/n and $F(n)\equiv n/p^s \pmod p$; on the basis of the properties of these functions, he determined E(n!), F(n!), $E\left(\binom{n}{k}\right)$, $F\left(\binom{n}{k}\right)$, and so generalized the results of Lucas, Legendre, and Kummer. In [348], he obtained the least n for which the multinomial coefficient $(n;\ n_1,n_2,\ldots,n_r)$, $n_1,n_2,\ldots,n_r$ given, is divisible by p^s. In [349] he showed that "almost all" binomial coefficients are divisible by any positive whole number d. The notion of "almost all" appears in four versions, using the definitions: $A(\alpha,m)$ is the number of pairs (j,k) for which $0\le j$, $k<m$, $p^s\,|\binom{j+k}{k}$; $B(\alpha,m)$ is the number of pairs (j,k) for which $0\le j+k<m$, $p^s\,|\binom{j+k}{k}$; $C(\alpha,n)$ is the number of values of k for which $0\le k\le n$, $p^s\,|\binom{n}{k}$; $D(\alpha,k)$ is the density of j's for which $p^s\,|\binom{j+k}{k}$; $s=\alpha$. He showed, then, that

$$\lim_{m\to\infty} A(\alpha,m)/m^2 = 0, \quad \lim_{m\to\infty} B(\alpha,m)/(m(m+1)/2) = 0,$$

$$\lim_{n\to\infty} \frac{1}{n+1}\sum_{i=0}^{n} C(\alpha,i)/(i+1) = 0, \quad \lim_{k\to\infty} \frac{1}{k+1}\sum_{i=0}^{k} D(\alpha,i) = 0.$$

In [350] he discussed the greatest common divisor of corresponding triples of binomial coefficients in the Pascal triangle, and in [351] he considered the equation $\binom{n+1}{k+1}=\binom{n}{k+2}$ and showed there are infinitely many solutions of the form $n=F_{2\ell+2}F_{2\ell+3}-1$, $k=F_{2i}F_{2i+3}-1$, where F_k is a Fibonacci number. Finally, [352] is a systematic review of more than seventy papers by various authors, and also contains some new results on divisibility of binomial and multinomial coefficients by a prime p and its powers.

R. Fray [143] posed the problem of determining the least positive number *a* for which

$$\binom{n+a}{m} \equiv \binom{n}{m} \pmod{p^r} \tag{2.15}$$

for all m=0,1,...,n, r=1,2,.... He showed that if $p^s \le n < p^{s+1}$, the least a satisfying (2.15) for all m is $a=p^{r+s}$, and if $p^s \le m < p^{s+1}$, the solution is again $a=p^{r+s}$.

H. Gupta [167] solved the problem of determining the smallest positive n so that for a given positive m the binomial coefficient $\binom{n}{m}$ will have at least m prime divisors.

In [234, 235] G.S. Kazandzidis worked on a method for obtaining the highest power of a prime p which will divide $\binom{n}{m}$ and $\binom{np}{mp}$.

H.B. Mann and D. Shanks [274], using the Pascal triangle, established a criterion that a natural number m be prime: m is a prime if and only if for $\frac{m}{3} \le n \le \frac{m}{2}$, n divides $\binom{n}{m-2n}$.

In [175, 178] H. Harborth, with the help of the Pascal triangle, generalized the criterion of [174], showing that m is a prime if and only if for $\frac{m}{c+1} \le n \le \frac{m}{c}$, n divides $\binom{n}{m-cn}$; here, for fixed $c \le 2$, $m \ge 2$, n is not a multiple of a prime less than or equal to c^2-c-1. The details for c=3 are given in [175], and for c=4 in [178].

J. Bernard and G. Letac [67] proved that if a and b are whole numbers satisfying $|a|<p$, $|b|<p$, where p is prime, $(a+b) \ge 0$, and $\binom{n+m}{m}$ is divisible by p^s, then

$$\binom{pn+pm+a+b}{pm+b} \equiv 0 \pmod{p^s}. \tag{2.16}$$

Lastly, E.F. Ecklund [127] proved that $\binom{n}{m}$ has a prime divisor $p \le \max\left\{\frac{n}{m}, \frac{n}{2}\right\}$, with the exception of $\binom{7}{3}$ for $n \ge 2m$.

The question of the divisibility of the binomial coefficients by powers of a prime p is discussed in the following section.

2.2 THE DISTRIBUTION OF THE BINOMIAL COEFFICIENTS IN THE PASCAL TRIANGLE MODULO p AND ITS POWERS.

Here we consider problems associated with the number and distribution of the binomial coefficients in the Pascal triangle with respect to their remainder, or residue, after division by the prime p, and also the distribution of these coefficients strictly divisible by a power of p. These problems are also connected with the quantities $G_j(n,p)$, $H_s(n,p)$ introduced in section 2.1.

C.T. Long [261, 262] introduced two triangular forms composed of binomial coefficients:

$$\begin{array}{ccc} & \binom{np^k}{mp^k} & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \binom{np^k+p^k-1}{mp^k} & \cdots & \binom{np^k+p^k-1}{mp^k+p^k-1} \end{array}, \quad 0 \le m \le n, \tag{2.17}$$

and

$$\begin{array}{ccc} \binom{np^k}{mp^k+1} & \cdots & \binom{np^k}{mp^k+p^k-1} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ & \binom{np^k+p^k-2}{mp^k+p^k-1} & \end{array}, \quad n \ge 1,\ 0 \le m \le n-1, \tag{2.18}$$

where m, n, k are whole numbers, and $k \ge 1$. If we replace the elements in (2.17) by their residues mod p, and denote the result by $\Delta_{n,m}$, the author discusses the triangle of triangles

$$
\begin{array}{ccccc}
 & & \Delta_{0,0} & & \\
 & \Delta_{1,0} & & \Delta_{1,1} & \\
\Delta_{2,0} & & \Delta_{2,1} & & \Delta_{2,2} \\
\cdot & \cdot & \cdot & \cdot & \cdot
\end{array}
\tag{2.19}
$$

"isomorphic" to the usual Pascal triangle. He shows that for practical purposes the triangle $\Delta_{n,m}$ may be taken to be the triangle

$$
\begin{array}{ccccc}
 & & \binom{n}{m}\binom{0}{0} & & \\
 & \binom{n}{m}\binom{1}{0} & & \binom{n}{m}\binom{1}{1} & \\
 \cdot & & & & \cdot \\
 \cdot & & & & \cdot \\
 \cdot & & & & \cdot \\
\binom{n}{m}\binom{p^k-1}{0} & & \cdots & & \binom{n}{m}\binom{p^k-1}{p^k-1}
\end{array}
\tag{2.20}
$$

consisting of the residues mod p, which by Lucas's Theorem are congruent mod p to the corresponding elements of the triangle (2.17). It is also shown that the triangle $\Delta_{n,m}$ satisfies the recurrence relation

$$\Delta_{n+1,m+1} = \Delta_{n,m} + \Delta_{n,m+1} \tag{2.21}$$

"isomorphic" to the ordinary one, $\binom{n+1}{m+1} = \binom{n}{m} + \binom{n}{m+1}$. On the right side of (2.21), addition is carried out mod p.

Between each pair of adjacent triangles $\Delta_{n,m}$ and $\Delta_{n,m+1}$, $m=0,1,\ldots,n-1$, in (2.19), we may place (the elements of) a triangle $\nabla_{n,m}$ consisting of the residues mod p of the binomial coefficients in (2.18). By Lucas's Theorem the residues of each of the binomial coefficients in (2.18) are all zeros, and so all the elements of $\nabla_{n,m}$ are null.

<u>First Problem</u>. Let p be a prime, $k \geq 0$, $n=0,1,\ldots,p-1$, and $m=0,1,\ldots,n$. Then among the $p(p+1)/2$ triangles $\Delta_{n,m}$ constituting p^k rows, as in (2.19), there will be only $p-1$ distinct types. Their general form, which we denote by $\Delta_k^{(i)}$, $k \geq 0$, $i=1,2,\ldots,p-1$, is

$$\begin{array}{ccccccccc} & & & & i & & & & \\ & & & i & & i & & & \\ & & i & & & & i & & \\ & \cdot & & & & & & \cdot & \\ & \cdot & & & & & & \cdot & \\ & \cdot & & & & & & \cdot & \\ i & & & & & & & & i \\ i & p-i & & & \cdots & & & p-i & i \end{array} \tag{2.22}$$

The interior elements of the triangle are calculated by the usual rule as the sums of two elements in the preceding row, except here mod p. Also, the elements of the triangle $\nabla_{n,m}$ are zeros independently of n and m, and for a given k compose p^k-1 rows, these triangles are all the same and we denote them by $\nabla_k^{(0)}$.

Using the "geometric" (Pascal rule) interpretation, we can establish that each triangle $\Delta_k^{(i)}$ is itself a "geometric" sum of $p(p+1)/2$ triangles $\Delta_{k-1}^{(i)}$ and $p(p-1)/2$ triangles $\nabla_{k-1}^{(0)}$. Thus, we can write out the "geometric" equations

$$\left.\begin{aligned}
\Delta_k^{(1)} &= a_{1,1}\Delta_{k-1}^{(1)} + a_{1,2}\Delta_{k-1}^{(2)} + \cdots + a_{1,p-1}\Delta_{k-1}^{(p-1)} + \frac{p(p-1)}{2}\nabla_{k-1}^{(0)},\\
\Delta_k^{(2)} &= a_{2,1}\Delta_{k-1}^{(1)} + a_{2,2}\Delta_{k-1}^{(2)} + \cdots + a_{2,p-1}\Delta_{k-1}^{(p-1)} + \frac{p(p-1)}{2}\nabla_{k-1}^{(0)},\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
\Delta_k^{(p-1)} &= a_{p-1,1}\Delta_{k-1}^{(1)} + a_{p-1,2}\Delta_{k-1}^{(2)} + \cdots\\
&\quad + a_{p-1,p-1}\Delta_{k-1}^{(p-1)} + \frac{p(p-1)}{2}\nabla_{k-1}^{(0)},
\end{aligned}\right\} \tag{2.23}$$

where $a_{i,j}$ is the number of triangles $\Delta_{k-1}^{(j)}$ occurring in $\Delta_k^{(i)}$. These coefficients $a_{i,j} \equiv a_{i,j}^{(p)}$ depend on p but not on k, and their values coincide with the numbers of ones, twos, ..., up to $(p-1)$ inclusive, contained in the triangles $\Delta_1^{(1)}, \Delta_1^{(2)}, \ldots, \Delta_1^{(p-1)}$, whose general form is that of (2.22), and which have p rows.

Let $P_k^{(i)}(r)$ be the number of occurrences of r in the triangle $\Delta_k^{(i)}$. Thus, $P_k^{(1)}(1)$ is the number of ones in $\Delta_k^{(1)}$, $P_k^{(2)}(3)$ is the number of threes in $\Delta_k^{(2)}$, and so on. Using (2.23) and taking into account that the $\nabla_{k-1}^{(0)}$ contain only zeros and may be neglected, we can form the system of first order recurrence relations

$$\left.\begin{aligned}
P_k^{(1)}(r) &= a_{1,1}P_{k-1}^{(1)}(r) + a_{1,2}P_{k-1}^{(2)}(r) + \cdots + a_{1,p-1}P_{k-1}^{(p-1)}(r),\\
P_k^{(2)}(r) &= a_{2,1}P_{k-1}^{(1)}(r) + a_{2,2}P_{k-1}^{(2)}(r) + \cdots + a_{2,p-1}P_{k-1}^{(p-1)}(r),\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
P_k^{(p-1)}(r) &= a_{p-1,1}P_{k-1}^{(1)}(r) + a_{p-1,2}P_{k-1}^{(2)}(r) + \cdots + a_{p-1,p-1}P_{k-1}^{(p-1)}(r),
\end{aligned}\right\} \tag{2.24}$$

where $r=1,2,\ldots,p-1, a_{i,j} \equiv a_{i,j}^{(p)}$.

