CHAPTER 1

THE PASCAL TRIANGLE AND ITS PLANAR AND SPATIAL GENERALIZATIONS

In this chapter we outline some of the history of the Pascal triangle and the binomial coefficients, and also describe some modern results obtained by mathematicians in recent decades. We consider, as well, generalized Pascal triangles of s^{th} order, Pascal pyramids and hyperpyramids, and triangles associated with the Fibonacci, Lucas, and Catalan numbers. Finally, we discuss generalized binomial coefficients of s^{th} order, multinomial coefficients, and Gauss-, Fibonacci-, and other analogs of the binomial coefficients.

1.1 THE PASCAL TRIANGLE AND ITS PROPERTIES

One of the most familiar objects in the history of mathematics is the so-called "arithmetical triangle", more commonly known today as the Pascal triangle in honor of the seventeenth century French mathematician and philosopher Blaise Pascal (1623-1662), who set forth his results in this area in his Traité du triangle arithmetique [303] (published after the author's death). Pascal generalized known results, and gave a number of new properties of the arithmetic triangle, which he formulated in nineteen theorems. [Figure 1 is an example from Pascal's work.] The various properties of the numbers generated in the arithmetic triangle were given by Pascal in descriptive form, rather than algebraically, but he made direct and significant use of the principles he had discovered, e.g., in the method of induction and the application of the arithmetic triangle to problems in the theory of probability.

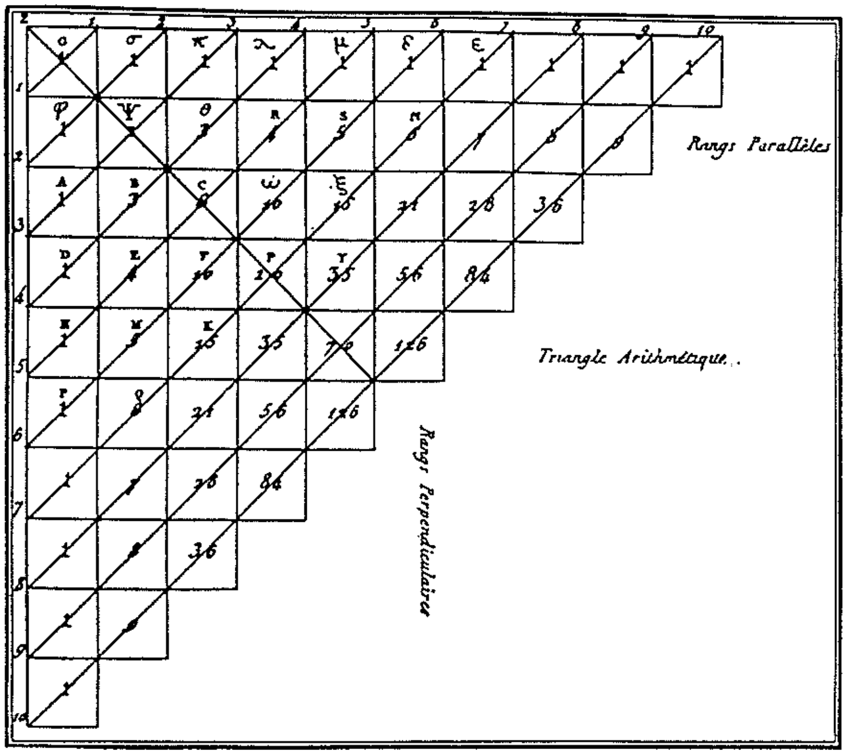

Figure 1

The arithmetic triangle and the additive rules for the formation of its entries were known in India virtually as we know them today. Its structure was also known to Omar Khayyám, the Persian mathematician, poet, and philosopher (c.1100). Later, the triangle appeared in China, and was depicted in a book of Chu Shin-Chien (1303).

In Europe, the arithmetic triangle had been known long before the publication of Pascal's work. It appeared, for example, on the title page of a book by A. Apian in 1529, and was used by many other mathematicians, among them M. Stifel (1544), G. Peletier (1549), K. Rudolph (1533), N. Tartaglia (1556), J. Cardan (1570), S. Stevin (1585), A. Girard (1629), W. Oughtred (1631), and G. Briggs (1633). More on the history of the Pascal triangle may be found in [17, 28, 31, 44, 50, 89, 90, 122, 141, 147, 241, 257, 265, 291, 292, 320, 379].

The familiar form of the table,

1	1	1	1	1	1	...
1	2	3	4	5	6	...
1	3	6	10	15	21	...
1	4	10	20	35	56	...
1	5	15	35	70	126	...
1	6	21	56	126	252	...

was published more than a century before Pascal's treatise in a work of the outstanding Italian mathematician Nicolo Tartaglia (1556). Subsequent investigations on the Pascal triangle and the binomial coefficients, and their connection with the origins and development of combinatorial analysis are connected with the names of Leibnitz, Bernoulli, Euler, Lucas, Legendre, and other prominent eighteenth- and nineteenth-century mathematicians.

Interest in the Pascal triangle has not diminished even up to the present, which accounts for the discovery of new and often unexpected properties related to divisibility and the distribution of the triangle's elements modulo a prime p, the construction and study of its fractals and graphs, and its application to important practical problems. We also depend on the triangle for a model in considering new types of arithmetic triangles, and rectangular, pyramidal, and other arithmetic tables.

The Pascal triangle is often presented in the form of an isosceles triangle whose sides are bordered by ones (Figure 2), and such that the remaining elements are the sums of the two entries just above to the left and right. The line numbered n consists of the coefficients in the binomial expansion of $(1+x)^n$. These coefficients are denoted in various ways in the literature, but here we will use the notation $\binom{n}{m}$, introduced as far back as Euler's time, and/or the notation C_m^n, which appeared in the nineteenth century.

								1								
							1		1							
						1		2		1						
					1		3		3		1					
				1		4		6		4		1				
			1		5		10		10		5		1			
		1		6		15		20		15		6		1		
	1		7		21		35		35		21		7		1	
.		.		.		.		.		.		.		.		.

Figure 2

The Pascal triangle may also be presented in right triangular form, as for instance,

	a							**b**		
1									1	
1	1							1	1	
1	2	1					1	2	1	
1	3	3	1			1	3	3	1	
	. . .						. . .			
	c							**d**		
1	1	1	1	...	...	1	1	1	1	
1	2	3		.	.		1	2	3	
1	3		.			.		3	1	
1		.					.		1	.
...							...			

Most common is the form

n \ m	0	1	2	3	4	.
0	1					
1	1	1				
2	1	2	1			
3	1	3	3	1		
4	1	4	6	4	1 .	
.	.	.	.	.	.	

Results on the properties of the Pascal triangle, including some questions of divisibility, may be found in Uspensky [50], in the literature on combinatorial analysis and number theory [4, 17, 21, 40, 41, 44, 45, 52, 54, 141, 255, 300, 316], and in mathematical reference books. The most complete description of the numerous elementary properties of the Pascal triangle is that of Green and Hamberg [162], with its many tables, figures, and diagrams, and interesting problems for independent study. Included, for example, is a table of prime factors of the binomial coefficients up through the 54^{th} row of the triangle.

We should also mention some results connected with direct applications of the Pascal triangle. T.M. Green [161] considered recurrent sequences connected with the triangle in the following way. Let the vertex of the triangle coincide with the origin of the usual coordinate system, and its elements with the lattice points of the first quadrant. This establishes a relation between the lattice points (x,y) and the elements of the Pascal triangle:

$$\binom{n}{r} = \frac{(x+y)!}{x!\ y!},$$

where $n=x+y$, $r=y$. Then, any set of parallel diagonals of the triangle having rational slopes gives rise to a recurrent sequence, the elements of which are sums of the triangle elements lying on the corresponding diagonals. That is, comparison of successive diagonals with the straight lines $ax+by=n$, where $k=-a/b$ is the given slope, $n=0,1,2,\ldots$, and $a,b=1,2,\ldots$, leads to the sequence T_0, T_1, T_2, ... satisfying the relation

$$T_n = T_{n-a} + T_{n-b},$$

where T_n is the sum of the numbers on the n^{th} diagonal. The case $a=2$, $b=1$ gives the Fibonacci sequence.

In a series of works, the Pascal triangle has also been directly employed in problems involving the expansion of functions. Thus, M. Bicknell [71], using the column elements of the triangle, found an expansion for an exponential generating function; the result is used to construct the series expansion for some specific functions.

D.C. Duncan [126] showed that the n^{th} diagonal of the isosceles Pascal triangle gives the coefficients in the McLaurin series expansion of $(1-x)^{-n}$ for all positive n and $|x| < 1$. This expansion was also obtained in the work of A.R. Pargeter [302]. We also note that this interesting expansion allows us to find with any degree of precision the value of $(1+x)^{-n}$ for $x < 1$ and n a positive integer.

Power series with coefficients situated in the vertical columns of the isosceles Pascal triangle are considered in the work of A.A. Fletcher [142]. The general expression of these expansions has the form

$$S_r = 1 + rx + \binom{r+2}{2} x^2 + \binom{r+4}{3} x^3 + \dots + \binom{2n+r-2}{n} x^n + \dots.$$

It can be shown that S_r satisfies the recurrence relation

$$S_{r+2} = \frac{1}{x}\left(S_{r+1} - S_r\right),\ r \geq 2,$$

where S_2 is the series corresponding to the central vertical column with elements $\binom{2n}{n}$, $n=1,2,\dots$, which are the coefficients in the expansion of $(1-4x)^{-1/2}$ for $x < 1/4$.

L.K. Jones [230] estimated the magnitude of the sums of the reciprocals of the elements of the Pascal triangle. For the n^{th} row, if we write

$$a_n = \sum_{k=0}^{n} \binom{n}{k}^{-1},$$

he established an upper estimate of the form $2+O(n)$, and a lower estimate of 2; $2+O(n^{-1})$?

consequently, $\lim_{n\to\infty} a_n = 2$. He also proved that for the k^{th} diagonal,

$$\sum_{n=k}^{\infty} \binom{n}{k}^{-1} = \frac{k}{k-1}.$$

In the work of A.R. Turguette [380, 381] the Pascal triangle is used in the study of Post sets and the solution of problems of many-valued logic.