The solution of the system depends on the initial data, which are determined by the occurrences of r in the triangles $\Delta_0^{(j)}$, $1 \le j \le p-1$. Since each of the $\Delta_0^{(j)}$ contains but one element equal to j, and does not contain $r \ne j$, the data take the form of $(p-1)$ groups:

$$\left.\begin{array}{l} P_0^{(1)}(1) = 1,\ P_0^{(2)}(1) = 0,\ P_0^{(3)}(1) = 0, \cdots ,\ P_0^{(p-1)}(1) = 0, \\ P_0^{(1)}(2) = 0,\ P_0^{(2)}(2) = 1,\ P_0^{(3)}(2) = 0, \cdots ,\ P_0^{(p-1)}(2) = 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ P_0^{(1)}(p-1) = 0,\ P_0^{(2)}(p-1) = 0,\ P_0^{(3)}(p-1) = 0, \cdots,\ P_0^{(p-1)}(p-1) = 1. \end{array}\right\} \qquad (2.25)$$

From (2.24) and (2.25) we may determine the number of occurrences of r in any triangle $\Delta_k^{(i)}$, $1 \le i \le p-1$. The triangle $\Delta_k^{(1)}$ itself represents the Pascal triangle mod p, and so $P_k^{(1)}(r)$ gives the value of $G_r(p^k-1,p)$; the values of $P_k^{(j)}(r)$, $2 \le j \le p-1$, are the number of occurrences of r in $\Delta_k^{(j)}$, as shown schematically in (2.22).

The matrix of the system (2.24) is

$$A_p = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,p-1} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,p-1} \\ \cdots & \cdots & \cdots & \cdots \\ a_{p-1,1} & a_{p-1,2} & \cdots & a_{p-1,p-1} \end{bmatrix}, \quad a_{i,j} \equiv a_{i,j}^{(p)}$$

which, except for $p=3$, is not symmetric. The element $a_{i,j}$, as indicated before, is the number of occurrences of i in $\Delta_1^{(j)}$ for a given value of p. We give below some examples.

Let p=3. Then

$$\Delta_1^{(1)} = \begin{matrix} & & 1 & & \\ & 1 & & 1 & \\ 1 & & 2 & & 1 \end{matrix} \qquad \Delta_1^{(2)} = \begin{matrix} & & 2 & & \\ & 2 & & 2 & \\ 2 & & 1 & & 2 \end{matrix},$$

and so

$$a_{1,1}^{(3)} = 5,\ a_{1,2}^{(3)} = 1,\ a_{2,1}^{(3)} = 1,\ a_{2,2}^{(3)} = 5.$$

The system corresponding to (2.24) is

$$\left.\begin{aligned} P_k^{(1)}(r) &= 5P_{k-1}^{(1)}(r) + P_{k-1}^{(2)}(r), \\ P_k^{(2)}(r) &= P_{k-1}^{(1)}(r) + 5P_{k-1}^{(2)}(r), \end{aligned}\right\} \tag{2.26}$$

where r=1,2, and the matrix is

$$A_3 = \begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix}.$$

The initial data are obtained from $\Delta_0^{(1)}$ and $\Delta_0^{(2)}$, and we have

$$P_0^{(1)}(1) = 1,\ P_0^{(2)}(1) = 0,\ P_0^{(1)}(2) = 0,\ P_0^{(2)}(2) = 1.$$

The solution of (2.26), with these initial values is

$$\left.\begin{aligned} P_k^{(1)}(1) &= \tfrac{1}{2}(6^k+4^k), \quad & P_k^{(1)}(2) &= \tfrac{1}{2}(6^k-4^k) \\ P_k^{(2)}(1) &= \tfrac{1}{2}(6^k-4^k), \quad & P_k^{(2)}(2) &= \tfrac{1}{2}(6^k+4^k). \end{aligned}\right\} \tag{2.27}$$

It follows from (2.27) that for p=3 the number of ones in the Pascal triangle whose base is row 3^k-1 is

$$G_1(3^k-1,3) = P_k^{(1)}(1) = \tfrac{1}{2}(6^k+4^k),$$

and the number of twos is

$$G_2(3^k-1,3) = P_k^{(1)}(2) = \tfrac{1}{2}(6^k-4^k).$$

It also follows that the total number of coefficients in this triangle which are not divisible by 3 is

$$G(3^k-1,3) = G_1 + G_2 = 6^k,$$

which agrees with Theorem 2.5. If $3^k<n<3^{k+1}$, then for the enumerations $G_1(n,3)$, $G_2(n,3)$ we need to use the appropriate "geometric" equations and the formulas for $P_\ell^{(1)}(1)$, $P_\ell^{(1)}(2)$, where $\ell<k$ and k and ℓ must be specified.

Let p=5. Then

$$\Delta_1^{(1)} = \begin{array}{ccccccccc} &&&&1&&&& \\ &&&1&&1&&& \\ &&1&&2&&1&& \\ &1&&3&&3&&1& \\ 1&&4&&1&&4&&1 \end{array} \qquad \Delta_1^{(2)} = \begin{array}{ccccccccc} &&&&2&&&& \\ &&&2&&2&&& \\ &&2&&4&&2&& \\ &2&&1&&1&&2& \\ 2&&3&&2&&3&&2 \end{array}$$

$$\Delta_1^{(3)} = \begin{array}{ccccccccc} &&&&3&&&& \\ &&&3&&3&&& \\ &&3&&1&&3&& \\ &3&&4&&4&&3& \\ 3&&2&&3&&2&&3 \end{array} \qquad \Delta_1^{(4)} = \begin{array}{ccccccccc} &&&&4&&&& \\ &&&4&&4&&& \\ &&4&&3&&4&& \\ &4&&2&&2&&4& \\ 4&&1&&4&&1&&4 \end{array},$$

and so we have

$$a_{1,1}^{(5)} = 10, \quad a_{1,2}^{(5)} = 1, \quad a_{1,3}^{(5)} = 2, \quad a_{1,4}^{(5)} = 2;$$

$$a_{2,1}^{(5)} = 2, \quad a_{2,2}^{(5)} = 10, \quad a_{2,3}^{(5)} = 2, \quad a_{2,4}^{(5)} = 1;$$

$$a_{3,1}^{(5)} = 1, \quad a_{3,2}^{(5)} = 2, \quad a_{3,3}^{(5)} = 10, \quad a_{3,4}^{(5)} = 2;$$

$$a_{4,1}^{(5)} = 2, \quad a_{4,2}^{(5)} = 2, \quad a_{4,3}^{(5)} = 1, \quad a_{4,4}^{(5)} = 10,$$

and the matrix A_5

$$A_5 = \begin{bmatrix} 10 & 1 & 2 & 2 \\ 2 & 10 & 2 & 1 \\ 1 & 2 & 10 & 2 \\ 2 & 2 & 1 & 10 \end{bmatrix}.$$

In vector form the system is

$$\begin{bmatrix} P_k^{(1)}(r) \\ P_k^{(2)}(r) \\ P_k^{(3)}(r) \\ P_k^{(4)}(r) \end{bmatrix} = \begin{bmatrix} 10 & 1 & 2 & 2 \\ 2 & 10 & 2 & 1 \\ 1 & 2 & 10 & 2 \\ 2 & 2 & 1 & 10 \end{bmatrix} \begin{bmatrix} P_{k-1}^{(1)}(r) \\ P_{k-1}^{(2)}(r) \\ P_{k-1}^{(3)}(r) \\ P_{k-1}^{(4)}(r) \end{bmatrix}$$

for r=1,2,3,4, and the initial conditions are

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

The solution of this system with the given initial data has the form

$$P_k^{(1)}(1) = P_k^{(2)}(2) = P_k^{(3)}(3) = P_k^{(4)}(4)$$

$$= \frac{1}{4}(15^k + 9^k) + \frac{1}{2}H_0^k(1,8),$$

$$P_k^{(1)}(2) = P_k^{(2)}(4) = P_k^{(3)}(1) = P_k^{(4)}(3)$$

$$= \frac{1}{4}(15^k - 9^k) - \frac{1}{2}H_1^k(1,8),$$

$$P_k^{(1)}(3) = P_k^{(2)}(1) = P_k^{(3)}(4) = P_k^{(4)}(2)$$

$$= \frac{1}{4}(15^k - 9^k) + \frac{1}{2}H_1^k(1,8),$$

$$P_k^{(1)}(4) = P_k^{(2)}(3) = P_k^{(3)}(2) = P_k^{(4)}(1)$$

$$= \frac{1}{4}(15^k + 9^k) - \frac{1}{2}H_0^k(1,8),$$

where

$$H_\alpha^k(x, y) = \sum_{i=0}^{\left[\frac{k-\alpha}{2}\right]} (-1)^i \binom{k}{2i+\alpha} x^{2i+\alpha} y^{k-2i-\alpha}, \quad \alpha = 0, 1,$$

is itself known to be a harmonic polynomial in two variables [6]. With this information we can find the distribution of the residues 1,2,3,4 mod 5 in the Pascal triangle whose base is the row numbered $5^k - 1$:

$$\left.\begin{aligned} G_1(5^k - 1,5) &= \frac{1}{4}(15^k + 9^k) + \frac{1}{2}H_0^k(1,8), \\ G_2(5^k - 1,5) &= \frac{1}{4}(15^k - 9^k) - \frac{1}{2}H_1^k(1,8), \\ G_3(5^k - 1,5) &= \frac{1}{4}(15^k - 9^k) + \frac{1}{2}H_1^k(1,8), \\ G_4(5^k - 1,5) &= \frac{1}{4}(15^k + 9^k) - \frac{1}{2}H_0^k(1,8). \end{aligned}\right\}$$

In like fashion for p=7,11, the matrices will be

$$A_7 = \begin{bmatrix} 15 & 2 & 2 & 1 & 4 & 4 \\ 1 & 15 & 4 & 2 & 4 & 2 \\ 4 & 2 & 15 & 4 & 1 & 2 \\ 2 & 1 & 4 & 15 & 2 & 4 \\ 2 & 4 & 2 & 4 & 15 & 1 \\ 4 & 4 & 1 & 2 & 2 & 15 \end{bmatrix} \qquad A_{11} = \begin{bmatrix} 27 & 5 & 5 & 4 & 3 & 3 & 2 & 4 & 4 & 9 \\ 3 & 27 & 2 & 5 & 4 & 5 & 4 & 4 & 9 & 3 \\ 4 & 4 & 27 & 3 & 4 & 5 & 3 & 9 & 5 & 2 \\ 5 & 3 & 4 & 27 & 4 & 2 & 9 & 5 & 3 & 4 \\ 4 & 2 & 3 & 5 & 27 & 9 & 4 & 3 & 4 & 5 \\ 5 & 4 & 3 & 4 & 9 & 27 & 5 & 3 & 2 & 4 \\ 4 & 3 & 5 & 9 & 2 & 4 & 27 & 4 & 3 & 5 \\ 2 & 5 & 9 & 3 & 5 & 4 & 3 & 27 & 4 & 4 \\ 3 & 9 & 4 & 4 & 5 & 4 & 5 & 2 & 27 & 3 \\ 9 & 4 & 4 & 2 & 3 & 3 & 4 & 5 & 5 & 27 \end{bmatrix}$$

These matrices for any prime p, as here for p=3,5,7,11, have a property which we might call quasi-symmetry, in which the elements satisfy three types of conditions:

$$a_{1,1} = a_{2,2} = a_{3,3} = \cdots = a_{p-1,p-1},$$

$$a_{1,p-1} = a_{2,p-2} = a_{3,p-3} = \cdots = a_{p-1,1},$$

$$a_{i,j} = a_{p-i,p-j}, \; i \neq j, \; i+j \neq p.$$

The first and second of these require the equality of the elements on the main diagonal, and of those on the counterdiagonal, respectively. The third requires, in effect, the equality of elements in positions above and below the main diagonal, related to one another by a 180° rotation about an axis perpendicular to the center of the array. Without going into the matter here, we mention that these quasi-symmetric matrices which arise in connection with the Pascal triangle mod p have a number of interesting properties.

Questions connected with the distribution in the Pascal triangle of the binomial coefficients mod p are discussed in papers by A. Fadini [137], J.B. Roberts [325], M. Sved and J. Pitman [369], among others. In particular, [137] uses a triangle of triangles like that of Long [262]; [325] gives the distribution of the binomial coefficients mod 3 and mod 5; in [369] are tables of the distribution mod 3 up to the 50th row, and also for the composite modulus $9=3^2$ up to the 60th row. In the last, there are also tables of values α, β for the expression of the binomial coefficients in the forms $\alpha \cdot 3^2 + \beta \cdot 3$ and $\alpha \cdot 7^2 + \beta \cdot 7$, and other tables.