Others have employed the Pascal triangle in the solution of various problems. Thus, D.A. Holton [214] showed that the dimensions of stable orbits are the coefficients in the polynomial $[1+(r-1)x]^n$, where in the case of the n-dimensional cube $r=2$, and the orbit dimensions are found in the Pascal triangle. In the work of H. Gorenflo [153], it is used to obtain the lifting force of pulley blocks. R.L. Morton [288] suggested a simple method of obtaining certain powers of 11 with the aid of the rows of the triangle. J. Wlodarski [395] showed that certain multiples of the elements of the triangle are related to two well-known numerical sequences in nuclear physics. G. Hoyer [226] suggested ways of deriving various formulas and relations among the binomial coefficients directly from the Pascal triangle. C.W. Trigg [378] considered properties of the sequence of elements of the fifth column of the triangle, as for example the length of the period of the sequence of low order digits, the sums of the digits, and so on.

In references [63, 76, 97, 110, 170, 193, 229, 242, 263, 294, 329] are discussions of elementary properties of the Pascal triangle, alternate versions of its development, and geometric interpretations.

The numbers of Fibonacci, Lucas, Catalan, Fermat, Stirling, and others may be derived and investigated by making use of the Pascal triangle directly [96, 112, 184, 192, 295, 298, 321, 327-329, 337, 394, 396].

1.2 BINOMIAL COEFFICIENTS AND THEIR GENERALIZATIONS

As we know, the elements of the Pascal triangle are the binomial coefficients, which were already known before the appearance of the Pascal triangle. However, Pascal was the first to define and to apply them [303]. Some references on the history of the binomial coefficients and the binomial theorem are [17, 36, 40, 41, 44, 50, 111, 122, 141, 241, 242, 268, 292].

The binomial coefficients are the simplest combinatorial objects, being defined as the number of distinct combinations of m elements out of n. They may be obtained from the generating function as the coefficients in the expansion of the expression

$$(1+x)^n = \sum_{m=0}^{n} \binom{n}{m} x^m, \tag{1.1}$$

where

$$\binom{n}{m} = \frac{n!}{m!\,(n-m)!}, \quad n = 0,1,2,\dots, m \le n.$$

The binomial coefficients satisfy the recurrence relation

$$\binom{n+1}{m} = \binom{n}{m-1} + \binom{n}{m}, \quad \binom{0}{0} = 1, \tag{1.2}$$

as well as the simple equalities

$$\binom{n}{0} = \binom{n}{n} = 1, \quad \binom{n}{m} = \binom{n}{n-m},$$
$$\sum_{m=0}^{n} \binom{n}{m} = 2^n, \quad \sum_{m=0}^{n} (-1)^m \binom{n}{m} = 0, \tag{1.3}$$

$$\binom{n+m}{l} = \sum_{k=0}^{l} \binom{n}{k}\binom{m}{l-k}, \binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2. \tag{1.4}$$

Hundreds of identities and relations among the binomial coefficients have been established; many of these may be found in [4, 21, 29, 42, 46, 52, 158, 233, 248, 292]. The greatest numbers of identities are collected in the books of J. Riordan [42], B.N. Sachkov [46], H.W. Gould [158], and E. Netto [292]. In recent decades, new relations among the binomial coefficient have also been obtained, some of which we mention below.

M. Boscarol [88] obtained for nonnegative integers m and n the relation

$$\sum_{i=0}^{m} \frac{\binom{n+i}{i}}{2^{n+i}} + \sum_{h=0}^{n} \frac{\binom{m+n-h}{m}}{2^{m+n-h}} = 2.$$

H. Scheid [338] proved that the number of distinct prime factors of the binomial coefficient $\binom{n}{m}$ is not less than (m log 2)/(log 2m) for $2 \le 2m \le n$.

S.M. Tanny and M. Zuker [371] studied the sequence of binomial coefficients of the form $\binom{n-r}{r}$ for $n \ge 0$, $0 \le r \le [n/2]$, and pointed out its importance for many combinatorial problems.

G. Zirkel [405] discussed a method for numerically approximating the binomial coefficients with the help of a table of areas under the normal curve approximating the corresponding binomial distribution.

In [376], C.A. Tovey discussed the problem of the existence of infinite sets of natural numbers N, each element of which is equal to t distinct binomial coefficients $\binom{n}{m}$, where n=0,1,2,..., and $1<m<[n/2]$. He showed that for t=2, the least such value is the number 120, which equals $\binom{10}{3}$ and $\binom{16}{2}$; the number 210, which equals $\binom{10}{4}$ and $\binom{21}{2}$, would also be in this set. The number 3003, for example, has three representations: $\binom{14}{6}$, $\binom{15}{5}$, $\binom{78}{2}$. For t=2, this problem is solved, i.e., it is known that there are infinitely many natural numbers N which have two representations as binomial coefficients.

G.H. Weiss and M. Dishon [391] proved that in the expansion

$$\frac{1}{2}\left[1 - u - v - \sqrt{1-2(u+v)+(u-v)^2}\right] = \sum_{r=1}^{\infty}\sum_{s=1}^{\infty} C_{r,s}\, u^r v^s$$

the values of the $C_{r,s}$ may be expressed in terms of binomial coefficients:

$$C_{r,s} = (r+s-1)^{-1}\binom{r+s-1}{r}\binom{r+s-1}{s}.$$

Various other new properties appear in references [49, 56, 91, 118, 128, 163, 183, 358, 393]. The binomial coefficients and their various identities and relations play a major role in the solution of many problems in mathematics, mechanics, and physics. They also serve as a model for various generalized binomial coefficients. Two of these, the generalized binomial coefficients of s^{th} order, $\binom{n}{m}_s$, and the multinomial coefficients, $(n;\, n_1, n_2, \ldots, n_s)$, will be discussed in detail in sections 1.3 and 1.4 of the present chapter; a few other generalizations we mention below.

S.W. Golomb [151] introduced the so-called "iterated binomial coefficients" by the scheme

$$(a_1) = a_1;\ (a_1;\ a_2) = \binom{a_1}{a_2};\ (a_1;\ a_2;\ a_3) = \left(\begin{matrix}\binom{a_1}{a_2}\\ a_3\end{matrix}\right),\ \ldots,$$

$$(a_1;\ a_2;\ \cdots;\ a_{k-1};\ a_k) = \left((a_1;\ a_2;\ \cdots;\ a_{k-1});\ a_k\right).$$

For these iterated binomial coefficients, for specified values k and a_i, $i = 1,2,\ldots,k$, the author establishes various identities, inequalities, transformation formulas, and asymptotic and other formulas and relations.

M. Sved [366] introduced a different kind of generalized binomial coefficient as follows. Let $S = [a_1, a_2, \ldots, a_n]$ be a set of n distinct elements. The "sequence" $A = a_1^{(m_1)}\, a_2^{(m_2)} \cdots a_n^{(m_n)}$ is formed from the elements of S taken with multiplicities $(m_1,m_2,\ldots,m_n)$, and the degree of A is the number $|m| = m_1 + m_2 + \cdots + m_n$. If we take the "subsequence" $B = a_1^{(k_1)}\, a_2^{(k_2)} \cdots a_n^{(k_n)}$, where $0 \le k_i \le m_i$, to be a subsequence of A, then the generalized binomial coefficient $G_r^n(m)$ is the number of such subsequences B of A ($G_r^n(m) = 0$ for $r < 0$ and $r > n$). In elementary number theory the introduction of these coefficients has the following meaning. From the factorization of a natural number into its prime factors we form a sequence, starting with the set of distinct prime divisors, and the degree of the sequence is the sum of the divisors occurring in the factorization. Then $G_r^n(m)$ enumerates the set of all divisors of fixed degree; this generalizes a known property of the binomial coefficients to the coefficients $G_r^n(m)$.

The Gaussian binomial coefficients, also known as the q-binomial coefficients are defined [48] by:

$$\begin{bmatrix} n \\ m \end{bmatrix}_q = \prod_{k=1}^{m} \frac{q^{n-k+1}-1}{q^k-1}, \quad 0<m\leq n \tag{1.5}$$

$$\begin{bmatrix} n \\ 0 \end{bmatrix}_q = 1, \quad \begin{bmatrix} n \\ m \end{bmatrix}_q = 0, \quad m<0, \ m>n, \tag{1.6}$$

where m,n are nonnegative integers and q is a real number. We know that the q-binomial coefficients occur in the expansion

$$\prod_{m=1}^{n} (1+q^{m-1}x) = \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix}_q q^{\frac{1}{2}m(m-1)} x^m, \tag{1.7}$$

from which it follows that the q-binomial coefficient is itself a polynomial in q, which for $q \to 1$ reduces to the ordinary binomial coefficient. These coefficients satisfy the recurrence

$$\begin{bmatrix} n+1 \\ m \end{bmatrix}_q = \begin{bmatrix} n \\ m \end{bmatrix}_q + \begin{bmatrix} n \\ m-1 \end{bmatrix}_q q^{n-m+1}, \quad \begin{bmatrix} n \\ 0 \end{bmatrix}_q = 1. \tag{1.8}$$

In [315] G. Polya and G.L. Alexanderson discuss various combinatorial interpretations and properties of the q-binomial coefficients, and construct their multinomial analogs.

M. Sved in [367] discusses known and new properties of the q-binomial coefficients, including their geometric significance, and gives for q=2,3,4,5 the triangular tables of these coefficients analogous to the Pascal triangle. Equations (1.5)-(1.8) summarize the basic relations for the q-binomial coefficients; these and others are to be compared with the corresponding formulas for ordinary binomial coefficients.

L. Carlitz [101] generalized various theorems for the q-binomial coefficients to the multinomial case. R.D. Fray [143] and F.T. Howard [224] studied the question of the divisibility of the q-binomials by prime divisors; we will take up divisibility questions at length in the next chapter.

Another generalization of the binomial coefficients is given by the so-called Fibonomial coefficients [57],

$$\left(\!\!\binom{n}{m}\!\!\right)_F = \frac{F_n F_{n-1} \cdots F_{n-m+1}}{F_m F_{m-1} \cdots F_1}, \tag{1.9}$$

where the F_n are the Fibonacci numbers [20], n and m are nonnegative integers, and

$$\left(\!\!\binom{n}{0}\!\!\right)_F = \left(\!\!\binom{n}{n}\!\!\right)_F = 1 \quad \text{for all } n = 0, 1, 2, \ldots .$$

In [57] G.L. Alexanderson and L.F. Klosinski also introduce the Gaussian Fibonomial coefficients

$$\left[\!\left[\begin{matrix} n \\ k \end{matrix}\right]\!\right] = \frac{(x^{F_n}-1)(x^{F_{n-1}}-1) \cdots (x^{F_{n-k+1}}-1)}{(x^{F_k}-1)(x^{F_{k-1}}-1) \cdots (x^{F_1}-1)}, \tag{1.10}$$

where n,k are nonnegative integers, and

$$\left[\!\left[\begin{matrix} n \\ 0 \end{matrix}\right]\!\right]_F = \left[\!\left[\begin{matrix} n \\ n \end{matrix}\right]\!\right]_F = 1, \; n = 0, 1, 2, \ldots .$$

These Gaussian Fibonomials satisfy a recurrence relation which for $x \to 1$ includes that of the Fibonomial coefficients, and similarly for other relations. They also examine the case of a more general Fibonacci sequence

$$g_{n+2} = p g_{n+1} + q g_n, \; n \geq 0, \tag{1.11}$$

where $g_0=0$, $g_1=1$, and p and q are arbitrary.

Other analogs and generalizations of the binomial coefficients will be discussed in 1.4 and 1.5, along with the corresponding analogs of the Pascal triangle.

1.3 GENERALIZED PASCAL TRIANGLES AND GENERALIZED BINOMIAL COEFFICIENTS

The generalized Pascal triangle of s^{th} order is the table of coefficients of powers of x in the expansion

$$(1+x+x^2+\dots+x^{s-1})^n = \sum_{m=0}^{(s-1)n} \binom{n}{m}_s x^m \quad, s \geq 2. \tag{1.12}$$

The coefficients $\binom{n}{m}_s$ are known as the generalized binomial coefficients of order s. For s=2, they become the ordinary binomial coefficients, $\binom{n}{m}_2 = \binom{n}{m}$, and the corresponding triangular table is <u>the</u> Pascal triangle. (We note that some authors speak of triangles of "kind" s rather than triangle of "order" s.) In the literature, the generalized Pascal triangle is sometimes referred to as the s-arithmetic triangle.

The generalized Pascal triangle of order s may be written, as is the Pascal triangle, in the form of a right triangle or an isosceles triangle. For example, we give the generalized Pascal triangles of order 3 and 4 in right triangle form:

n \ m	0	1	2	3	4	5	6	7	8	.
0	1									
1	1	1	1							
2	1	2	3	2	1					
3	1	3	6	7	6	3	1			
4	1	4	10	16	19	16	10	4	1	
.	.	.	.	.	.	.	.	.	.	.

n \ m	0	1	2	3	4	5	6	7	8	9	10	11	12	.
0	1													
1	1	1	1	1										
2	1	2	3	4	3	2	1							
3	1	3	6	10	12	12	10	6	3	1				
4	1	4	10	20	31	40	44	40	31	20	10	4	1	.
.	.	.	.	.	.	.	.	.	.	.	.	.	.	.

Figure 3a

In isosceles form, these are:

				1				
			1	1	1			
		1	2	3	2	1		
	1	3	6	7	6	3	1	
1	4	10	16	19	16	10	4	1

. . . .

						1						
					1	1	1	1				
			1	2	3	4	3	2	1			
	1	3	6	10	12	12	10	6	3	1		
1	4	10	20	31	40	44	40	31	20	10	4	1

. . . .

Figure 3b

In the first triangle (s=3) of Fig. 3a every element is equal to the sum of three elements in the preceding row: the number just above and its two neighbors to the left. In the zero-th column, all elements are ones, and we assume any missing elements to the left are zeros. Similarly, in the second triangle (s=4) each element is the sum of four elements in the preceding row: the number just above and its three neighbors to the left. In like fashion we fill in the rows of the generalized Pascal triangle of any order.

Dozens of papers have been devoted to the properties and applications of the generalized Pascal triangle and generalized binomial coefficients of order s. We will give some of these references after we list some of the basic properties of the generalized binomial coefficients of order s.

The generalized binomial coefficient $\binom{n}{m}_s$ is the number of different ways of distributing m objects among n cells where each cell may contain at most s-1 objects.

We also note the recurrence relation for the generalized binomial coefficients:

$$\binom{n+1}{m}_s = \sum_{k=0}^{s-1} \binom{n}{m-k}_s, \binom{n}{0}_s = 1. \tag{1.13}$$

For s=2, this coincides with the recurrence relation (1.2) for the ordinary binomial coefficients. The generalized binomial coefficients satisfy many equalities, identities, and other relations analogous to those for the binomial coefficients. For example,

$$\left.\begin{aligned} &\binom{n}{0}_s = \binom{n}{n}_s = 1, \quad \binom{n}{m}_s = \binom{n}{(s-1)n-m}_s, \\ &\sum_{m=0}^{(s-1)n} \binom{n}{m}_s = s^n, \quad \sum_{m=0}^{(s-1)n} (-1)^m \binom{n}{m}_s = \begin{cases} 0, & s=2t \\ 1, & s=2t+1. \end{cases} \end{aligned}\right\} \tag{1.14}$$

The relation among the generalized binomial coefficients in successive triangles has the form:

$$\left.\begin{aligned} &\binom{n}{m}_{s+1} = \sum_{k=0}^{n} \binom{n}{k} \binom{k}{m-k}_s, \text{ where } s \geq 2, \\ &\text{and} \quad \binom{k}{m-k}_s = 0 \quad \text{for } k < \frac{m}{s}. \end{aligned}\right. \tag{1.15}$$

The generalized binomial coefficient of order s may be expressed in terms of the binomial coefficients as:

$$\binom{n}{m}_s = \sum_{k=0}^{[m/s]} (-1)^k \binom{n}{k} \binom{n+m-sk-1}{n-1}. \tag{1.16}$$

We introduce for the multinomial coefficient the notation

$$(n;m_1,m_2,\ldots,m_{s-1}) = \frac{n!}{(n-m_1)!\,(m_1-m_2)!\ldots(m_{s-1}-m_{s-2})!\,m_{s-1}!}$$

in place of the usual $(n;\, n_1,\, n_2,\, \ldots,\, n_s)$; more detail will appear in section 1.6. Then it is true that

$$\binom{n}{m}_s = \sum (n;m_1,m_2,\ldots,m_{s-1}), \tag{1.17}$$

where $n\geq 0$, $0\leq m\leq(s-1)n$, $s\geq 3$, and the summation is over all m_k such that $m_1+m_2+\cdots+m_{s-1}=m$, $m_k\leq m_{k-1}$.

Let $C_{n,s} = \sup_m \binom{n}{m}_s$. Then for any n and $s\geq 2$, the correct asymptotic formula is

$$\lim_{n\to\infty} C_{n,s}\frac{\sqrt{n}}{s^n} = \sqrt{\frac{6}{\pi(s^2-1)}}. \tag{1.18}$$

The derivation of (1.13)-(1.18) is fairly straightforward and is omitted here.

In the Pascal triangle of order s, denote by $N_{n,s}$ the number of generalized binomial coefficients in the row numbered n, and by $Q_{n,s}$ the total number of coefficients in the triangle up to and including row n; then

$$N_{n,s} = n(s-1)+1, \quad Q_{n,s} = \frac{1}{2}(n+1)[(s-1)n+2]. \tag{1.19}$$

For $s=2$, $N_{n,2}=n+1$, and $Q_{n,2} = \frac{1}{2}(n+1)(n+2)$.

The generalized binomial coefficients have other interesting properties, as well; in succeeding chapters we will consider their divisibility properties, and the construction of their fractals and graphs. The applications of these triangles and coefficients in various mathematical contexts originated in the 1950's, and below we list in chronological order some works which were fundamental in this period and up to the present.

In considering these works, we must emphasize the original articles of J.E. Freund [144], and J.E. Freund and A.N. Pozner [145], in which they construct the generalized Pascal triangle, set forth the recurrence (and other) relations for the generalized binomial coefficients (which they denote by $N_m(r,k)$), and apply the results to some occupancy problems. J.D. Bankier [64] also used the results of [144] to find the coefficients in the expansion of $(x^2-x)(1+x+x^2)^k$.

V.E. Hoggatt and M. Bicknell [200,203] obtained difference relations and derived formulas for the sums of the elements in the generalized Pascal triangle which lie on the diagonals. A.K. Gupta in [164] explicitly expressed generalized binomial coefficients of arbitrary order by means of binomial coefficients. J.M. Deshouillers [117] derived asymptotic formulas for the generalized binomial coefficients, with integral estimates of their increase with increasing n.

V.E. Hoggatt and G.L. Alexanderson [197] worked out a method for determining partial sums of generalized binomial coefficients:

$$S(n,s,q,r) = \sum_{i=0}^{N} \binom{n}{r+iq}_s, \quad N=\left[\frac{(s-1)n-r}{q}\right].$$

In the special cases s=2, q=3,4,5,8; s=3, q=5, the expressions for the sums take the form of simple formulas involving the Lucas numbers, or the Pell-Lucas numbers, or their powers. These partial sums are also considered, for s=2,3,4,6, in C. Smith and V.E. Hoggatt [354-356].

T.B. Kirkpatrick [239] took the ascending diagonals of the generalized Pascal triangle to be the lines of a new triangle; iterating this operation R times, he obtains the additive triangle of order s and "degree" R. He then shows that if the diagonal sums of the elements of this triangle form the sequence $\{T_i\}_1^\infty$, this sequence has the recurrence relation

$$T_{N+(k-1)R+1} = T_N + T_{N+R} + T_{N+2R} + \cdots + T_{N+(k-1)R},$$

where $k \geq 2$, $R \geq 1$, and $T_1 = T_2 = \cdots = T_{R+1} = 1$.