As examples of the properties mentioned in this section, we show the distribution of the binomial coefficients mod 2 in Figure 18, and mod 3 in Figure 19 (the dots stand for zeros).

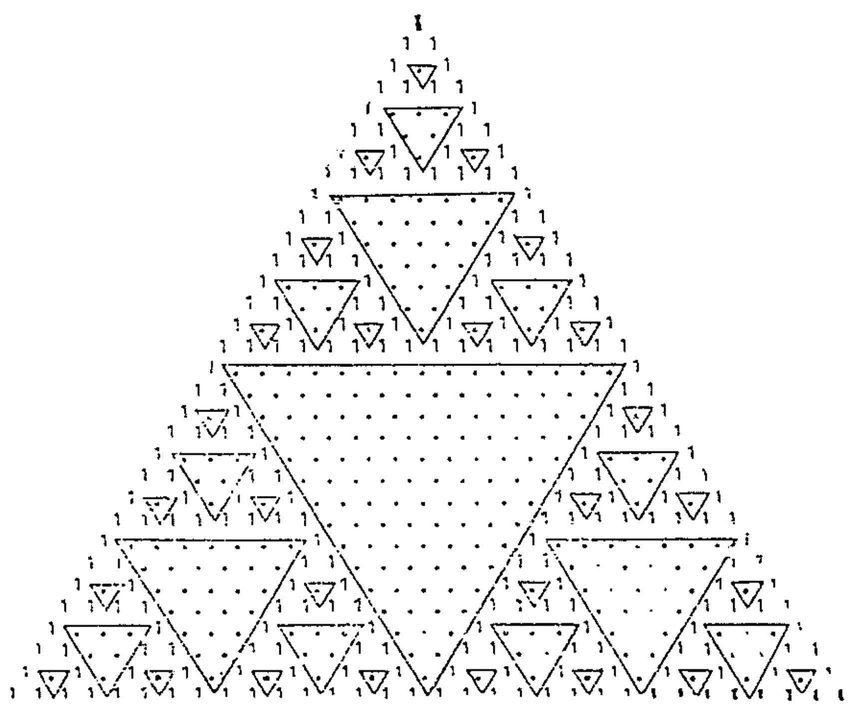


Figure 18

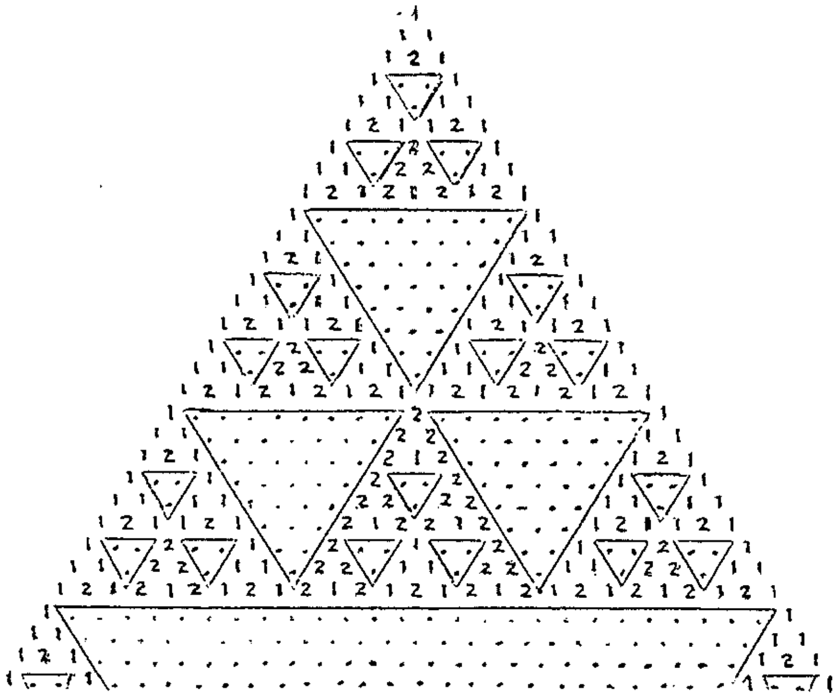


Figure 19

Second Problem. Here again we are interested in the distribution of the binomial coefficients in the Pascal triangle, but now the criterion (for forming the distribution) is that of strict divisibility by a power of the prime p. That is, using the notation introduced by Long [261], we denote by $\begin{bmatrix} n \\ m \end{bmatrix}$ the exponent of the highest power of p which divides $\binom{n}{m}$, and consider the triangle whose elements are the values $\begin{bmatrix} n \\ m \end{bmatrix}$, $0 \leq m \leq n$, called the p-index Pascal triangle. This triangle has a number of interesting properties, and was first discussed by K.R. McLean [277].

We list some of these properties, formulated and proved in [261]. Let p be a prime, and N and n natural numbers; then

$$\begin{bmatrix} np^k-1 \\ m \end{bmatrix} = 0, \text{ if } 1 \le n < p,\ 0 \le m < np^k-1;$$

$$\begin{bmatrix} p^k \\ m \end{bmatrix} \ge 1, \text{ if } 1 \le m < p^k,\ \begin{bmatrix} p^k \\ 0 \end{bmatrix} = \begin{bmatrix} p^k \\ p^k \end{bmatrix} = 0;$$

$$\begin{bmatrix} Np^k+np^{k-1}-1 \\ m \end{bmatrix} = \begin{cases} 0 \text{ for } rp^k \le m \le rp^k+np^{k-1}-1, & 0 \le r \le N, \\ 1 \text{ for } rp^k+np^{k-1} \le m \le (r+1)p^k, & 0 \le r < N, \end{cases}$$

where $1 \le n < p$, $1 \le N < p$.

Let p be a prime, k a natural number, and n,m integers, $0 \le m \le n$. Denote by $T^{(k)}_{n,m}$ the p-index triangle formed from the triangle (2.17), and which will be of the form

$$\begin{matrix} & \begin{bmatrix} np^k \\ mp^k \end{bmatrix} & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \begin{bmatrix} np^k+p^k-1 \\ mp^k \end{bmatrix} & \cdots & \begin{bmatrix} np^k+p^k-1 \\ mp^k+p^k-1 \end{bmatrix} \end{matrix} \qquad (2.28)$$

In [261] it is shown that $T^{(k)}_{n,m} = \begin{bmatrix} n \\ m \end{bmatrix} + T^{(k)}_{0,0}$, where the symbolic addition on the right-hand side indicates that the element $\begin{bmatrix} n \\ m \end{bmatrix}$ is added to each entry of $T^{(k)}_{0,0}$. Thus, it follows that if we know the distribution of elements in the triangle $T^{(k)}_{0,0}$, we can easily find the distribution in the triangle $T^{(k)}_{n,m}$, and moreover we obtain the distribution of the binomial coefficients according to the criterion of the strict divisibility by a power of the prime p in the triangle (2.17). From the triangles $T^{(k)}_{n,m}$ we can form an indefinitely increasing triangle.

We introduce the p-index triangle $R^{(k)}_{n,m}$ corresponding to the triangle (2.18):

$$\begin{bmatrix} np^k \\ mp^k+1 \end{bmatrix} \quad \cdots \quad \begin{bmatrix} np^k \\ mp^k+p^k-1 \end{bmatrix}$$
$$\cdot \qquad \cdot$$
$$\cdot \qquad \cdot \tag{2.29}$$
$$\cdot \qquad \cdot$$
$$\begin{bmatrix} np^k+p^k-2 \\ mp^k+p^k-1 \end{bmatrix}$$

where $n \geq 1$, $0 \leq m \leq n-1$.

It is not difficult to show that the correct equation for this triangle is $R_{n,m}^{(k)} = \begin{bmatrix} n \\ m \end{bmatrix} + R_{1,0}^{(k)}$. It follows that if we know the distribution of the entries in the triangle $R_{1,0}^{(k)}$, we can find the distribution of the elements in $R_{n,m}^{(k)}$ for any n,m, and moreover we will obtain the distribution of the binomial coefficients strictly divisible by a power of p in the triangle (2.18) for any n,m.

Consider the Pascal triangle whose base is the row numbered $N=p^k-1$. The corresponding p-index triangle $T_{0,0}^{(k)}$ consists of $p(p+1)/2$ p-index triangles $T_{n,m}^{(k-1)}$, where $n=0,1,\ldots,p-1$, and $0 \leq m \leq n$. The distributions of the elements in all the triangles $T_{n,m}^{(k-1)}$ are identical, and so we may replace each of them by $T_{0,0}^{(k-1)}$. Besides the triangles $T_{n,m}^{(k-1)}$, $T_{0,0}^{(k)}$ also contains $p(p-1)/2$ p-index triangles $R_{n,m}^{(k-1)}$, where $n=1,2,\ldots,p-1$, $0 \leq m \leq n-1$. The distributions in these are also identical, and so we may replace each of them by the triangle $R_{1,0}^{(k-1)}$.

Consider now the p-index triangle (2.29) for $n=1$, $m=0$, i.e., $R_{1,0}^{(k)}$. It may be shown that $R_{1,0}^{(k)}$ itself consists of $p(p-1)/2$ triangles $\overline{T}_{n,m}^{(k-1)}$, $1 \leq n \leq p-1$, $0 \leq m \leq p-n-1$, and $p(p+1)/2$ triangles $\overline{R}_{n,m}^{(k-1)}$, $1 \leq n \leq p$, $0 \leq m \leq p-n$. As before, each of the $\overline{T}_{n,m}^{(k-1)}$ may be replaced by

$\overline{T}^{(k-1)}_{1,1}$, and each of the $\overline{R}^{(k-1)}_{n,m}$ may be replaced by $\overline{R}^{(k-1)}_{1,0}$. Figure 20 shows $T^{(k)}_{0,0}$ and $R^{(k)}_{1,0}$, denoted by T_k and R_k, and the arrangement of the triangles $T^{(k-1)}_{0,0}$, $R^{(k-1)}_{1,0}$, $\overline{T}^{(k-1)}_{1,1}$, $\overline{R}^{(k-1)}_{1,0}$, denoted by T_{k-1}, R_{k-1}, $T^{(1)}_{k-1}$, $R^{(1)}_{k-1}$.

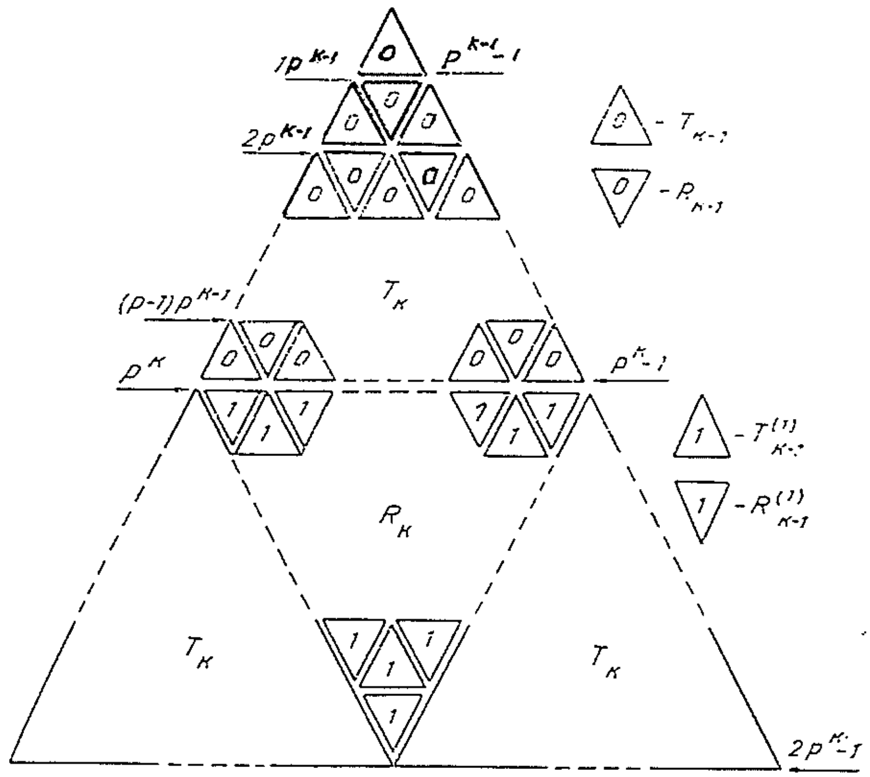


Figure 20

Using the equation given earlier and taking in account that $n<p$, we find that

$$\overline{T}^{(k-1)}_{1,1} = 1 + T^{(k-1)}_{0,0}, \qquad \overline{R}^{(k-1)}_{1,0} = 1 + R^{(k-1)}_{1,0}.$$