In [78-80], R.C. Bollinger considers a number of properties of generalized Pascal triangles (there called Pascal-T triangles) and their coefficients. In [78] he constructs (modified) Fibonacci sequences of order k and uses them to solve various enumeration problems, which he calls "k-in-a-row" problems. In [79] the connection between the generalized binomial coefficients and the multinomials is found to have the form

$$C_m(n,k) = \sum \binom{n}{n_1, n_2, \cdots, n_m},$$

where the sum is taken over all $n_1, n_2, \ldots, n_m$ satisfying $n_1+n_2+\cdots+n_m=m$ and $0n_1+1n_2+\cdots+(m-1)n_m=k$. Also given is the recurrence relation

$$C_m(n,k) = \sum_{j=0}^{n} \binom{n}{j} C_{m-1}(j,k-j). \tag{1.20}$$

In [80] Bollinger generalized the Mann-Shanks primality condition [274] for a natural number m to be a prime. Mann and Shanks gave this novel criterion in terms of the displaced entries of the Pascal triangle as follows. Each row of the Pascal triangle is displaced successively two places to the right, so that the n+1 entries in row n occupy columns m=2n to m=3n. Also, we underline the entries in row n which are divisible by n. Then the criterion is: column number m is a prime if and only if all the entries in column m are underlined. The table below shows how this works for the Pascal triangle (the triangle of order 2).

n \ m	0	1	2*	3*	4	5*	6	7*	8	9	10	11*	12	13*	14	15	16	17*	.
0	1																		
1			<u>1</u>	<u>1</u>															
2					1	<u>2</u>	1												
3							1	<u>3</u>	<u>3</u>	1									
4									1	<u>4</u>	6	<u>4</u>	1						
5											1	<u>5</u>	<u>10</u>	<u>10</u>	<u>5</u>	1			
6													1	<u>6</u>	15	20	15	<u>6</u>	.
7															1	<u>7</u>	<u>21</u>	<u>35</u>	.
8																	1	<u>8</u>	.
.																			.

Using (1.20) and a theorem of G. Ricci [322], Bollinger in [80] shows that if the same displacements are applied to the generalized Pascal triangle of order three (so that the 2n+1 elements in row n occupy columns m=2n to m=4n), and the entries are underlined in the same way, then it is again true that the column number m is a prime if and only if all the entries in column m are underlined. The table below shows how this works for the triangle of order three. He also conjectured that the criterion is true for the generalized Pascal triangle of any order.

n \ m	0	1	2*	3*	4	5*	6	7*	8	9	10	11*	12	13*	14	15	16	17*	·
0	1																		
1			$\underline{1}$	$\underline{1}$	$\underline{1}$														
2					1	$\underline{2}$	3	$\underline{2}$	1										
3							1	$\underline{3}$	$\underline{6}$	7	$\underline{6}$	$\underline{3}$	1						
4									1	$\underline{4}$	10	$\underline{16}$	19	$\underline{16}$	10	$\underline{4}$	1		
5											1	$\underline{5}$	$\underline{15}$	$\underline{30}$	$\underline{45}$	51	$\underline{45}$	$\underline{30}$	·
6													1	$\underline{6}$	21	50	$\underline{90}$	$\underline{126}$	·
7															1	$\underline{7}$	28	$\underline{77}$	·
8																	1	$\underline{8}$	·
·																			·

R.C. Bollinger and C.L. Burchard in [81] showed there is, for the generalized binomial coefficients, an analog of Lucas's Theorem for the binomial coefficients, namely,

$$C_m(n,k) \equiv \sum_{(s_0,\ldots,s_r)} \prod_{i=0}^{r} C_m(n_i,s_i) \ (mod\ p),$$

where p is a prime, $n = (n_r n_{r-1} \cdots n_1 n_0)_p$, $k = (k_t k_{t-1} \cdots k_1 k_0)_p$, $0 \le n_i < p$, $0 \le k_i < p$, $0 \le k \le (m-1)n$, and the summation is over all s_i for which $s_0 + s_1 p + \cdots + s_r p^r = k$, $0 \le s_i \le (m-1)n$. If we denote by $N_m(n,p)$ the number of generalized binomial coefficients for which $C_m(n,k) \not\equiv 0$ (mod p) and apply the extended Lucas's Theorem, the authors found exact formulas for $N_m(n,p)$ in the cases m=p and $m=p^\ell$. Let $(p-1)n = (a_r a_{r-1} \cdots a_1 a_0)_p$; then

$$N_m(n,p) = (1+a_0)(1+a_1)\cdots(1+a_r),$$

$$N_m(n,p^\ell) = N_p(n(p^\ell-1)/(p-1),p).$$

They also established, for the generalized Pascal triangle of order p, that for large n "almost all" coefficients $C_p(n,k)$ are divisible by p.

Other questions connected with the application of the generalized binomial coefficients and generalized Pascal triangle of order s are discussed in [119, 154, 164, 212, 231, 232, 243, 287, 308, 314, 357].

1.4 LUCAS, FIBONACCI, CATALAN, AND OTHER ARITHMETIC TRIANGLES

In sections 1.1 and 1.3 we discussed Pascal triangles and generalized Pascal triangles of order s. We now turn our attention to the construction and application of other forms of arithmetic triangles: the triangles associated with the names of Lucas, Fibonacci, Catalan, Stirling, and others.

M. Feinberg [138] constructed the arithmetic triangle whose elements are the coefficients in the expansion of $(a+2b)(a+b)^{n-1}$; the result is what might be called the Lucas triangle, in which the sums of the elements on the ascending diagonals give the sequence of Lucas numbers 1,3,4,7,11,18,29,....

The Lucas triangle and its properties were studied in detail by H.W. Gould and W.E. Greig [160]. In this triangle (nine rows of which are shown below), the elements satisfy

n \ k	0	1	2	3	4	5	6	7	8	9	·
1	1	2									
2	1	3	2								
3	1	4	5	2							
4	1	5	9	7	2						
5	1	6	14	16	9	2					
6	1	7	20	30	25	11	2				
7	1	8	27	50	55	36	13	2			
8	1	9	35	77	105	91	49	15	2		
9	1	10	44	112	182	196	140	64	17	2	
·	·	·	·	·	·	·	·	·	·	·	·

see pg 39

the recurrence relation

$$A(n+1,k) = A(n,k) + A(n,k-1), \tag{1.21}$$

with initial conditions A(1,0) = 1, A(1,1) = 2, and A(n,k) = 0 for k<0 or k>n. The relation between the numbers A(n,k) and the binomial coefficients is

$$A(n,k) = \binom{n}{k} + \binom{n-1}{k-1}. \tag{1.22}$$

There are also four criteria given, the proofs being based on the properties of the Lucas triangle and its elements, for deciding whether a given natural number $d \geq 2$ is a prime.

V.E. Hoggatt [194] constructed a new triangle from the Lucas triangle by shifting the i^{th} column down k places (k=1,2,3,...), and derived various results, including the Lucas numbers, for the elements of this triangle.

H. Hosoya [216] constructed the arithmetic triangle (Figure 4) for the numbers $\{f_{m,n}\}$ satisfying the equations

$$\left.\begin{aligned} f_{m,n} &= f_{m-1,n} + f_{m-2,n} \\ f_{m,n} &= f_{m-1,n-1} + f_{m-2,n-2}\ , \quad m \geq 2,\ m \geq n \geq 0 \end{aligned}\right\}, \tag{1.23}$$

with initial conditions $f_{0,0} = f_{1,0} = f_{1,1} = f_{2,1} = 1$. He showed that $f_{m,n} = f_n f_{m-n}$ $(m \geq n \geq 0)$, where f_n is the n^{th} Fibonacci number, and called the resulting triangle a Fibonacci triangle. He studied the topological properties of its graph, obtained using the triangle, and applied the results to the classification of chemical formulas.

1

1 1

2 1 2

3 2 2 3

5 3 4 3 5

8 5 6 6 5 8

13 8 10 9 10 8 13

21 13 16 15 15 16 13 21

. . . .

Figure 4

J. Turner [382] suggested and studied what he called the Fibonacci-T triangle.

J. Šána in [335] considered a sequence $\{g_{m,n}\}$ like that of Hosoya [216],

$$\left.\begin{aligned} g_{m,n} &= g_{m-1,n} + g_{m-2,n} \\ g_{m,n} &= g_{m-1,n-1} + g_{m-2,n-2}, \quad m \geq 2,\ m \geq n \geq 0 \end{aligned}\right\}, \tag{1.24}$$

with initial conditions $g_{0,0}=2$, $g_{1,0}=1$, $g_{1,1}=1$, $g_{2,1}=2$, and constructed the arithmetic triangle in Figure 5, which he called a Lucas triangle. It has properties analogous to those obtained in [216]; some of these are investigated, and also the graph equivalent to the Lucas triangle is constructed.

							2							
						1		1						
					3		2		3					
				4		3		3		4				
			7		5		6		5		7			
		11		8		9		9		8		11		
	18		13		15		14		15		13		18	
29		21		24		23		23		24		21		29

. . . .

Figure 5

The elements of these Fibonacci and Lucas triangles have a recurrence relation of the form (1.23) or (1.24), in which each element is the sum of two preceding elements on an ascending or descending diagonal. Other relevant references here are [11, 138, 207].

M. Sved [367] also discussed the arithmetic triangle whose elements are the Gaussian binomial coefficients $\begin{bmatrix} n \\ r \end{bmatrix}_q$, and obtained the Gaussian triangles for $q = 2,3,4,5$.

L.W. Shapiro [342] constructed the arithmetic triangle whose elements are the numbers $B_{n,k}$ satisfying the recurrence relation

$$B_{n,k} = B_{n-1,k-1} + 2B_{n-1,k} + B_{n-1,k+1}$$

with the conditions $B_{1,1}=1$, $B_{n,0}=0$, $B_{n,m}=0$, $m>n+1$. The first several rows are shown below.

n \ k	1	2	3	4	5	6	.
1	1						
2	2	1					
3	5	4	1				
4	14	14	6	1			
5	42	48	27	8	1		
6	132	165	110	44	10	1	
.	.	.	.	.	.	.	.

The sequence of numbers $\{C_n\} = \{1,2,5,14,42,132,\ldots\}$ in the first column are the Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$. It is not difficult to show that the solution of the recurrence relation is $B_{n,k} = \frac{k}{n}\binom{2n}{n-k}$, and for k=1,

$$B_{n,1} = C_n = \frac{1}{n}\binom{2n}{n-1} = \frac{1}{n+1}\binom{2n}{n}. \tag{1.25}$$

The article also shows that the $B_{n,k}$ may be expressed as a sum of products of Catalan numbers by means of the formula $B_{n,k} = \sum C_{i_1} C_{i_2} \cdots C_{i_k}$, where the summation is over values for which $i_1+i_2+\cdots+i_k=n$. As a result, each element of the Catalan triangle may be expressed in terms of the Catalan numbers; the name arises because of this connection. Various properties of the Catalan triangle, analogous to those of the Pascal triangle, are also discussed.

D.G. Rogers [328] studied questions connected with renewal sequences which led to various generalized Pascal and Catalan triangles. These are connected with the introduction of the generalized Catalan sequence $\{C_t(n)\}$, where

$$C_t(n) = \frac{1}{tn+1}\binom{(t+1)n}{n}, \quad n \geq 0, \ t \geq 0. \tag{1.26}$$

For $t=1$, we have $C_1(n) = C_n$, the Catalan numbers. The introduced sequence and the related generalized Catalan triangle are applied in the solution of some combinatorial problems.

A number of authors have constructed arithmetic triangles by choosing as their elements the numbers which satisfy a recurrence relation of the form

$$f(n+1,m) = p(n,m)f(n,m-1) + q(n,m)f(n,m) \tag{1.27}$$

with appropriate coefficients p, q and initial conditions.

C. Cadogan [98] considered the case of this equation where $p, q \in R$ and with initial conditions $f(0,k) = d_k \in R$; he found then,

$$f(n,k) = \sum_{m=0}^{n} \binom{n}{m} p^{n-m} q^m f(0,n-m). \tag{1.28}$$

By choosing as the values of the d_k the cases: $d_0=1$, $d_k=0$ $(k\neq 0)$; $d_0=a$, $d_{-1}=d$, $d_k=0$ $(k\neq 0,-1)$; $d_k = a(m-1)^{-k}$, $k\leq 0$ $(d_k=0,\ k>0)$, the author constructs the corresponding Pascal triangle, a triangle with elements which form an arithmetic progression, and a triangle with elements which form a geometric progression. The results are also generalized to the three-dimensional case.

In [240] M. Klika considered (1.27) for integer-valued functions $p(m)$, $q(n)$ and initial conditions $f(0,0)=1$, $f(i,j)=0$ for $i<j$, $j<0$, where i,j are nonnegative whole numbers, and constructed the corresponding generalized Pascal triangle $P(p,q)$. For $p=q=1$ we get the Pascal triangle itself, and for $p=m+1$, $q=1$ the triangle whose elements are the Stirling numbers of the second kind; the author also discusses the triangle $P(p,q)$ for various other conditions. In [227], S.K. Janardan and K.G. Janardan also investigate this kind of Stirling triangle.

H. Ouellette and G. Bennett [301] considered the triangle whose elements are the absolute values of the Stirling numbers of the first kind.

In a dissertation [24] V.N. Dokina studied the special cases of (1.27) consisting of: $p=1$, $q=\mu_n$; $p=1$, $q=\mu_m$; $p=1$, $q=\mu_n+\mu_m$ and initial conditions equal to unity. He formed the corresponding triangles consisting of generalized Stirling numbers of the first and second kind, and Lah numbers. He also extended the discussion to the case when $p(n,m)$ and $q(n,m)$ are not merely numerical, but are operators operating on a linear space of polynomials in t with real coefficients. In these cases the elements of the generalized Pascal triangle are functions of t. The results are applied to various probability problems, problems connected with population growth, and others.

V.L. Jannelli [228] constructed and studied the triangle formed from the coefficients in the expansion of $(x+a_1)(x+a_2)\cdots(x+a_n)$. For $a_1=0$, $a_2=1$, ..., $a_n=-(n-1)$ the author arrives at the triangle of Stirling numbers of the first kind; other cases, when $a_k=k$, are discussed in [120, 133].

In [333], M. Rumney and E.J. Primrose studied the triangle whose rows are the coefficients in the expansions of 1, 1+x, (1+x)(2+x), (1+x)(2+x)(3+x),...; a portion of this triangle is:

n \ m	0	1	2	3	4	5	.
0	1						.
1	1	1					
2	2	3	1				
3	6	11	6	1			
4	25	50	35	10	1		
5	120	274	225	85	15	1	
.	.	.	.	.	.	.	.

The elements, denoted by $e_{n,m}$, satisfy the recurrence

$$e_{n+1,m} = e_{n,m-1} + (n+1)e_{n,m}, \tag{1.29}$$

which gives a simple rule for forming the triangle. It is also not difficult to show that

$$\sum_{m=0}^{n} e_{n,m} = (n+1)!,$$

and other relations are given. The authors also study in great generality the triangle composed of the numbers in the harmonic series.

C.W. Puritz [318] generalized the binomial coefficient $\binom{n}{m}$ to the case of n negative, using the notation C(n,m). He used the arithmetic and symmetry properties of the recurrence

$$C(n,m) = C(n+1,m) - C(n,m-1)$$

and found that

$$C(-n,m) = (-1)^m \binom{n+m-1}{m},$$

writing out a portion of the complementary Pascal triangle as below.

n \ m	0	1	2	3	4	.
.	.	.	.	.	.	.
-4	1	-4	10	-20	35	.
-3	1	-3	6	-10	15	.
-2	1	-2	3	-4	5	.
-1	1	-1	1	-1	1	.
0	1	0	0	0	0	.
1	1	1	0	0	0	.
2	1	2	1	0	0	.
3	1	3	3	1	0	.
4	1	4	6	4	1	.
.	.	.	.	.	.	.

Other variants of the Pascal triangle, in which the elements come from the coefficients in the expansion of

$$(a\mp b)(a\pm b)(a\mp b)\ \cdots\ (a\pm(-1)^n b),$$

were considered by P. Sahmel [334]. For n=2m and n=2m+1, we obtain the corresponding expansions of $(a^2-b^2)^n$ and $(a\mp b)(a^2-b^2)^m$.

a

a b

a a b

a a a+b b

a a 2a+b a+b b

a a 3a+b 2a+b a+2b b

a a 4a+b 3a+b 3a+3b a+2b b

. . . .

Figure 6

H.W. Gould [156] constructed and studied the Pascal triangle (Fig. 6) in which the elements are defined by the recurrence relation

$$C_m^{n+1} = C_{m-1}^n + \frac{1+(-1)^m}{2} C_m^n,\ n\geq 1,\ m\geq 0, \tag{1.30}$$

and the conditions

$$C_0^0 = C_0^1 = a,\ C_1^1 = b,\ C_m^n = 0 \text{ for } m>n,\ m<0.$$

The cases a=b=1 and a=1, b=2 are studied in detail. The coefficients are denoted by A_m^n in the first case and by B_m^n in the second, m=0,1,2,...,n. Using (1.30), the following values are calculated for n=0,1,2,....

$$A_{2k}^n = \binom{n-k}{k},\ A_{2k+1}^n = \binom{n-k-1}{k},\ A_0^n = 1,\ A_1^1 = 1,$$

$$B_{2k}^n = \frac{n}{n-k}\binom{n-k}{k},\ B_{2k+1}^n = \frac{n-1}{n-k-1}\binom{n-k-1}{k},\ B_0^n = 1,\ B_1^1 = 2.$$

The A_m^n and B_m^n can be used to express the Fibonacci and Lucas numbers as

$F_{n+2} = \sum_{m=0}^{n} A_m^n$ and $L_{n+1} = \sum_{m=0}^{n} B_m^n$, for $n \geq 0$. It should be noted that

$$\sum_{m=0}^{n} C_m^n = aF_{n+1} + bF_n,\ n\geq 0,$$

$$\sum_{m=0}^{n} (-1)^m C_m^n = aF_{n-2} + bF_{n-3},\ n\geq 1.$$

In [77] M.B. Boisen considers two tables A and B,

$$\begin{array}{rrrr} & & & a_{44}\ , \\ & & a_{33} & a_{34}\ , \\ & a_{22} & a_{23} & a_{24}\ , \\ a_{11} & a_{12} & a_{13} & a_{14}\ , \end{array}$$

$$\begin{array}{llll} b_{11} & & & \\ b_{12} & b_{22} & & \\ b_{13} & b_{23} & b_{33} & \\ b_{14} & b_{24} & b_{34} & b_{44} \\ \cdot & \cdot \quad \cdot & \cdot & \cdot \end{array}$$

where the a's and b's are integers, and defines the superposition of A on B, which then generates the sequence $C = \{c_1, c_2, \ldots\}$ with elements of general form

$$c_i = \sum_{k=0}^{\left[\frac{i}{2}\right]} \sum_{l=k+1}^{i-k} a_{k+1,l}\, b_{l,i-k}. \tag{1.31}$$

With the sequence $\{c_i\}$ defined he takes the following approach. Let

$$P_k(x) = a_{1k} + a_{2k}x + \cdots + a_{kk}x^{k-1}$$

and let $G_k(x)$ be the generating function of the k^{th} column of table B, $k=0,1,2,\ldots$. Then $\sum_{i=0}^{\infty} P_i(x)G_i(x)$ is the generating function of $\{c_i\}$. Several examples are considered in which A and B are chosen to be the Pascal triangle or its generalizations; in one of these, for example, the sequence $\{c_i\}$ turns out to be the Fibonacci sequence.

C.K. Wong and T.W. Maddocks [399] studied the numbers $M_{k,r}$ satisfying the recurrence relation

$$M_{k+1,r+1} = M_{k+1,r} + M_{k,r+1} + M_{k,r} \tag{1.32}$$

with initial conditions $M_{0,0} = M_{1,0} = M_{1,1} = 1$. The numbers $M_{k,r}$, for which the condition $M_{k,r} = M_{r,k}$ clearly holds, constitute an analog of the Pascal triangle (Fig. 7).

1

1 1

1 3 1

1 5 5 1

1 7 13 7 1

1 9 25 25 9 1

1 11 41 63 41 11 1

. . . .

Figure 7

In this triangle, k is the number of the line parallel to the right side of the triangle, k=0,1,2,..., and r is the number of the line parallel to the left side of the triangle, r=0,1,2,.... If we introduce the number of the line parallel to the base of the triangle and denote it by n=0,1,2,..., then the law of formation of the elements is simple: any value in the n^{th} row is the sum of the two elements above in the $(n-1)^{st}$ row and the element directly above in the $(n-2)^{nd}$ row. Thus, 41 is the sum 9+25+7. The author also shows that the sums of the elements on the ascending diagonals form the "Tribonacci" numbers, 1, 1, 2, 4, 7, 13, 24, 44, ...