It follows that we can form the "geometric" equations

$$T^{(k)}_{0,0} = \tfrac{1}{2}p(p+1)T^{(k-1)}_{0,0} + \tfrac{1}{2}p(p-1)R^{(k-1)}_{1,0},$$

$$R^{(k)}_{1,0} = \tfrac{1}{2}p(p-1)\overline{T}^{(k-1)}_{1,1} + \tfrac{1}{2}p(p+1)\overline{R}^{(k-1)}_{1,0}.$$

Denote now by $P_k(s)$ and $Q_k(s)$ the number of occurrences of the value s in the respective triangles $T^{(k)}_{0,0}$ and $R^{(k)}_{1,0}$, where s is the greatest exponent of p such that p^s divides the corresponding binomial coefficient. From the geometric equations above, we can write the recurrence relations

$$\left.\begin{aligned} P_k(s) &= \tfrac{1}{2}p(p+1)P_{k-1}(s) + \tfrac{1}{2}p(p-1)Q_{k-1}(s), \\ Q_k(s) &= \tfrac{1}{2}p(p-1)P_{k-1}(s-1) + \tfrac{1}{2}p(p+1)Q_{k-1}(s-1), \end{aligned}\right\} \tag{2.30}$$

where $k=2,3,\ldots$, and $s=0,1,\ldots,k-1$ in the first equation and $s=1,2,\ldots,k$ in the second. For the initial conditions ($P_1(s)$, $Q_1(s)$ for $s=0,1$), we enumerate the numbers of zeros and ones in $T^{(1)}_{0,0}$, $R^{(1)}_{1,0}$, and find that

$$\begin{aligned} P_1(0) &= p(p+1)/2, \quad & P_1(1) &= 0 \\ Q_1(0) &= 0, & Q_1(1) &= p(p-1)/2. \end{aligned}$$

The system (2.30) is a special case of the system

$$\left.\begin{aligned} X_{k,s} &= aX_{k-1,s} + bY_{k-1,s}, \\ Y_{k,s} &= bX_{k-1,s-1} + aY_{k-1,s-1}, \end{aligned}\right\} \tag{2.31}$$

which we solve by the method discussed in [6]. If we further choose

$$X_{1,0} = a, \quad X_{1,1} = 0, \quad Y_{1,0} = 0, \quad Y_{1,1} = b,$$

then it may be shown by complete induction that the solution takes the form

$$X_{k,s} = a^k \sum_{i=0}^{\mu(k,s)} \left(\frac{b}{a}\right)^{2i+2} \binom{s-1}{i} \binom{k-s}{i+1}, \; s=0,\, 1,\ldots,\, k-1,$$

$$Y_{k,s} = a^k \sum_{i=0}^{\nu(k,s)} \left(\frac{b}{a}\right)^{2i+1} \binom{s-1}{i} \binom{k-s}{i}, \; s=1,\, 2,\ldots,\, k,$$

where $k=2,3,\ldots$; $\mu(k,s)=\min\{s-1,\ k-s-1\}$; $\nu(k,s)=\min\{s-1,k-s\}$.

It follows then that

$$\left.\begin{aligned} P_k(s) &= \binom{p+1}{2}^k \sum_{i=0}^{\mu(k,s)} \left(\frac{p-1}{p+1}\right)^{2i+2} \binom{s-1}{i} \binom{k-s}{i+1}, \\ Q_k(s) &= \binom{p+1}{2}^k \sum_{i=0}^{\nu(k,s)} \left(\frac{p-1}{p+1}\right)^{2i+1} \binom{s-1}{i} \binom{k-s}{i}. \end{aligned}\right\} \tag{2.32}$$

In particular,

$$P_k(0) = \binom{p+1}{2}^k, \; P_k(1) = (k-1)\, \frac{p-1}{p+1} \binom{p}{2} \binom{p+1}{2}^{k-1},$$

$$Q_k(0) = 0, \; Q_k(1) = \binom{p}{2} \binom{p+1}{2}^{k-1}.$$

Here $P_k(0)$ is the number of binomial coefficients not divisible by p in the Pascal triangle up through row p^k-1, i.e., the quantity $G(p^k\text{-}1,p)$ discussed previously in 2.1. The value of $P_k(s)$ is the number of coefficients exactly divisible by p^s, i.e,, the quantity $H_s(p^k-1,p)$.

Consider the Pascal triangle up through row N, where $p^k \le N \le p^{k+1}-2$. Then, putting $N=np^k+\ell$ for $0 \le n \le p-1$, $0 \le \ell \le p^k-2$, if we know the distribution of the p-index triangle for the given n,ℓ in the part of the triangle above row p^k-1, we can find the number of binomial coefficients exactly divisible by p^s, $1 \le s \le k$.

The determination of the number of coefficients exactly divisible by p^j in the Pascal triangle up through row p^r-1 is discussed in [103, 349, 369]. Let

$$S_j(r) = \sum_{a=0}^{p^r-1} \theta_j(a), \qquad S_j'(r) = \sum_{a=0}^{p^r-1} \psi_j(a),$$

where $\theta_j(a)$ is the number of binomial coefficients $\binom{a}{b}$ exactly divisible by p^j, and $\psi_j(a)$ is the number of products $(a+1)\binom{a}{b}$ exactly divisible by p^j. L. Carlitz [103] has shown, using generating functions, that

$$S_j(r) = \sum_{0<2k\le r} \binom{j-1}{k-1}\binom{r-j}{k}\binom{p}{2}^{2k}\binom{p+1}{2}^{r-2k}, \quad (0<j\le r),$$

$$S_j'(r) = \sum_{0\le 2k<r} \binom{j}{k}\binom{r-j-1}{k}\binom{p}{2}^{2k+1}\binom{p+1}{2}^{r-2k-1}, \quad (0\le j<r).$$

By a simple transformation, $S_j(r)$ and $S_j'(r)$ may be written in the form (2.32).

Analogous expressions for the number exactly divisible by p^j were introduced by D. Singmaster [349], who used the notations $A(\alpha,m)$, $B(\alpha,m)$; it can be shown, for example, that $B(j,p^r) = S_j(r)$. The problem was also studied by M. Sved and J. Pitman [369], who obtained the formulas

$$D(\alpha,m) = \sum_{h=1}^{m-\alpha}\sum_{j=0}^{\alpha-1} \binom{\alpha-1}{j}\binom{m-h-\alpha}{j}\binom{p}{2}^{2j+1}\binom{p+1}{2}^{m-h-(2j+1)}\binom{p^k}{2},$$

$$E(\alpha,m) = \sum_{j=0}^{\alpha-1} \binom{\alpha-1}{j}\binom{m-\alpha}{j+1}\binom{p}{2}^{2j+2}\binom{p+1}{2}^{m-2j-2},$$

where, in the Pascal triangle up through row $p^m - 1$, $D(\alpha,m)$ is the number of binomial coefficients divisible by p^α, and $E(\alpha,m)$ is the number exactly divisible by p^α. These formulas can be transformed into the form of $S_j(r)$ or $P_k(s)$.

The distribution of binomial coefficients exactly divisible by 2^s is shown in Figure 21, and that for 3^s in Figure 22. It is interesting to represent the p-index Pascal triangles by colors of various shades for $s=0,1,2,\ldots$; a fragment of a colored p-index triangle for $p=2$ appears in C.K. Abachiev [1,2].

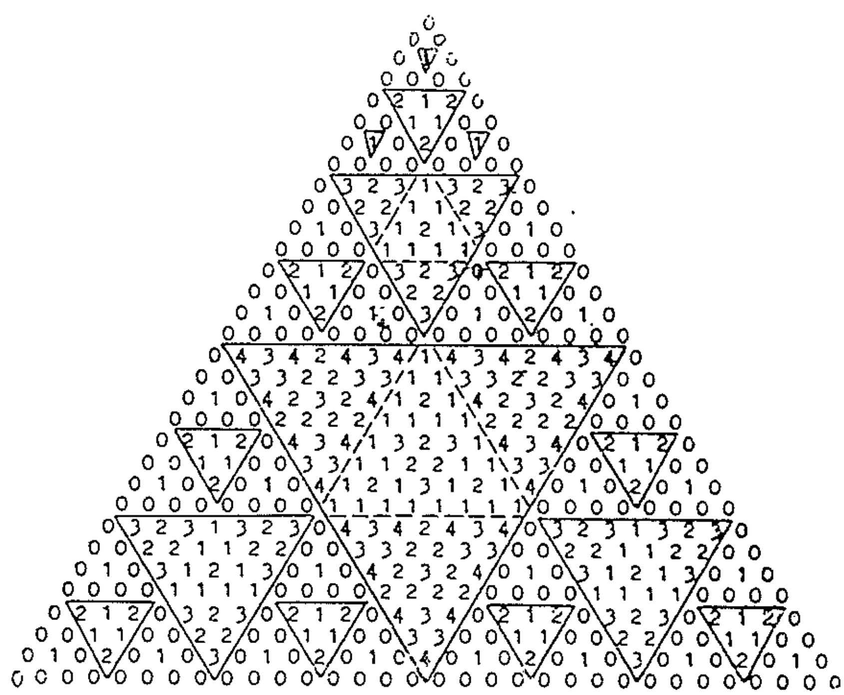


Figure 21

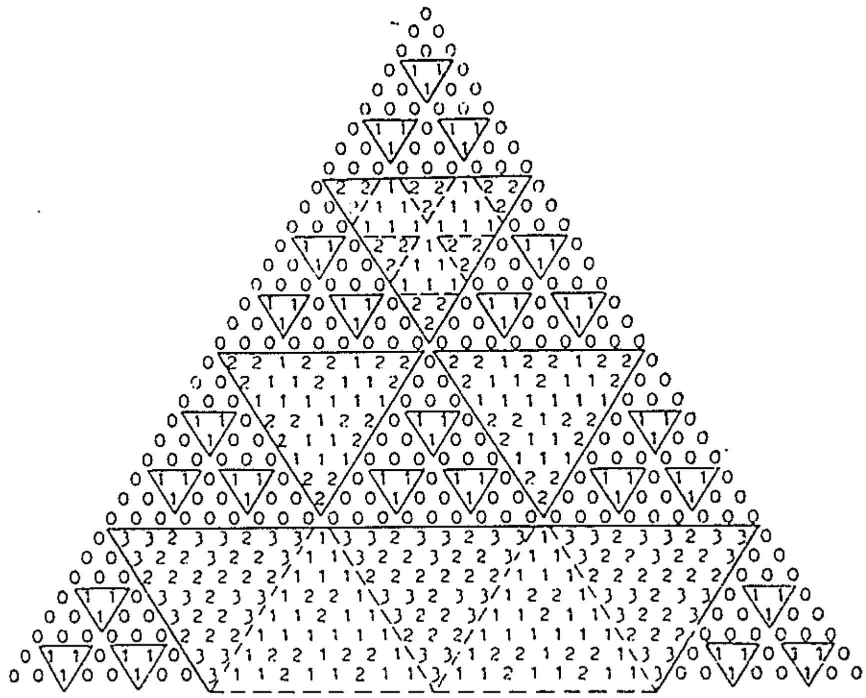


Figure 22

2.3 DIVISIBILITY OF TRINOMIAL COEFFICIENTS AND THEIR DISTRIBUTION MODULO THE PRIME p, AND ITS POWERS, IN THE PASCAL PYRAMID

To study divisibility questions for the trinomial coefficients $(n; m_1,m_2)$, discussed in 1.5, we need to extend some theorems established for the binomial coefficients in 2.1. We first note the analog of Lucas's Theorem, the generalization of which to the multi-dimensional case is given in [121].

Theorem 2.7. Let p be a prime, n,m_1,m_2 nonnegative whole numbers, $m_1 \le n$, $m_2 \le m_1$, and let the p-ary representations of these be $n=(a_r a_{r-1} \cdots a_0)_p$, $m_1=(b_r^1 b_{r-1}^1 \cdots b_0^1)_p$, $m_2=(b_r^2 b_{r-1}^2 \cdots b_0^2)_p$, where $a_r \ne 0$, $0 \le a_k < p$, $0 \le b_k^i < p$. Then

$$(n;\ m_1,m_2) \equiv \left(a_0;\ b_0^1,b_0^2\right)\left(a_1;\ b_1^1 b_1^2\right) \cdots \left(a_r;\ b_r^1 b_r^2\right) \pmod p, \tag{2.33}$$

where $(a_k;\ b_k^1 b_k^2) = 0$ if $b_k^1 > a_k$ or $b_k^2 > b_k^1$, $0 \le k \le r$.

Consider now the Pascal pyramid (cf. Fig. 15). We denote by g(n,p,3) the number of trinomial coefficients not divisible by p in the n^{th} cross section, and by h(n,p,3) the number divisible by p. Also let $g_j(n,p,3)$ denote the number of these coefficients for which $(n;\ m_1,m_2) \equiv j \pmod p$, $1 \le j \le p-1$, and let $h_v(n,p,3)$ denote the number of coefficients exactly divisible by p^v; again, these are for the n^{th} cross section. When the whole pyramid down to the n^{th} cross section, inclusive, is considered, the total numbers of coefficients satisfying the corresponding divisibility conditions will be denoted by G(n,p,3), H(n,p,3) $G_j(n,p,3)$, and $H_v(n,p,3)$.

Theorem 2.8. Let $n = (a_r a_{r-1} \cdots a_0)_p$ be the number of a cross section in the Pascal pyramid, and p a prime. Then

$$g(n,p,3) = \prod_{k=1}^{p-1} \binom{k+2}{2}^{f_k}, \tag{2.34}$$

where f_k is the number of digits k, $1 \le k \le p-1$, among $a_0, a_1, \ldots, a_r$.

The proof of Theorem 2.8 follows from Theorem 2.7. Note that if the cross section number $n = p^r$, $r = 1,2,\ldots$, then it follows from Theorem 2.8 that in this cross section only the three coefficients (n; 0,0), (n; n,0), (n; n,n) are ones, and not divisible by p.

Theorem 2.9. With the same hypothesis as Theorem 2.8, we have

$$G(n,p,3) = \frac{1}{3} \sum_{i=0}^{r} b_{r-i} \binom{p+2}{3}^{r-i} \prod_{j=0}^{i} \binom{b_{r-j}+2}{2}, \tag{2.35}$$

where $n+1 = (b_r b_{r-1} \cdots b_0)_p$.

This theorem is proved in the same way as Theorem 2.5. From Theorem 2.9, if $n=p^r-1$, we note that

$$G(p^r-1,p,3) = \binom{p+2}{3}^r. \tag{2.36}$$

Since the total number of coefficients in the n^{th} cross section is $\binom{n+2}{2}$, we have that

$$h(n,p,3) = \binom{n+2}{2} - g(n,p,3),$$

and, since the total number of coefficients in the pyramid down through the n^{th} cross section is $\binom{n+3}{3}$, by the same token,

$$H(n,p,3) = \binom{n+3}{3} - G(n,p,3).$$

Theorem 2.10. Let p be a prime. Then for $n\to\infty$

$$\lim[G(n,p,3)/H(n,p,3)] = 0.$$

The proof of this theorem uses (2.36), and is like the proof of Theorem 2.6. As for the binomial coefficients, we may formulate two principal problems for the trinomial coefficients. The first is to obtain the value of $g_j(n,p,3)$, the number of trinomial coefficients in the n^{th} cross section with residue j (mod p), and the value of $G_j(n,p,3)$, the total number of coefficients with residue j (mod p) in the whole pyramid down through the n^{th} cross section. The second problem is that of obtaining the distributions of the coefficients with respect to strict divisibility by p^v, for both the cross section and the pyramid, as above.

The solution of the first problem we may think of as depending on determining the residues of the three elements in the corners of the triangular elements in the $(n-1)^{st}$ cross section. Obtaining $g_j(n,p,3)$ and $G_j(n,p,3)$ then reduces to the formulation of the corresponding recurrence relations and their solutions. As examples, we show the distributions of the trinomials mod 2 in Figure 23, and mod 3 in Figure 24, through the 12th cross section.

An algorithm for constructing the distributions of the trinomial coefficients with respect to strict divisibility by p^v for any cross section is as follows. Let n be the cross section number, and construct the Pascal triangle for $\binom{n}{m_1}$, $0 \le m_1 \le n$. Using the algorithm of section 2.2, construct the "triangular distribution" of the binomial coefficients (for strict divisibility by p^v) for this triangle. Then, based on the equation $(n; m_1, m_2) = \binom{n}{m_1}\binom{n}{m_2}$, add to each of the elements of the rows of the "triangular distribution" the elements of the base row rotated counterclockwise by 90°. The result is the desired distribution.

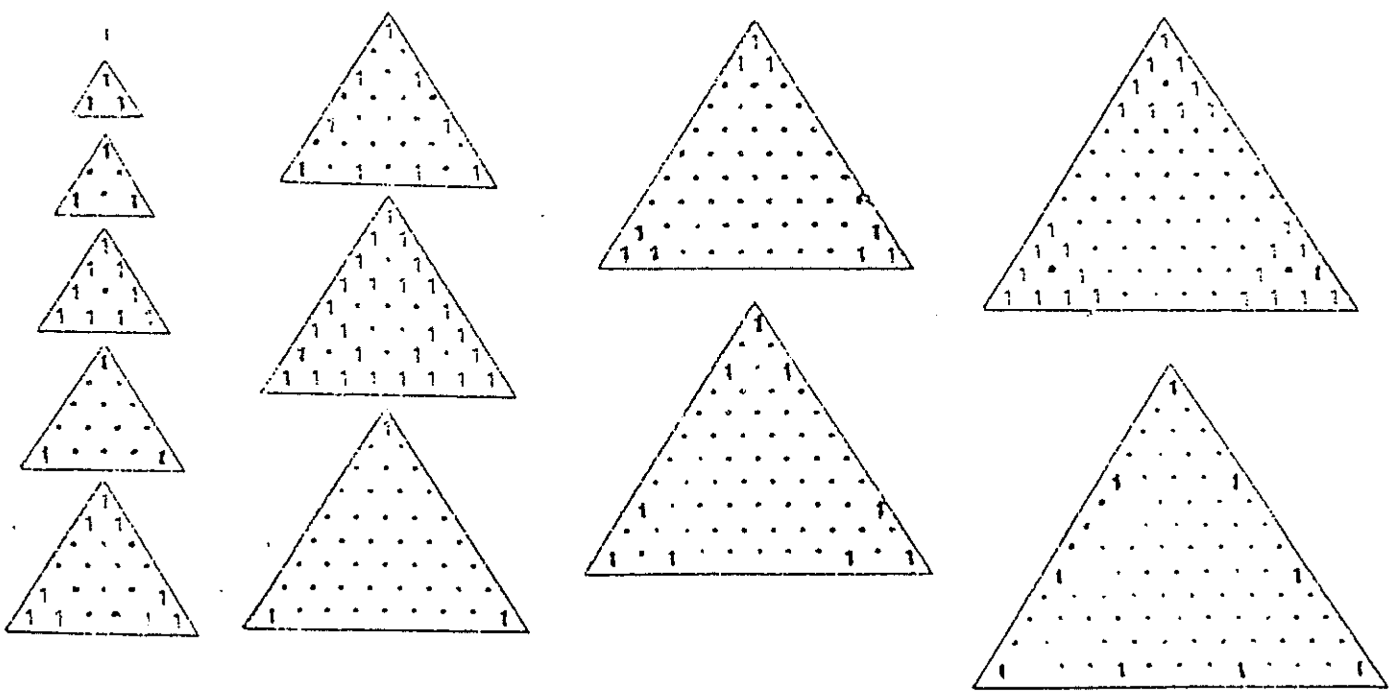


Figure 23

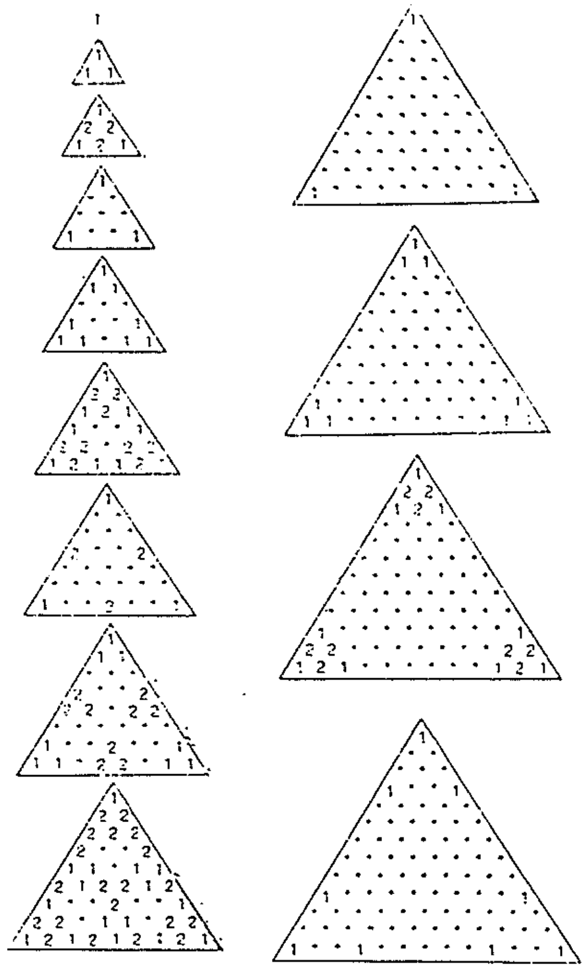
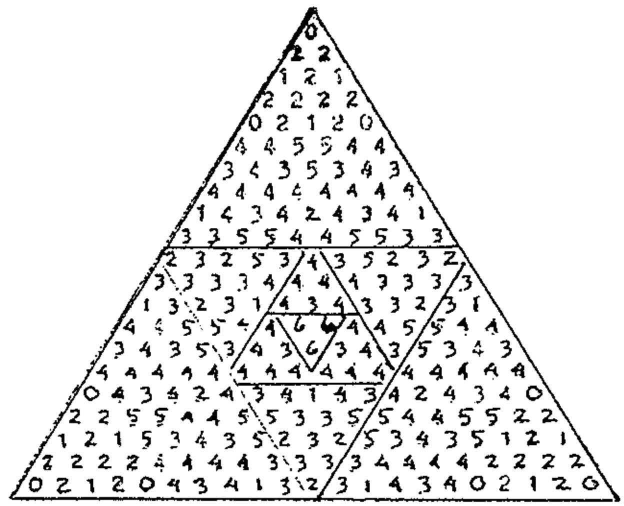


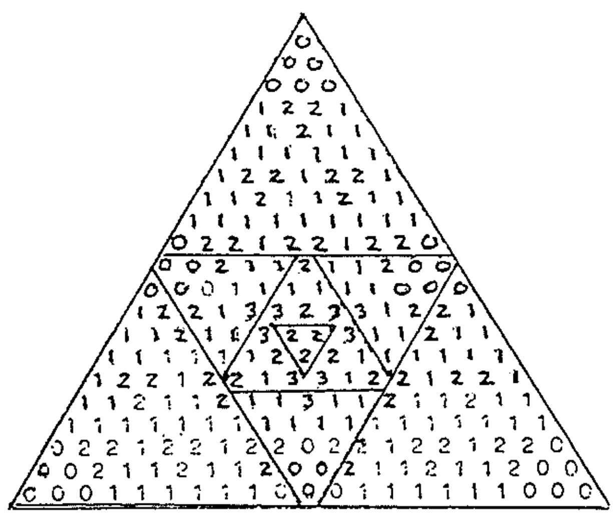
Figure 24

We note that the distribution of the trinomial coefficients with respect to divisibility by p^v in the cross section $n=p^r-1$ coincides with the corresponding distribution of the binomial coefficients in the Pascal triangle whose base is row $n=p^r-1$, for any r.