M. Bicknell-Johnson in [73] writes on the Leibnitz harmonic triangle (Fig. 8), whose diagonals are the products of the reciprocals of the n^{th} row elements by the reciprocals of the row numbers (assumed to begin with one) in the Pascal triangle. The sums of the row elements, and of the ascending diagonal elements are found.

$$\frac{1}{1} \quad \frac{1}{2} \quad \frac{1}{3} \quad \frac{1}{4} \quad \frac{1}{5} \quad \frac{1}{6} \quad \cdot \cdot$$

$$\frac{1}{2} \quad \frac{1}{6} \quad \frac{1}{12} \quad \frac{1}{20} \quad \frac{1}{30} \quad \cdot$$

$$\frac{1}{3} \quad \frac{1}{12} \quad \frac{1}{30} \quad \frac{1}{60} \quad \cdot$$

$$\frac{1}{4} \quad \frac{1}{20} \quad \frac{1}{60} \quad \cdot$$

$$\frac{1}{5} \quad \frac{1}{30} \quad \cdot$$

$$\frac{1}{6} \quad \cdot$$

$$\cdot$$

Figure 8

D. Logothetti [259] formed a new, truncated triangle (Fig. 9) (without a vertex) by taking groups of four elements at the vertices of a rhombus in the Pascal triangle and forming the numbers

$$I(n,k) = \binom{n-2}{k-1} + \binom{n-2}{k} + \binom{n-1}{k-1} + \binom{n-1}{k},$$

where $n=1,2,3,\ldots$, $k=0,1,2,\ldots,n$.

2	1							
2	3	1						
2	5	4	1					
2	7	9	5	1				
2	9	16	14	6	1			
2	11	25	30	20	7	1		
2	13	36	55	50	27	8	1	
.	.	.	.	.	.	.	.	.

see pg 25

Figure 9

Although there is no symmetry, the triangle and its elements have interesting properties, as, e.g.,

$$I(n,k) = I(n-1,k-1) + I(n-1,k),$$

$$\sum_{k=0}^{n} I(n,k) = 3 \cdot 2^{n-1}, \sum_{k=0}^{n} (-1)^k I(n,k) = 0,$$

$$(2x+1)(x+1)^{n-1} = \sum_{k=0}^{n} I(n,k) x^{n-k}.$$

The truncated triangle of Fig. 9 may be considered as a special case of a more general triangle (Fig. 10).

$$\begin{array}{ccccccccccc} & & & & a & & 1 & & & & \\ & & & a & & a+1 & & 1 & & & \\ & & a & & 2a+1 & & a+2 & & 1 & & \\ & a & & 3a+1 & & 3a+3 & & a+3 & & 1 & \\ a & & 4a+1 & & 6a+4 & & 4a+6 & & a+4 & & 1 \\ . & & . & & . & & . & & . & & . \end{array}$$

Figure 10

Here, the elements G(n,k) are the coefficients of x^{n-k} in the expansion of $(ax+1)(x+1)^{n-1}$, and satisfy the recurrence G(n,k) = G(n-1,k-1) + G(n-1,k), G(n,0)=a, G(n,n)=1.

H. Harborth [173] considered triangles composed of plus and minus signs, to every pair of which is assigned a (+) or a (-) sign according to Pascal's rule. Such a triangle for a given n contains $N = \frac{1}{2}n(n+1)$ signs and we assign for that n the signs in the first row. His results solve the Steinhaus problem [53] on the existence of numbers n, where $n \equiv 0,3 \pmod 4$, for which the generated triangle has plus signs as half of its elements. For example, for n=11 Figure 11 shows such a triangle, with 33 of its 66 elements being plus signs. Variants of this problem were also solved and studied by M. Bartsch [65].

\+ − + + − + − − + + +

− − + − − − + − + +

\+ − − + + − − − +

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Figure 11

Arithmetic triangles of Stirling numbers of the first kind $S_n^{(m)}$, satisfying the recurrence $S_{n+1}^{(m)} = S_n^{(m-1)} - nS_n^{(m)}$, where $S_1^{(1)} = 1$, and $S_n^{(m)} = 0$ for $n<1$, $m<1$, $m<n$, have the form

$$
\begin{array}{cccccccc}
1 & & & & & & \\
\underline{1} & 1 & & & & & \\
2 & \underline{3} & 1 & & & & \\
\underline{6} & 11 & \underline{6} & 1 & & & \\
24 & \underline{50} & 35 & \underline{10} & 1 & & \\
\underline{120} & 274 & \underline{225} & 85 & \underline{15} & 1 & \\
. & . & . & . & . & . & .
\end{array}
$$

pg 32

where the negative elements are underlined.

Triangles of Stirling numbers of the second kind $\sigma_n^{(m)}$, satisfying $\sigma_{n+1}^{(m)} = \sigma_n^{(m-1)} + m\sigma_n^{(m)}$, where $\sigma_1^{(1)} = 1$ and $\sigma_n^{(m)} = 0$ for $n<1$, $m<1$, $m<n$, have the form

1							
1	1						
1	3	1					
1	7	6	1				
1	15	25	10	1			
1	31	90	65	15	1		
1	63	301	350	140	21	1	
.	.	.	.	.	.	.	.

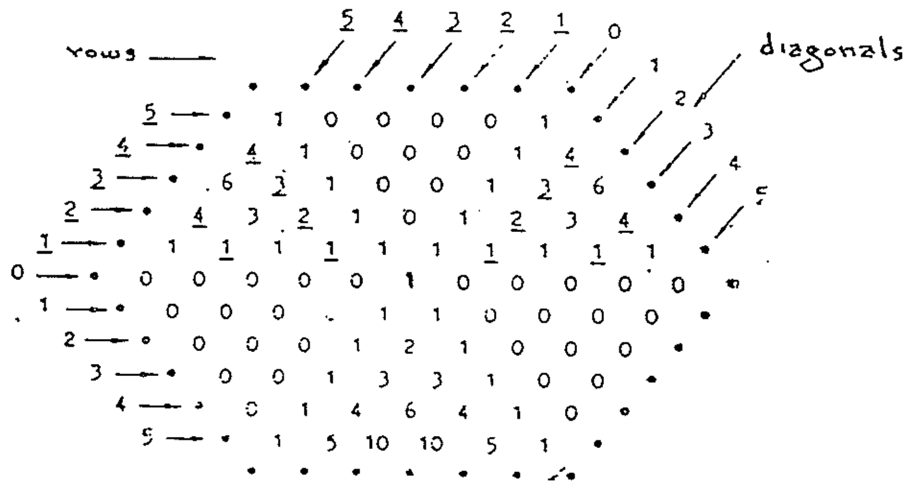
P. Hilton and J. Pederson [188-190] obtained new arithmetic and geometric properties of the binomial coefficients $\binom{n}{m}$, including the case of negative values of m and n, by extending the definition as follows:

$$\binom{n}{r} = 0 \text{ for } n \geq 0,\ r>n,\ r<0,$$

$$\binom{-n}{r} = (-1)^r \binom{n+r-1}{r} \text{ for } n>0,\ r\geq 0,$$

$$\binom{-n}{-r} = (-1)^{r-n} \binom{r-1}{n-1} \text{ for } n>0,\ r>0.$$

As a result of this generalization the authors construct the hexagon (Fig. 12) consisting of the binomial coefficients for both positive and negative values of m and n, and call it the Pascal hexagon.



They considered the geometric properties of the Pascal hexagon and other figures such as the arrangement in Figure 13.

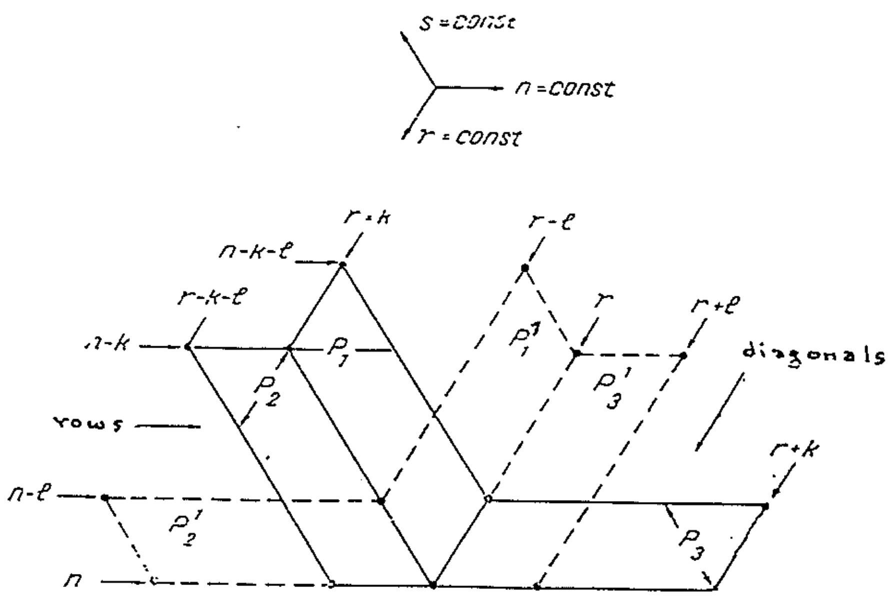


Figure 13

In [191] they also discuss the Leibnitz harmonic coefficients and the q-binomials for positive and negative values of m and n, as well as describing the properties of the Pascal hexagon and constructing the generalized star of David, the harmonic triangle, and the Pascal hexagon.

K. Dilcher [123] replaced the partial differential equation $u_{xx} = u_{tt} + u_t$ by a difference equation and, after appropriate normalization, constructed the triangle in Figure 14, which is a kind of generalized Pascal triangle of order three (discussed in 1.3). The elements $C_{n,m}$ of this triangle (cf. Fig. 14) in the n^{th} row are combinations of three elements in the $(n-1)^{st}$ row and one in the $(n-2)^{nd}$ row, according to the recurrence

$$C_{n,m} = C_{n-1,m-1} + C_{n-1,m} + C_{n-1,m+1} - 2C_{n-2,m}, \; C_{0,0} = 1.$$

					1					
				1	1	1				
			1	2	1	2	1			
		1	3	2	3	2	3	1		
	1	4	4	4	5	4	4	4	1	
1	5	7	6	9	7	9	6	7	5	1
.	.	.	.	.	.	.	.	.	.	.

Figure 14

These coefficients may also be generalized by introducing the parameters λ, ν, in which case they satisfy

$$C_{n,m}^{\lambda,\nu} = \left(1 + \frac{\nu-1}{n}\right)\left(C_{n-1,m-1}^{\lambda,\nu} + C_{n-1,m}^{\lambda,\nu} + C_{n-1,m+1}^{\lambda,\nu}\right)$$

$$- \left(1 + 2\,\frac{\nu-1}{n}\right) \lambda\, C_{n-2,m}^{\lambda,\nu} ,$$

where $C_{n,m}^{\lambda,\nu} = C_{n,n-m}^{\lambda,\nu}$; for $\lambda=2$, $\nu=1$, $C_{n,m}^{2,1} = C_{n,m}$. The properties of the $C_{n,m}^{\lambda,\nu}$ and their arithmetic triangles are considered in detail.

Arithmetic triangles also appear in the references [69, 109, 120, 125, 281, 357, 385, 396].

1.5 PASCAL PYRAMIDS AND TRINOMIAL COEFFICIENTS

As we have seen, the binomial coefficients $\binom{n}{m}$ arise as a result of the expansion of $(1+x)^n$, and can be written in the form of a Pascal triangle of one sort or another. If we write the binomial in terms of x_0, x_1, the expansion takes the form

$$(x_0 + x_1)^n = \sum_{m=0}^{n} \binom{n}{m} x_0^{n-m}\, x_1^{m}.$$

If we denote the trinomial coefficients by $(n;\, m_1,\, m_2)$, where n, m_1, m_2 are nonnegative integers, and set

$$(n;\; m_1,m_2) = \frac{n!}{(n-m_1)!\;(m_1-m_2)!\;m_2!} , \tag{1.33}$$

we can write the expansion of $(x_0+x_1+x_2)^n$ in the form

$$(x_0 + x_1 + x_2)^n = \sum_{m_1=0}^{n} \sum_{m_2=0}^{n} (n;\ m_1, m_2)\ x_o^{n-m_1}\ x_1^{m_1-m_2}\ x_2^{m_2}\ . \tag{1.34}$$

The trinomial coefficients are often written as

$$(n;\ n_1,\ n_2,\ n_3) = \frac{n!}{n_1!\,n_2!\,n_3!},\ n_1 + n_2 + n_3 = n\ ; \tag{1.35}$$

however, in many contexts in which one constructs and uses multi-harmonic, multi-wave, and other polynomials, the representation (1.33) is more convenient than (1.35), since (1.33) orders the polynomial terms and trinomial coefficients of the Pascal pyramid and its cross sections (Figure 15).

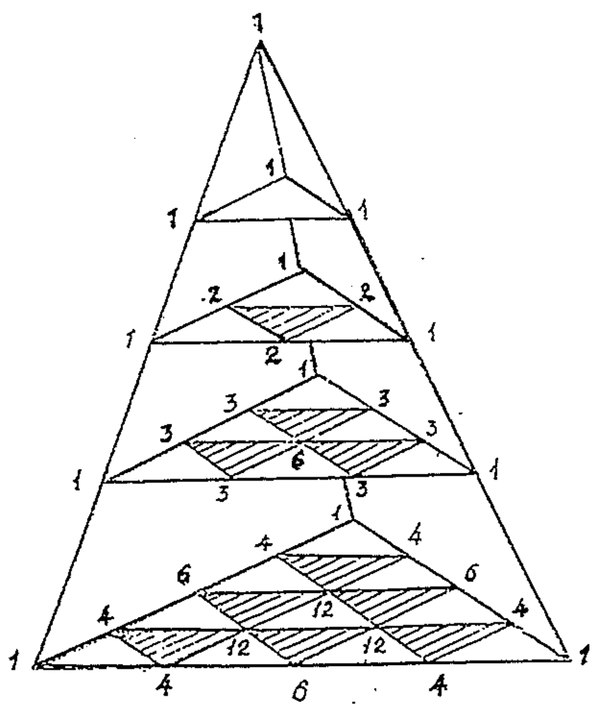


Figure 15

It is not difficult to show that the trinomial coefficients of (1.33) satisfy the recurrence relation

$$(n+1;\ m_1,m_2) = (n;\ m_1,m_2) + (n;\ m_1-1,\ m_2) + (n;\ m_1-1,\ m_2-1) \tag{1.36}$$

with initial conditions (0; 0,0) = 1, and where $(n;\ m_1,m_2) = 0$ for $n<0$, m_1 or $m_2<0$, $m_1>n$, $m_2>m_1$. We can also verify from Figure 15 the presence of three axes of symmetry.

Much like the binomial coefficients, the trinomial coefficients satisfy the conditions (n; 0,0) = (n; n,0) = (n; n,n) = 1, and the equations

$$\left.\begin{aligned} (n;\ m_1,m_2) &= (n;\ m_1,m_1-m_2), \\ (n;\ m_1,m_2) &= (n;\ n-m_1+m_2,m_2), \\ (n;\ m_1,m_2) &= (n;\ n-m_2,\ n-m_1). \end{aligned}\right\} \tag{1.37}$$

Some special sums are

$$\sum_{m_1=0}^{n}\sum_{m_2=0}^{m_1} (n;\ m_1,m_2) = 3^n,\ \sum_{m_1=0}^{n}\sum_{m_2=0}^{m_1} (-1)^{m_2}(n;\ m_1,m_2) = 1 \tag{1.38}$$

and the analog of the Cauchy summation formula is

$$\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{k_1} (n_1;\ k_1,k_2)(n_2;\ m_1-k_1,m_2-k_2) = (n_1+n_2;\ m_1,m_2). \tag{1.39}$$

The Pascal pyramid can be considered as a regular tetrahedron, or as a pyramid with unequal dihedral angles as shown. In the n^{th} cross section (n=0,1,2,...) parallel to the base, which is itself a triangle, we arrange the $\frac{1}{2}(n+1)(n+2)$ coefficients $(n;\ m_1,m_2)$. At the outer edges the entries are ones, and each of the sides (faces) is itself a Pascal triangle. The relation (1.36) allows us to conclude that each interior element of a cross section is the sum of three elements in the triangular element which forms the (n-1)st cross section.

The rule for constructing the elements in the nth cross section can also be thought of in terms of the equation

$$(n;\ m_1,m_2) = \binom{n}{m_1}\binom{m_1}{m_2}, \tag{1.40}$$

where $n=0,1,2,\ldots$; $m_1=0,1,2,\ldots,n$; $m_2=0,1,2,\ldots,m_1$. This says, in effect, that we get the entries in the nth cross section by taking the ordinary Pascal triangle for that n, rotating its last row counterclockwise through the angle $\pi/2$, and then multiplying the resulting row entries on the rows of the triangle, as shown for n=4 by the example in Figure 16(a); the result is Figure 16(b). If the cross section is considered an equilateral triangle its axes of symmetry are as shown in Figure 17.

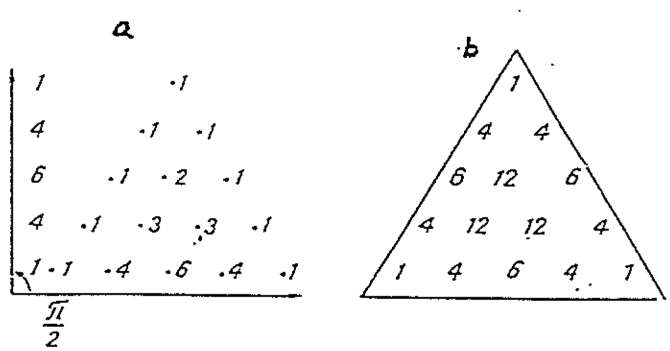


Figure 16

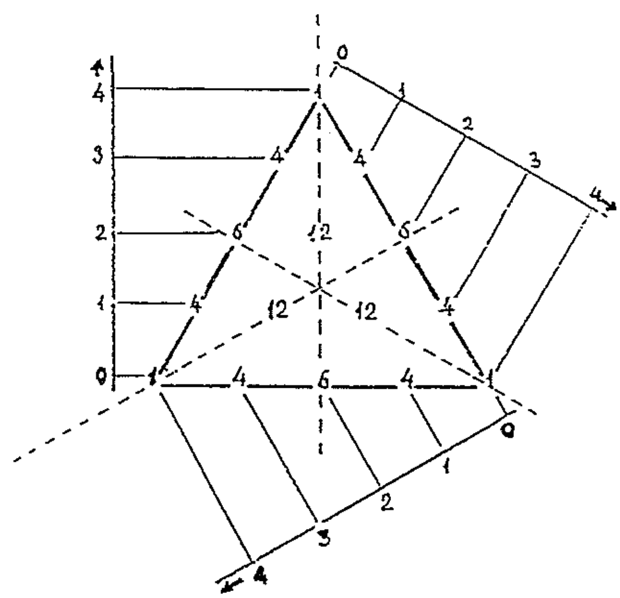


Figure 17

When the cross sections are taken to be right triangles, an algorithm for constructing the entries is given in [6].

These ideas can be extended to the multi-dimensional case. In particular, the coefficients in the expansion of $(x_0+x_1+x_2+x_3)^n$ form a four-dimensional Pascal pyramid, bounded by five tetrahedrons. Analogously, we can think of multi-dimensional Pascal pyramids bounded by Pascal pyramids of dimension one less.

Pascal pyramids and hyperpyramids have been used in the solution of problems on probability theory, polyharmonic polynomials, generalized Fibonacci sequences, and so on. The ideas of the construction and use of these objects appear in the works of many authors, and below we give a brief chronological survey of some of these papers and the results obtained.

One of the first occurrences of the Pascal pyramid, apparently, is in the work of E.B. Rosenthal [330], who suggested and wrote out the trinomial coefficients in an array which he called a Pascal pyramid.

The author of the present volume worked out an algorithm for constructing the cross sections of the Pascal pyramid, discussed the multi-dimensional case, and applied the results to the construction of harmonic and polyharmonic polynomials, and polynomial solutions to some problems in elasticity theory [5,6].

G. Garcia [146] geometrically formed the Pascal pyramid in the course of considering the coefficients in the expansion of $(a+b+c)^n$, and discussed the possibility of extending the example to the four-dimensional case.

A note of M. Basil [66] considers some properties of the trinomial coefficients written in the form of a Pascal pyramid.

R.L. Keeney [236] derived an algorithm for the construction of the elements in the cross sections, noted their symmetry, and the possibility of extension to the multi-dimensional case.

A note of S. Mueller [289] discusses the relations among the trinomial coefficients by means of the Pascal pyramid.

J. Staib and L. Staib [359] gave an algorithm for constructing the cross section elements in the trinomial case, and discussed the question of extension to the multi-dimensional case.