As examples we show the distributions of the trinomial coefficients with respect to strict divisibility by 2^v in Figure 25a, and by 3^v in Figure 25b, for the 20th cross section of the Pascal pyramid.



a



b

Figure 25

2.4 DIVISIBILITY OF THE MULTINOMIAL COEFFICIENTS BY THE PRIME p AND ITS POWERS

Questions of divisibility specifically for the multinomial coefficients, the determination of the number divisible, or not divisible, by a prime or power of a prime, in a cross section of the hyperpyramid or the whole hyperpyramid, and problems related to these topics are treated in [14-16, 18, 67, 121, 225, 275, 348]. The discussion of these results again begins with the extension of Lucas's Theorem [266] to the multinomial case, which is given in L.E. Dickson [121]; it will also be useful to represent the multinomial coefficients in the form (1.41), and denote them by $(n; n_1,n_2,\ldots,n_s)$.

We write the p-ary representations

$$n = (a_r a_{r-1} \cdots a_0)_p, \quad n_i = (b_r^i b_{r-1}^i \cdots b_0^i)_p, \tag{2.37}$$

where $a_r \neq 0$, $0 \leq a_k < p$, $0 \leq b_k^i < p$, $0 \leq k \leq r$, $1 \leq i \leq s$.

<u>Theorem 2.11</u>. Let p be a prime, n and n_i nonnegative whole numbers with p-ary representations (2.37). Then

$$(n;\ n_1,n_2,\ldots,n_s) \equiv \prod_{k=0}^{r} \left(a_k;\ b_k^1 b_k^2 \cdots b_k^s\right) \pmod{p}, \tag{2.38}$$

in which $(a_k;\ b_k^1 b_k^2 \cdots b_k^s) = 0$ if $b_k^1 + b_k^2 + \cdots + b_k^s \neq a_k$.

It follows from Theorem 2.11 that $(n; n_1,\ldots,n_s) \not\equiv 0 \pmod{p}$ if and only if $b_k^1 + b_k^2 + \cdots + b_k^s = a_k$ for all values of k.

Known and new methods for determining the highest power of the prime p which divides a multinomial coefficient are presented in detail in the article of R.J. Martin and G.L. Mullen [275].

Theorem 2.12. If p is a prime, the multinomial coefficient $(n; n_1,\ldots,n_s)$ is divisible by p^ν if and only if

$$E(p,n!) - \sum_{i=1}^{s} E(p,n_i!) \geq \nu,$$

where E(p,n!) denotes the highest power of p which divides n!.

Let n and n_i be written in p-ary form (2.37), and form the system of equations:

$$\left.\begin{array}{l} b_0^1 + \cdots + b_0^s = \varepsilon_0 p + a_0, \\ \varepsilon_0 + b_1^1 + \cdots + b_1^s = \varepsilon_1 p + a_1, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \varepsilon_{r-2} + b_{r-1}^1 + \cdots + b_{r-1}^s = \varepsilon_{r-1} p + a_{r-1}, \\ \varepsilon_{r-1} + b_r^1 + \cdots + b_r^s = a_r. \end{array}\right\} \quad (2.39)$$

Then R.D. Fray in [143] proved the following result.

Theorem 2.13. The highest power of the prime p which divides $(n; n_1,n_2,\ldots,n_s)$ is given by

$$\nu = \varepsilon_0 + \varepsilon_1 + \cdots + \varepsilon_{r-1},$$

where the values $\varepsilon_0,\ldots,\varepsilon_{r-1}$ are chosen from among $0,1,\ldots,s-1$ so as to satisfy (2.39).

If we put $S(n) = a_0+a_1+\cdots+a_r$ and $S(n_i) = b_0^i+b_1^i+\cdots+b_r^i$, then it may be shown [225] that

$$\nu = \frac{1}{p-1}\left[S(n_1)+S(n_2)+\cdots+S(n_s)-S(n)\right]. \tag{2.40}$$

In [275] Martin and Mullen worked out a new, more effective method for calculating ν, based on obtaining the residues of $n_1,n_2,\ldots,n_s$ modulo distinct powers of p. Denote by n_i^j the residue of n_i (mod p^j), for $1\le i\le s$, $1\le j\le h$, where $p^h\le n<p^{h+1}$, $h=[\log n/\log p]$. Then the following theorem is from [275].

Theorem 2.14. The multinomial coefficient $(n; n_1,n_2,\ldots,n_s)\equiv 0 \pmod{p^\nu}$ if and only if

$$\sum_{j=1}^{h} \frac{1}{p^j}\left(n_1^j+n_2^j+\cdots+n_s^j\right) \ge \nu.$$

It follows from Theorem 2.14 that for $\nu=1$, $(n;n_1,\ldots,n_s)\equiv 0 \pmod p$ if and only if for some value of j, $(n_1^j+n_2^j+\cdots+n_s^j)\ge p^j$.

D. Singmaster [348] discussed the question of the least value of n for which $(n; n_1\ldots,n_s)$ is divisible by p^ν, and obtained the following theorem.

Theorem 2.15. Let the power ν of the prime $p\ge s$ be represented in the form $\nu=a(s-1)+b$, where $0<b\le s-1$. Then the least value of n for which $(n; n_1,\ldots,n_s)$ is divisible by p^ν is $n=bp^{a+1}$.

He also considered [352] various properties of the multinomials and proved the following result.

Theorem 2.16. The multinomial coefficient $(n; n_1,\ldots,n_s)$ in which n is strictly divisible by p^v, and n_i is strictly divisible by p^{t_i}, is divisible by p^{s-t} if $t \le v$, where $t=\min\{t_i\}$.

The problems of determining the number of multinomial coefficients not divisible by p, or divisible by p^v, are discussed in [15, 16, 18, 225]; in these problems it is sometimes convenient to use the form (1.43) for the multinomial coefficients.

Let g(n,p,s) be the number of multinomial coefficients $(n; n_1,\ldots,n_s)$ not divisible by p in the n^{th} cross section of the Pascal hyperpyramid, and h(n,p,s) the number divisible by p. Also, let $g_j(n,p,s)$ be the number congruent to j (mod p); then we have

$$g(n,p,s) = g_1(n,p,s) + g_2(n,p,s) + \cdots + g_{p-1}(n,p,s).$$

Likewise

$$h(n,p,s) = h_1(n,p,s) + h_2(n,p,s) + \cdots + h_q(n,p,s),$$

where $h_v(n,p,s)$ denotes the number of multinomial coefficients $(n; m_1,m_2,\ldots,m_{s-1})$ in the n^{th} cross section divisible by p^v, and $q=\max\{v\}$. For the total numbers of coefficients in the hyperpyramid satisfying the corresponding conditions we use the notations G(n,p,s), H(n,p,s), $G_j(n,p,s)$, and $H_v(n,p,s)$. Then

$$G(n,p,s) = G_1(n,p,s) + \cdots + G_{p-1}(n,p,s),$$

$$H(n,p,s) = H_1(n,p,s) + \cdots + H_q(n,p,s).$$

Theorem 2.17. Let $n = (a_r a_{r-1}\ldots a_0)_p$ be a cross section number in the hyperpyramid, and p a prime. Then

$$g(n,p,s) = \prod_{k=1}^{p-1} \binom{k+s-1}{s-1}^{f_k}, \tag{2.41}$$

where f_k is the number of digits k among $a_r,\ldots,a_0$. In the proof of Theorem 2.17 we use Theorem 2.11 and a corresponding transformation; as a result, (2.41) differs from the representation obtained in [18, 225].

In [225], F.T. Howard extended the results in [100, 103, 222, 223] and obtained formulas for the quantities $\theta_0(s,n)$, $\theta_1(s,n)$, $\theta_2(s,n)$, where these are the numbers of multinomial coefficients strictly divisible by p^0, p^1, p^2. For $\theta_\nu(s,n)$, $\nu>2$, he constructed the corresponding generating function and found explicit expressions for $\theta_\nu(s,n)$ for certain values of n.

Denote by C(i) the coefficients in the expansion of $(1+x+x^2+\cdots+x^{p-1})^s$ in powers of x, where p is a given prime, and s is the "dimension" of the multinomial $(n; n_1,\ldots,n_s)$. Howard proved that

$$C(a+bp) = \sum_{k=0}^{b} (-1)^k \binom{s}{k} \binom{s+a+bp-kp-1}{s-1}. \tag{2.42}$$

In (2.42), which is analogous to (1.16) for generalized binomial coefficients, a and b satisfy $0\le a<p$, $0\le b$. Using only the coefficients C(i) and the p-ary representation $n=(a_r a_{r-1}\cdots a_0)_p$, Howard [225] proved a theorem containing the following formulas:

$$\theta_0(s,n) = C(a_0)C(a_1)\ldots C(a_r),$$

$$\theta_1(s,n) = \sum_{i=0}^{r-1} C(a_0)\ldots C(a_{i-1})C(a_i+p)C(a_{i+1}-1)C(a_{i+2})\ldots C(a_r),$$

$$\theta_2(s,n) = \sum_{i=0}^{r-2} C(p+a_i)C(p+a_{i+1}-1)C(a_{i+2}-1)A_i + \sum_{i=0}^{r-1} C(2p+a_i)C(a_{i+1}-2)B_i$$

$$+ \sum_{i=0}^{r-3} \sum_{k=i+2}^{r-1} C(p+a_i)C(a_{i+1}-1)C(p+a_k)C(a_{k+1}-1)H_{i,k}.$$

The values A_i, B_i, $H_{i,k}$, mentioned in [225], may be written in the form

$$A_i = P_n/Q_iC(a_{i+2}), \qquad B_i = P_n/Q_i, \qquad H_{i,k} = P_n/Q_iQ_k,$$

where

$$P_n = \prod_{j=0}^{r} C(a_j), \qquad Q_i = C(a_i)C(a_{i+1}), \qquad Q_k = C(a_k)C(a_{k+1}).$$

Also given is $\theta_v(n)$ for the values $n=a+bp$, $n=a+p^2$, $n=a+2p^2$, $n=a+bp+p^2$.

N.A. Volodin [18] developed a formula for the number of multinomial coefficients not divisible by p, and for the number divisible by p, in the form of a sum of products of binomial coefficients. Methods for obtaining the number of multinomial coefficients not divisible by p in the Pascal hyperpyramid, are given in [15, 16, 18].

Theorem 2.18. Let the base of the Pascal hyperpyramid of dimension s be the n^{th} cross section, and p a prime. Then

$$G(n,p,s) = \frac{1}{s}\sum_{i=0}^{r} b_{r-i}\binom{p+s-1}{s}^{r-i}\prod_{j=0}^{i}\binom{b_{r-j}+s-1}{s-1}, \tag{2.43}$$

where $n+1=(b_rb_{r-1}\cdots b_0)_p$.

The proof of Theorem 2.18 is analogous to the proofs of Theorem 2.5 and Theorem 2.9. If $n=p^r-1$, $r\geq 1$, from (2,43) we find that

$$G(p^r-1,\ p,\ s) = \binom{p+s-1}{s}^r. \tag{2.44}$$

The total number of multinomial coefficients in the n^{th} cross section of the Pascal hyperpyramid of dimension s is $\binom{n+s-1}{s-1}$. Thus,

$$h(n,p,s) = \binom{n+s-1}{s-1} - g(n,p,s).$$

Likewise,

$$H(n,p,s) = \binom{n+s}{s} - G(n,p,s),$$

since $\binom{n+s}{s}$ is the total number of coefficients in the Pascal hyperpyramid of dimension s and whose base is numbered n.

Theorem 2.19. If p is a prime, then for $n\rightarrow\infty$,

$$\lim[G(n,p,s)/H(n,p,s)] = 0.$$

The proof of Theorem 2.19 is like those of Theorem 2.6 and Theorem 2.10, and uses (2.44).

The problems of determining $G_j(n,p,s)$ and $H_v(n,p,s)$ (using the usual notation) may also be formulated for the multinomial coefficients.

2.5 GREATEST COMMON DIVISORS AND LEAST COMMON MULTIPLES OF BINOMIAL COEFFICIENTS. FACTORIZATION

We consider here some questions associated with the greatest common divisor (GCD) and the least common multiple (LCM) of those binomial coefficients which are arranged in some definite pattern in the Pascal triangle.

On the basis of the known equation

$$\binom{n-1}{m}\binom{n}{m-1}\binom{n+1}{m+1} = \binom{n-1}{m-1}\binom{n}{m+1}\binom{n+1}{m}$$

H.W. Gould [159] stated the conjecture that

$$GCD\left\{\binom{n-1}{m}, \binom{n}{m-1}, \binom{n+1}{m+1}\right\} = GCD\left\{\binom{n-1}{m-1}, \binom{n}{m+1}, \binom{n+1}{m}\right\},$$

which was proved in various ways by A.P. Hillman and V.E. Hoggatt [187], D. Singmaster [350], and E.G. Straus [364].

C.T. Long [262] discussed the triangle ∇_n of binomial coefficients of the form

$$\begin{matrix} \binom{n}{1} & \cdots & \binom{n}{n-1} \\ \cdot & & \cdot \\ & \cdot \quad \cdot & \\ & \cdot & \\ & \binom{2n-2}{n-1} & \end{matrix} .$$

If we denote the GCD of all elements in this triangle by d, and the GCD of the three corner elements by D, he proved that: $d=D=p$ if $n=p$; $d=p$, $D=p^s$ if $n=p^s$ $(s>1)$; $d=1$, $D=n$ for all $n\neq p^s$, where p is a prime and s is a whole number.

Let $n=p_1^{s_1}p_2^{s_2}\cdots p_r^{s_r}$ and k be whole numbers satisfying $1\leq k\leq \min\{p_i^{s_i}\}$, $1\leq i\leq r$, and denote by m the product of all divisors of n of the form p^s, where $p^s\leq k\leq p^{s+1}$. T. Tonkov [49] proved that

$$GCD\left\{\binom{n}{1},\binom{n}{2},\ldots,\binom{n}{m}\right\}=\frac{n}{m}.$$

In [345] G.J. Simmons showed that there are infinitely many values of m for which m! is a divisor of $\binom{n}{m}$, but m!p for $p\leq m$ does not divide this coefficient. He further proved that for given N,m, there exist infinitely many n such that $GCD\left\{\binom{n}{m}, N\right\}=1$.

J. Albree [56] proved that for $1\leq m\leq n-1$, if $GCD\{m,p\}=1$, then $GCD\left\{\binom{n}{m}\right\}=p^s$, where s is the highest power for which p^s divides n.

Let

$$a=\binom{n-1}{m-1}\binom{n+1}{m},\quad b=\binom{n+1}{m}\binom{n}{m+1},\quad c=\binom{n}{m+1}\binom{n-1}{m-1},$$
$$d=\binom{n}{m-1}\binom{n+1}{m+1},\quad e=\binom{n+1}{m+1}\binom{n-1}{m},\quad f=\binom{n-1}{m}\binom{n}{m-1}.$$

Then H.M. Edgar [128] proved that

$$LCM\{a,b,c\}=LCM\{d,e,f\}.$$

In [393], I.S. Williams showed that for powers of primes $p_i^{s_i}$, where $p_i^{s_i} \le n+1 \le p_i^{s_i+1}$,

$$LCM\left\{\binom{n}{m}\right\} = \left(\prod_i p_i^{s_i}\right) / n+1,$$

where the product is taken over all primes $p_i \le n+1$.

R. Meynieux [280] discussed questions connected with the LCM of binomial coefficients, and with determining the powers of primes which occur in the factorizations into prime factors of binomial coefficients. A typical result is as follows. Let λ_p be the power of the prime p in the factorization of $\binom{n}{m}$; let $\mu_p(n) = \sup_m \lambda_p$; for $m \le n/2$ let $\rho_p(n)$ be the largest prime such that $\lambda_p = \mu_p$; and let $\rho(n) = \inf \rho_p(n)$ for p belonging to the set of primes occurring in the factorization. Then $\rho(n) \ge (n-1)/3$ and $\lim[\rho(n)/n] = 1/3$ for $n \to \infty$.

Problems connected with the factorization into prime factors of the binomial coefficients, asymptotic estimates, and other topics are treated in the works of P. Erdos [132, 133], P. Erdös and R. Graham [134], P. Erdös, H. Gupta, and S.P. Khare [135], H. Gupta and S.P. Khare [168], and S.P. Khare [237], among others. Omitting details, we summarize three of these papers. Khare [135] proves a theorem on the factorization of binomial coefficients and gives tables of such factorizations for special conditions imposed on n and m. Included also is a discussion of the case where $\binom{n}{m}$ has m prime factors, e.g., $\binom{4}{2} = 2 \cdot 3$, $\binom{10}{4} = 2 \cdot 3 \cdot 5 \cdot 7$, and so on. In [168] it is shown that $\binom{n^2}{n}$ is greater than the product of the first n primes for $2 < n < 1794$, and less than this product for $n \ge 1794$. And [237] gives tables of factors of n! for $n \le 1000$.

The factorizations of the binomial coefficients $\binom{n}{m}$ up through n=54 are given in the book of T.M. Green and C.L. Hamberg [162]. Matters related one way or another to this topic are also discussed in [108, 130, 140, 166, 169, 279].

CHAPTER 3

DIVISIBILITY AND DISTRIBUTION MODULO p IN GENERALIZED PASCAL TRIANGLES, AND FIBONACCI, LUCAS, AND OTHER SEQUENCES

In this chapter we consider divisibility of generalized binomial coefficients $\binom{n}{m}_s$. We give the analog of Lucas's Theorem, and prove some theorems on the divisibility by a prime p of generalized binomial coefficients in a given row of the generalized Pascal triangle of order s for $s=3$ and $p=2,3$. We also discuss the distribution of these coefficients for the moduli 2 and 3, and the situation for $n\to\infty$.

Divisibility and distributions for a prime modulus are also considered for Fibonacci, Lucas, and other sequences, as well as periodicity of these sequences with respect to a prime modulus.

3.1 DIVISIBILITY AND THE DISTRIBUTION MODULO p OF GENERALIZED BINOMIAL COEFFICIENTS

In section 1.3 we discussed generalized Pascal triangles of order s, the elements of which are the generalized binomial coefficients $\binom{n}{m}_s$, and considered their recurrence and other relations analogous to those of the binomial coefficients. Not a great deal of work has appeared on questions of divisibility and distribution of these coefficients, but we first turn to the analog of Lucas's Theorem, and some related results, given by R.C. Bollinger and C.L. Burchard [81].

Theorem 3.1. Let p be a prime, and n and m nonnegative whole numbers, $0 \le m \le n(s-1)$ with p-ary representations $n=(a_r a_{r-1} \cdots a_0)_p$, $m=(b_r b_{r-1} \cdots b_0)_p$, where $a_r \ne 0$, $0 \le a_k$, $b_k < p$. Then

$$\binom{n}{m}_s \equiv \sum_{i_k} \prod_{k=0}^{r} \binom{a_k}{i_k}_s \pmod{p}, \tag{3.1}$$

where the summation is carried out over all indicies i_k for which $i_0+i_1p+i_2p^2+\cdots+i_rp^r=m$, $0 \le i_k \le (s-1)a_k$.

We note that if the latter two conditions are not satisfied, then $\binom{n}{m}_s \equiv 0 \pmod{p}$. The authors prove this theorem and give some related examples in [81]. They also discuss the question of the number $N_s(n,p)$ of generalized binomial coefficients $\binom{n}{m}_s \not\equiv 0 \pmod{p}$ for the two cases $s=p$ and $s=p^v$, where p is a prime.

Theorem 3.2. Let $(p-1)n$ have the p-ary representation $(c_r c_{r-1} \cdots c_0)_p$. Then in the generalized Pascal triangle of order p, the number of coefficients in row n which are not divisible by p is

$$N_p(n,p) = \prod_{k=0}^{r} (1+c_k). \tag{3.2}$$

From Theorem 3.2 it also follows that if $s=p^v$, then

$$N_{p^v}(n,p) = N_p[n(p^v-1)/p-1,p] = N_2[n(p^v-1),p], \tag{3.3}$$

and, further, that for $n \to \infty$ almost all coefficients $\binom{n}{m}_s$ are divisible by p.

Theorem 3.3. Let r be a natural number. Then in row $n=p^r$ of the generalized Pascal triangle of order s, we have

$$\binom{n}{m}_s \equiv 1 \pmod p,\ m = ip^r, \qquad \binom{n}{m}_s \equiv 0 \pmod p,\ m \neq ip^r. \tag{3.4}$$

The proof of this theorem is based on the analog of Lucas's Theorem in polynomial form, and the result is used in the construction of fractal generalized Pascal triangles.

Theorem 3.4. Let p be a prime. In the multinomial coefficient $(n; m_1, m_2, \ldots, m_{s-1})$ let n and m_k be written as

$$n = (a_r a_{r-1} \ldots a_1 a_0)_p, \quad m_k = \left(b_r^{(k)} b_{r-1}^{(k)} \ldots b_1^{(k)} b_0^{(k)}\right)_p,$$

where $a_r \neq 0$, $0 \leq a_i < p$, $0 \leq b_i^{(k)} \leq p$. Then

$$\binom{n}{m}_s \equiv \sum \prod_{i=0}^{r} \left(a_i;\ b_i^{(1)},\ b_i^{(2)},\ \ldots,\ b_i^{(s-1)}\right) \pmod p,$$

where the summation is over all $b_i^{(k)}$ satisfying

$$b_i^{(1)} + b_i^{(2)} + \cdots + b_i^{(s-1)} = m.$$

Unlike the Pascal triangle, in which the rule for forming the binomial coefficients mod p, and their distribution, depends only on p, the distribution of the generalized binomial coefficients depends on both p and s. Thus, we will only consider here the distribution of these coefficients for $s=3$, $p=2,3$; the method itself may be used for other values of s and p.

Let $p=2$. We introduce the following definition.

Definition 3.1. Let the natural number n be written in binary form. We will say that n contains a block of type k - denoted by $<1>_k$ - if its binary form contains a string of k consecutive ones which has at least one zero on both the left and the right.

Clearly, any natural number n written in binary form consists of $q_k \geq 0$ blocks of type k for k=1,2,...,t, where t=t(n). For example, the binary form of n=315837 is 1011001000110111101, which contains $q_1=3$ blocks of type 1, $q_2=2$ of type 2, $q_3=0$ of type 3, and $q_4=1$ of type 4. We note also that the binary forms of distinct natural numbers may contain identical numbers of the same kinds of blocks.

Theorem 3.5. In the generalized Pascal triangle of order 3, let the row number n be written in binary form, in which there are $q_k \geq 0$ blocks of type k, $1 \leq k \leq t$. Then the number of odd trinomial coefficients in row n is given by

$$P_1(n) = U_1^{q_1} U_2^{q_2} \dots U_t^{q_t}, \quad U_k = \tfrac{1}{3}[2^{k+2}-(-1)^k]. \tag{3.5}$$

The proof of this theorem follows from Theorem 3.4 and the solution of the recurrence relation $U_k=U_{k-1}+2U_{k-2}$ with the initial conditions $U_0=1$, $U_1=3$; it is not difficult to show that the solution is given by the expression for U_k in (3.5).

The total number of coefficients in row n is 2n+1, so that the number of even coefficients is $P_2(n)=(2n+1)-P_1(n)$. And, if there are N rows in the generalized Pascal triangle, there will be a total of $(N+1)^2$ coefficients, and the total number of even coefficients will be given by

$$Q_2(n) = (N+1)^2 - \sum_{n=0}^{N} P_1(n).$$

If we apply the elementary rule defining evenness/oddness to the sums of three terms occurring in the recurrence relation for the trinomial coefficients, and write out the triangle, we will have the distribution of even and odd coefficients in the Pascal triangle of order 3. We show this in Figure 26 for $N=2^4+1=17$ rows, where the odd coefficients are denoted by ones and the even coefficients by dots.