V.E. Hoggatt [195] discussed Pascal pyramids having as the elements of their cross sections the numbers in the expansion of $(a+b+c)^n$, and gave as the generating function of the columns

$$G^*_{m,n} = \frac{x^{pm+qn} b^m c^n \binom{m+n}{n}}{(1-ax)^{m+n+1}} .$$

He also showed that

$$G = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} G^*_{m,n} = \frac{1}{1-ax-bx^p-cx^q} ,$$

and particular choices of the parameters give the generating function for the Fibonacci numbers (a=1, b+c=1, p=q=2), the Tribonacci numbers (a=b=c=1, p=2, q=3), generalized Fibonacci numbers (a=b=c=1, p=t+1, q=2t+1), and other sequences.

In [341] A.G. Shannon used the Pascal pyramid to construct the Tribonacci numbers by summing the diagonal elements.

M. Alfonso and P. Hartung [58] emphasized the analogies between the Pascal pyramid and Pascal triangle, and used this approach to obtain some results in probability theory.

J.F. Putz [319, 320] discussed in detail the extension of the Pascal triangle, and the construction and properties of the pyramid, there called a Pascal polytope. He obtained a graphical representation, and also established the possibility of applying the polytope to the study of k-Fibonacci sequences. In [319], he generalized all 19 theorems of Pascal to the multi-dimensional case.

J. Shorter and F.M. Stein [343] constructed the Pascal tetrahedron, examined its properties, and discussed the possibility of extension to the multi-dimensional case.

The question of studying some special function values with the help of the pyramid is discussed in [267] by H.F. Lucas.

R.C. Bollinger [79] obtained some results on generalized binomial coefficients of order m, discussed the construction of the pyramid and its cross sections, and gave a method for computing the trinomial and multinomial coefficients.

1.6 MULTINOMIAL COEFFICIENTS AND PASCAL HYPERPYRAMIDS

As we know, the multinomial (also called polynomial) coefficients occur in the expansion of the polynomial $(x_0+x_1+\cdots+x_{s-1})^n$; the usual notation is $(n;\, n_1,n_2,\ldots,\, n_s)$, which stands for the form

$$(n;\ n_1,n_2,\cdots,n_s) = \frac{n!}{n_1!\,n_2!\cdots n_s!}\ , \tag{1.41}$$

$$\text{where}\quad n_1+n_2+\cdots+n_s = n\ . \tag{1.42}$$

The combinatorial sense of the multinomial coefficient may be expressed as: $(n; n_1,n_2,\ldots,n_s)$ gives the number of ways that n different objects may be distributed among s cells, where the number of objects in the k^{th} cell is n_k, $k=1,2,\ldots,s$.

Here we will denote the multinomial coefficients by $(n; m_1,m_2,\ldots,m_{s-1})$, defined as

$$(n;\ m_1,m_2,\cdots,m_{s-1}) = \frac{n!}{(n-m_1)!(m_1-m_2)!\cdots(m_{s-2}-m_{s-1})!m_{s-1}!}, \tag{1.43}$$

and introduced in [6]. Using this definition, condition (1.42) will be satisfied, and we have the ordered expansion

$$\begin{aligned} H_s(x,n) &\equiv (x_0 + x_1 + \cdots + x_{s-1})^n \\ &= \sum_{m_1=0}^{n} \sum_{m_2=0}^{m_1} \cdots \sum_{m_{s-1}=0}^{m_{s-2}} (n;\ m_1,\ m_2,\ \cdots,\ m_{s-1}) \cdot \\ &\qquad x_0^{n-m_1} x_1^{m_1-m_2} \cdots x_{s-2}^{m_{s-2}-m_{s-1}} x_{s-1}^{m_{s-1}} . \end{aligned} \tag{1.44}$$

We use (1.43) and (1.44) in the ordered construction of the Pascal hyperpyramid of multinomial coefficients, polyharmonic and other polynomial systems, and in discussing relations among the coefficients themselves. The multinomial expansion (1.44) appears in the literature of combinatorial analysis, algebra, statistics, and number theory [22, 23, 25, 38, 47].

We mention some basic formulas (omitting the proofs) for the multinomial coefficients (1.43), and then turn to a review of some references devoted to multinomial coefficients, the multinomial theorem, and connections with related matters.

The recurrence relation is

$$(n+1;\ m_1, m_2, \cdots, m_{s-1}) = (n;\ m_1, m_2, \cdots, m_{s-1}) + (n;\ m_1-1, m_2, \cdots, m_{s-1})$$
$$+ (n;\ m_1-1, m_2-1, \cdots, m_{s-1}) + \cdots$$
$$+ (n;\ m_1-1, m_2-1, \cdots, m_{s-2}-1, m_{s-1})$$
$$+ (n;\ m_1-1, m_2-1, \cdots, m_{s-1}-1), \qquad (1.45)$$

with initial condition (0; 0,0,...,0) = 1, and $(n;\ m_1,\ldots,m_{s-1}) = 0$ for $n<0$, or $m_k<0$, for at least one value of k, and for $m_1>n$, $m_k>m_{k-1}$. The coefficients also satisfy the conditions

$$(n;\ 0,0,\cdots,0) = (n;\ n,0,\cdots,0) = (n;\ n,n,0,\cdots,0) = \cdots$$
$$= (n;\ n,n,\cdots,n) = 1,$$

and s equalities, the first and last of which are

$$\left.\begin{aligned} (n;\ m_1,\cdots,m_{s-1}) &= (n;\ m_1,m_2,\cdots,m_{s-2},m_{s-2}-m_{s-1}) \\ &\ \vdots \\ (n;\ m_1,\cdots,m_{s-1}) &= (n;\ n-m_{s-1},n-m_{s-2},\cdots,n-m_2,n-m_1). \end{aligned}\right\} \qquad (1.46)$$

We have also the summation formulas

$$\sum_{m_1=0}^{n} \sum_{m_2=0}^{m_1} \cdots \sum_{m_{s-1}=0}^{m_{s-2}} (n;\ m_1,\cdots,m_{s-1}) = s^n, \qquad (1.47)$$

$$\sum_{m_1=0}^{n} \sum_{m_2=0}^{m_1} \cdots \sum_{m_{s-1}=0}^{m_{s-2}} \delta(m_1,\cdots,m_{s-1})(n;\ m_1,\cdots,m_{s-1})$$
$$= \begin{cases} 0, & s=2l \\ 1, & s=2l+1, \end{cases} \qquad (1.48)$$

where $\delta(m_1,\cdots,m_{s-1}) = (-1)^{m_1-m_2+m_3-m_4+\cdots+(-1)^s m_{s-1}}$.

We obtain (1.47) from (1.44) by taking $x_0=x_1=\cdots=x_{s-1}=1$, and (1.48) by taking

$x_0=x_2=\cdots=x_{r(s)}=1$ and $x_1=x_3=\cdots=x_{r(s)+1}=-1$, where $r(s) = 2\left[\frac{s-1}{2}\right]$.

The multi-dimensional analog of the Cauchy summation formula is

$$\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{k_1}\cdots\sum_{k_{s-1}=0}^{k_{s-2}}(n_1;\, k_1,\cdots,k_{s-1})\,(n_2;\, m_1-k_1, m_2-k_2,\cdots,m_{s-1}-k_{s-1})$$

$$= (n_1+n_2;\, m_1,\cdots,m_{s-1}), \tag{1.49}$$

where $(n;m_1,\cdots,m_{s-1})=0$ if at least one of the $m_k<0$.

As mentioned in the book of E. Netto [292], the multinomial theorem was first mentioned in a letter from Leibnitz to Johann Bernoulli in 1695. Its proof has been given by a number of authors using various methods, one of which is the combinatorial argument. There are many works devoted to the study of the multinomial coefficients, and reviews of earlier results may be found in [41, 122, 292, 322, 372]. Below, we give in chronological order a survey of some results from recent decades.

With the coefficients written in the form

$$\frac{n!}{i_1!\,i_2!\cdots i_r!\,(n-k)!}\;,\quad i_1+i_2+\cdots+i_r=k$$

for $i_1\le i_2\le\cdots\le i_r\le n-k\le n-2$, P. Erdős and I. Niven [136] obtained a formula for $f(x)$, the number of coefficients less than the positive number x, of the form

$$f(x) = (1+\sqrt{2})x^{1/2} + O(x^{3/2}).$$

Two works of S. Tauber [372, 373] are devoted to the study of the multinomial coefficients in the form (1.41). The first gives material of a historical nature and establishes basic summation formulas; the second contains proof of some summation formulas similar to those for binomial coefficients.

M. Abramson [55] discussed the multinomial coefficients in the form (1.41), and established the basic formulas and relations by using their combinatorial interpretations.

In the author's book [6], he studies the multinomial coefficients in the form (1.43), establishes their basic relations, and gives applications to the construction and study of multi-dimensional harmonic and polyharmonic polynomials.

V.E. Hoggatt and G.L. Alexanderson [196] defined for each multinomial coefficient (1.41) the s(s+1) neighboring coefficients for which their product is N^m, where N is an integer such that there exists a partition of these coefficients into s sets of (s+1) coefficients whose product equals N, and where any such set may be obtained from another such set by a cyclic permutation of indices.

In [296] A. Nishiyama discussed values f(n) which occur as sums of multinomial coefficients $(n; j_1,\ldots,j_p)$, when the j_n satisfy some condition. For example, for p=2 we have the binomial coefficients, and if the condition is that they should lie on the Pascal triangle diagonals, then f(n) is the Fibonacci sequence.

D.L. Hilliker [185] extended the binomial theorem for complex values, established by Abel in the binomial case, to the multinomial theorem, and gave [186] various representations of the expansion of $(a_1+a_2+\cdots+a_r)^n$ by means of binomial coefficients.

A.N. Philippou [309] proved a theorem on the representation of the terms of the Fibonacci sequence of order k, $\{f_n^{(k)}\}$ by means of multinomial coefficients. The result is

$$f_{n+1}^{(k)} = \sum (n_1+n_2+\cdots+n_k;\ n_1,n_2,\cdots,n_k),\ n\geq 0,$$

where the summation is over all nonnegative numbers $n_1,\ldots,n_k$ for which $n_1+2n_2+\cdots+kn_k=0$.

Further results on the multinomial coefficients connected with questions of divisibility and other properties may be found in [14-16, 18, 67, 121, 225, 261, 275, 348, 352], a review of which we turn to in the following chapter.