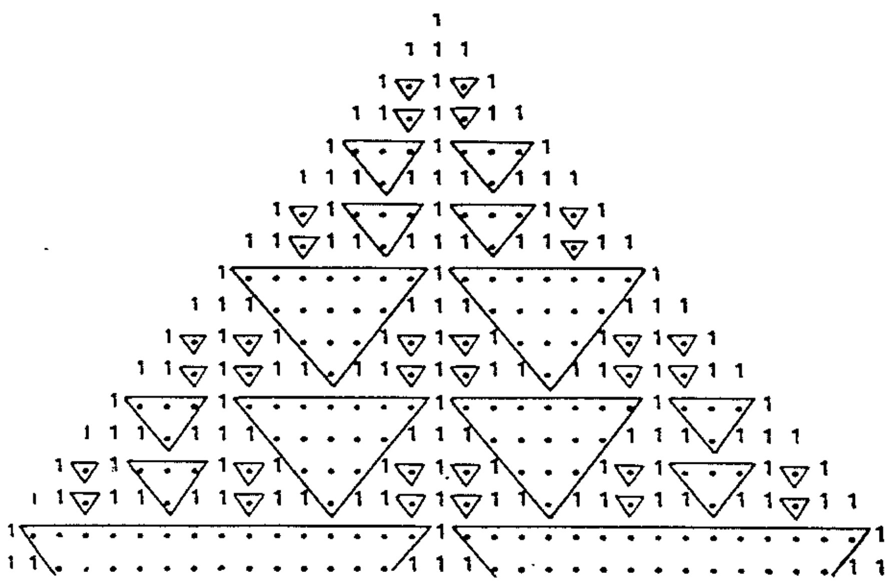


Figure 26

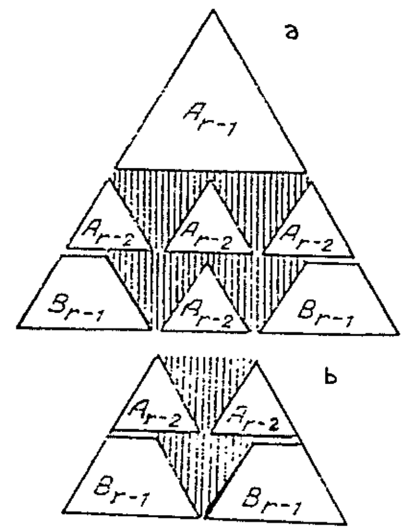


Figure 27

Let us denote by A_r the isosceles triangle (Figure 27a) whose altitude, measured by the number of rows from base to vertex inclusive, is $h_r=2^r$; the length of whose base is $d_r=2^{r+1}-1$; and the number of whose base row is $n=h_r-1=2^r-1$, $r\geq 0$.

Also, denote by B_r the isosceles trapezoid (Figure 27b) whose altitude is $h_r=2^{r-1}$; whose upper and lower bases have lengths $d_U=2^{r-1}+1$ and $d_L=2^r+2^{r-1}-1$; and the number of whose base row is $n=2^{r-1}-1$, $r\geq 1$.

In Figure 26, it is not difficult to see the triangles $A_0,\ldots,A_4$, and the trapezoids $B_1,\ldots,B_3$.

For the following theorem, we will need the Fibonacci numbers, which may be calculated by the known formula of Binet,

$$F_m = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^m - \left(\frac{1-\sqrt{5}}{2}\right)^m\right],$$

or expressed by means of the binomial coefficients

$$F_m = \sum_{i=0}^{\left[\frac{m}{2}\right]} \binom{m-i}{i}, \quad F_1 = F_2 = 1. \tag{3.6}$$

We note that the Binet formula [392] may be extended to the case of the sequence $\{G_m\}$, where

$$G_1 = G_2 = 1, \quad G_{m+2} = G_{m+1} + \frac{p-1}{4}G_m \quad (m \geq 1),$$

and

$$G_m = \frac{1}{\sqrt{p}}\left[\left(\frac{1+\sqrt{p}}{2}\right)^m - \left(\frac{1-\sqrt{p}}{2}\right)^m\right].$$

Theorem 3.6. Let the row number of the base of the generalized Pascal triangle of order 3 be $n=2^r-1$. Then for any natural number r, the number of odd trinomial coefficients in this triangle is given by

$$Q_1(2^r-1) = 2^r F_{r+2}. \tag{3.7}$$

Proof: It follows from Theorem 3.3 that in row $n=2^{r-1}$, which lies inside A_r, there are three odd coefficients: $\binom{n}{m}_3$ for $m=0,2^{r-1},2^r$. Each of these gives rise to a triangle A_{r-2}, the base of which has length $2^{r-2}-1$ and lies on the row $n=2^{r-1}+2^{r-2}-1$. The following row, $2^{r-1}+2^{r-2}$, according to Theorem 3.5, has five odd coefficients: for $m=0$, 2^{r-2}, $2^{r-1}+2^{r-2}$, 2^r+2^{r-2}, 2^r+2^{r-1}. From this, we can establish that the coefficient for $m=2^{r-1}+2^{r-2}$ gives rise to a triangle A_{r-2}, and each pair of coefficients for $m=0$, 2^{r-2} and $m=2^r+2^{r-2}$, 2^r+2^{r-1}, gives rise to a trapezoid B_{r-1}. Thus, the triangle A_r can be written as a "geometric" sum of the triangle A_{r-1} with base row $2^{r-1}-1$, four triangles A_{r-2}, and two trapezoids B_{r-1} (Figure 27a). Likewise, B_r is the geometric sum of two trapezoids B_{r-1} and two triangles A_{r-2} (Figure 27b).

If we denote by a_r the number of odd coefficients in A_r, and by b_r the number in B_r, we can, on the basis of the arguments given above, write the system of recurrence relations

$$\left.\begin{aligned} a_r &= a_{r-1} + 4a_{r-2} + 2b_{r-1} \\ b_r &= 2b_{r-1} + 2a_{r-2} \end{aligned}\right\} \tag{3.8}$$

where $r\geq 2$ and the initial data is $a_0=1$, $a_1=4$, $b_1=2$, determined by the number of odd coefficients in A_0, A_1, and B_1. The solution of (3.8) may be expressed in terms of the Fibonacci numbers as

$$a_r = 2^r F_{r+2}, \quad b_r = 2^{r-1} F_{r+2}. \tag{3.9}$$

Substituting (3.9) in (3.8), and using the fact that $F_{r+2}=F_{r+1}+F_r$, it is easy to show that (3.9) is correct. Thus, $Q_1=a_r=2^r F_{r+2}$, and the proof is complete.

If $n=2^r-1$, then the total number of trinomial coefficients in triangle A_r is 2^{2r}, and thus the number of even coefficients, using (3.7), is given by

$$Q_2(2^r-1) = 2^{2r} - 2^r F_{r+2}. \tag{3.10}$$

Then, using (3.7) and (3.10) we can show that from some r onward $Q_2(2^r-1) >> Q_1(2^r-1)$. Thus, $Q_2(2^4-1)=128$, $Q_1(2^4-1)=128$; $Q_2(2^7-1)=12032$, $Q_1(2^7-1)=4352$; $Q_2(2^{10}-1)=1048576$, $Q_1(2^{10}-1)=147456$.

Theorem 3.7. For $n \to \infty$, $\lim [Q_1(n)/Q_2(n)]=0$.

Proof: Since $Q_1(n)$ and $Q_2(n)$ are nondecreasing functions of n, then for $2^r-1 \le n < 2^{r+1}-1$,

$$Q_1(n)/Q_2(n) < Q_1(2^r-1)/Q_2(2^r-1).$$

Consequently,

$$\lim_{n\to\infty} Q_1(n)/Q_2(n) < \lim_{r\to\infty} Q_1(2^{r+1}-1)/Q_2(2^r-1).$$

Using (3.7) and (3.10) we find that

$$\begin{aligned} Q_1(2^{r+1}-1)/Q_2(2^r-1) &= 2^{r+1}F_{r+3}/(2^{2r}-2^rF_{r+2}) \\ &= 2F_{r+3}/(2^r-F_{r+2}) \\ &< F_{r+3}/(2^r-F_{r+3}) \\ &= 1/((2^r/F_{r+3})-1). \end{aligned}$$

But for $r\to\infty$, $\lim 2^r/F_{r+3}=\infty$, and so

$$\lim_{n\to\infty} Q_1(n)/Q_2(n) < \lim_{r\to\infty} Q_1(2^{r+1}-1)/Q_2(2^r-1) = 0,$$

which proves the theorem.

We consider now the distribution of the trinomial coefficients in the generalized Pascal triangle of order 3 with respect to the modulus $p=3$.

Definition 3.2. Let the natural number n be written in ternary form. We will say that n contains a 1-block of type k - denoted by $<1>_k$ - if its ternary form contains a string of k consecutive ones which is bounded on the left by at least one zero or one two, and on the right by at least one zero.

Definition 3.3. With n in ternary form, as above, we will say that n contains a 2-block of type i - denoted by $<2>_i$ - if it contains a string of i consecutive twos, ignoring imbedded ones, which is bounded on the left and right by at least one zero or by ones.

In connection with definition 3.2, note that strings of consecutive ones which precede twos are not considered. Thus, in $n=(211122)_3$ we ignore the three ones, and count what remains as a block of type $<2>_3$.

Example: Suppose n in ternary form is $n=20122102112022221101221$. To count blocks in n we first exclude ones which precede twos. As a result, we find the block form of n to be

$$<n> = <202210220222110221>,$$

and say that n contains two $<1>_1$ blocks, one $<1>_2$ block, one $<2>_1$ block, three $<2>_2$ blocks, and one $<2>_3$ block.

Theorem 3.8. In the generalized Pascal triangle of order 3, let the row number n when written in ternary form consist of $p_k \geq 0$ blocks $<1>_k$, $1 \leq k \leq s$, and q_i blocks $<2>_i$, $1 \leq i \leq t$. Then in row n the number of trinomial coefficients not divisible by three is

$$N_{1,2}(n) = \prod_{k=1}^{s} V_k^{p_k} \prod_{i=1}^{t} W_i^{q_i}, \quad V_k = 3^k, \quad W_i = 3^i + 3^{i-1}. \tag{3.11}$$

To prove Theorem 3.8, as in Theorem 3.5 we use the three-dimensional analog of Lucas's Theorem, and find the expressions for V_k and W_i as the solutions of the corresponding recurrence relations.

In Figure 28 we have written out, using the modulus p=3, the rows of the triangle up through row N=15, in which the coefficients not divisible by three appear as 1's and 2's, and those divisible by three are represented by dots.

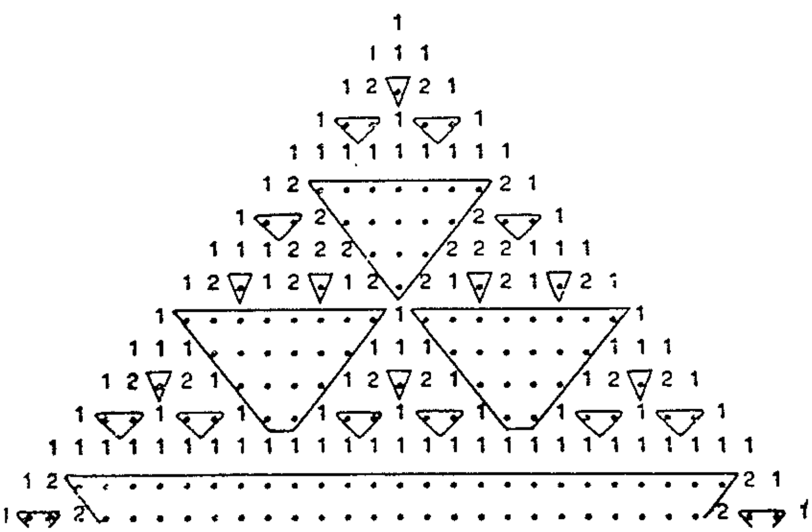


Figure 28

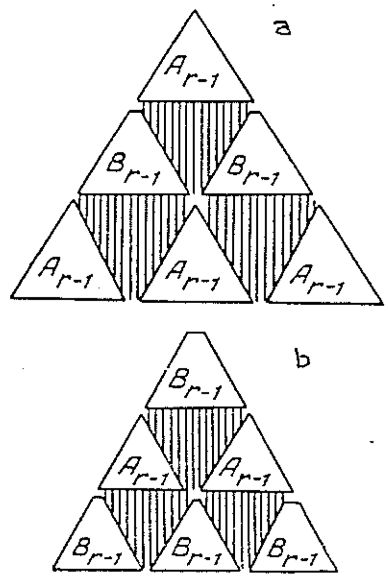


Figure 29

Denote by A_r the isosceles triangle (Figure 29a) whose height, width of base, and row number of the base are, respectively,

$$h_r = \frac{1}{2}(3^r+1), \quad d = 3^r, \quad n = \frac{1}{2}(3^r-1), \quad r \geq 0.$$